Math 256 – Final examination

University of British Columbia

April 28, 2015, 3:30 pm to 6:00 pm

Last name (print): First name:

ID number:

This exam is "closed book" with the exception of a single 8.5"x11" formula sheet. Calculators or other electronic aids are not allowed.

Page	Score	Max
2		8
3		8
4		8
5		12
6		8
7		6
8		10
Total		60

Multiple choice

1. [**2 pts**] How many distinct possibilities are there for $\lim_{t\to\infty} x(t)$ where $x(t)$ solves $x' - x = e^{-t}$?

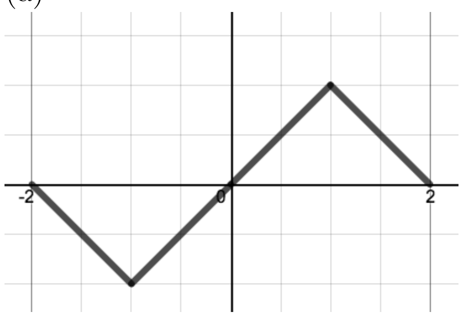
   (a) 1 (b) 2 (c) 3 (d) 4

2. [**2 pts**] Which of the following pairs of functions are not independent?

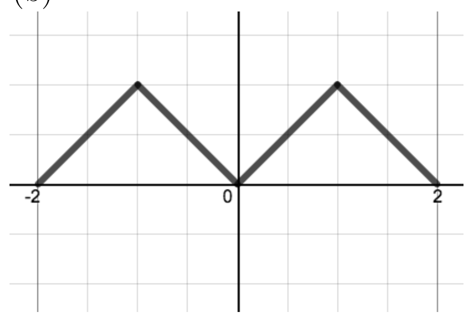
   (a) $\cos(2t), \sin(2t)$
   (b) e^t, e^{3t}
   (c) $t, \frac{1}{t}$
   (d) $\sin(t), \cos\left(t + \frac{\pi}{2}\right)$

3. [**2 pts**] Which of the following functions has a Fourier series with only sine terms of the form $b_n \sin(n\pi x/2)$ with $b_n = 0$ for all n even?

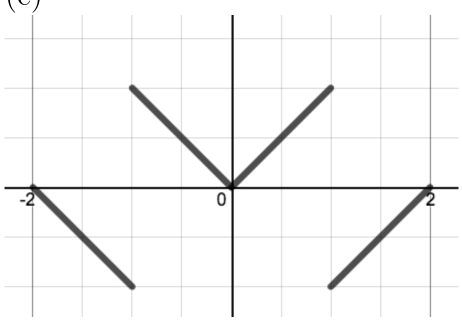
(a)



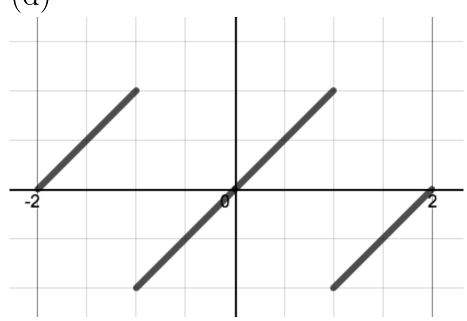
(b)



(c)



(d)



4. [**2 pts**] The solution to the equation $y'' + 16\pi^2 y = f(t)$, where $f(t)$ is a piecewise constant and periodic function with period T and $2T$ is a positive integer, ...

   (a) ...can be found by representing both $f(t)$ and the particular solution $y_p(t)$ as a Fourier series and using the Method of Undetermined Coefficients.
   (b) ...can be written as an infinite sum of sine functions.
   (c) ...will eventually (for t large) look like a sine function with period $1/2$.
   (d) ...is described by all of the options above.
   (e) ...is not described by any of the options above.

Written answers

1. **[4 pts]** A particular solution to the equation $y'' + \alpha y' + \beta y = f(t)$ is given by $y(t) = 2e^{-3t} + e^{-t} + 4\cos(2t)$. What are the constants α and β? What is $f(t)$?

2. **[4 pts]** Using Reduction of Order, find a second solution to the equation $y'' - y = 0$ given that $y_1(t) = e^t$ is a solution. No points will be given for getting the solution by other methods.

3. [**8 pts**]

(a) Find the general solution to the equation $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{pmatrix} a & a-2 \\ a-2 & a \end{pmatrix}.$$

Note: one of the eigenvalues is $\lambda = 2$.

(b) For what values of a is the steady state a saddle? Unstable node?

4. [**8 pts**] Consider the function $f(t) = u_2(t)(t-2) - u_3(t)(t-3) + u_3(t)(t-2)^2$.

   (a) Sketch the graph of $f(t)$.

   (b) Calculate the Laplace transform of $f(t)$.

5. [**4 pts**] Omar owns a restaurant and would like to estimate how many people will be in the restaurant at any time during an evening so he knows how many tables he'll need. Every time a train stops of the Skytrain station next door, 6 people enter the restaurant. Between trains, nobody comes in. The train comes through every 8 minutes with the first one arriving at 5 pm when the restaurant opens and last one arriving at 11 pm (46 trains). People finish eating and leave the restaurant at a rate proportional to the number of people currently in the restaurant with rate constant 1/120 min^{-1}. Write down a model (ODE and IC) for the number of people $P(t)$ in the restaurant at time t measured in minutes after 5 pm. You do not need to solve the equation.

6. [**8 pts**] The Laplace transform of the solution to a differential equation is given by

$$Y(s) = \frac{5}{(s+1)(s^2+4)}.$$

(a) Find the solution $y(t)$ by inverting $Y(s)$.

(b) Provide a differential equation and initial values that have a transformed solution $Y(s)$.

7. [**6 pts**]

(a) What is the form of the particular solution that you would use to solve the equation below using method of undetermined coefficients? You do not need to determine the coefficients.

$$y'' + 4y' + 4y = 2te^{-2t}$$

(b) Solve the equation $y'' + 2y' + 3y = e^{-t}$

8. **[5 pts]** Calculate the Fourier series for $f(x) = u_{-1}(x) - u_1(x)$ where $u_c(x)$ is the Heaviside function at $x = c$ and the domain of the function is $[-2, 2]$.

9. **[5 pts]** Solve the Diffusion equation $u_t = 7u_{xx}$ with boundary conditions $u(0, t) = 0$ and $u_x(4, t) = 5$ and initial condition $u(x, 0) = 5x + \sin(3\pi x/8)$.

This page is for rough work and doodles. It will not be marked.