Solving the Heat/Diffusion equation: $u_t = D u_{xx}$

To solve the Heat/Diffusion equation $u_t = D u_{xx}$ we guess that much like $\bar{x}' = A\bar{x}$ has a solution

$$\bar{x}(t) = c_1 e^{\lambda_1 t}\bar{v} + \cdots + c_n e^{\lambda_n t}\bar{v}_n$$ consisting of a sum of vectors scaled by an exponential function of time,

$u_t = D u_{xx}$ has a solution $u(x,t) = \sum_{n=1}^{\infty} c_n e^{\lambda_n t} \underset{\frown}{v_n}(x)$

Where vectors are replaced by functions of x.

Separation of variable reveals this. Assume $u(x,t)$ has a simple form like: $u(x,t) = T(t) \cdot Y(x)$. Plugging in to $u_t = D u_{xx}$:

$u_t = T' \cdot Y$, $D u_{xx} = D T \cdot Y''$ so $T' \cdot Y = D T \cdot Y''$ and

$$\frac{T'(t)}{D T(t)} = \frac{Y''(x)}{Y(x)}.$$ For a function of t to be equal to a function of x, both must constant. $\frac{T'}{DT} = \frac{Y''}{Y} = \beta$

Thus $T' = \beta D T$ so $T(t) = e^{\beta D t}$.

What about solving $Y'' = \beta Y$? ...

Solving $Y'' = \beta Y$ depends on the sign of β.

For $\beta > 0$, $Y(x) = c_1 e^{\sqrt{\beta}x} + c_2 e^{-\sqrt{\beta}x}$

For $\beta < 0$, $Y(x) = c_1 \sin(\sqrt{-\beta}\,x) + c_2 \cos(\sqrt{-\beta}\,x)$

For $\beta = 0$, we have $T(t) = 1$ so the corresponding $Y(x)$ will be steady states. And $Y''(x) = 0$ is solved by $Y(x) = ax + b$.

All solutions to the Heat/Diffusion equation that we'll deal with consist of linear combonations of these (usual infinite sums).

Boundary conditions dictate which ones get included. The exponentials $e^{\sqrt{\beta}x}$ and $e^{-\sqrt{\beta}x}$ will never appear for us.

How to determine acceptable values of β.

The H/D equation requires one IC (because of single time derivative) and two boundary conditions (because of two space derivatives).

BCs come in two types. (at least, two that we'll discuss)

(i) Dirichlet : $u(a,t) = 0$

(ii) Neumann : $u_x(a,t) = 0$

The BCs determine whether $\sin\sqrt{-\beta}\,x$, $\cos\sqrt{-\beta}\,x$ and/or $ax+b$ are involved and for what values of β. In general, we start with

$$u(x,t) = ax+b+\sum_n a_n e^{\lambda_n t}\cos\left(\sqrt{-\beta_n}\,x\right)+\sum_n b_n e^{\lambda_n t}\sin\left(\sqrt{-\beta_n}\,x\right)$$

where we assume there will be many β values and we list them with index n (hence the vague summation for now).

Ex.1 $u(0,t)=0=u(3,t)$. The first equation, $u(0,t)=0$, is satisfied easily if we set $a=b=a_n=0$ so that $u(x,t)=\sum_n b_n e^{\lambda_n t}\sin\left(\sqrt{-\beta_n}\,x\right)$.

To get $u(3,t)=0$ choose β_n:

$$u(3,t) = \sum_n b_n e^{\lambda_n t}\sin\left(3\sqrt{-\beta_n}\right) = 0$$

Here we make sure $\sin\left(3\sqrt{-\beta_n}\right)=0$ for each β_n so set

$3\sqrt{-\beta} = n\pi$ where n can be $1,2,3\cdots$. So $\beta = -\frac{n^2\pi^2}{3^2}$. This means

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{3^2}Dt}\sin\left(\frac{n\pi}{3}x\right).$$

Finally, we have to figure out the b_n. As always with the arbitrary coefficients, we look to the IC...

Figuring out the b_n.

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{3^2}Dt} \sin\left(\frac{n\pi}{3}x\right)$$

For a given IC "$u(x,0) = f(x)$ on $0 \le x \le 3$", we need

$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right) = f(x)$. This looks like a Fourier Series for $f(x)$ but it includes only sin terms (no cos, no const) and it is obviously periodic. But we only know the values of $f(x)$ on $[0,3]$. Let's compare ~~forms~~:

The Fourier series for some $g(x)$ that has period P looks like

$$g(x) = C + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{P} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{P}$$

where $a_n = \frac{2}{P}\int_{\text{one period}} g(x)\cos\frac{2n\pi x}{P}\,dx$, $b_n = \frac{2}{P}\int_{\text{one period}} g(x)\sin\frac{2n\pi x}{P}\,dx$

and $C = \frac{1}{P}\int_{\text{one period}} g(x)\,dx$

and we have $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right)$. We need to extend

$f(x)$ beyond $[0,3]$ to make $f_{ext}(x)$ so that

(i) $f_{ext}(x) = f(x)$ for $0 \le x \le 3$,

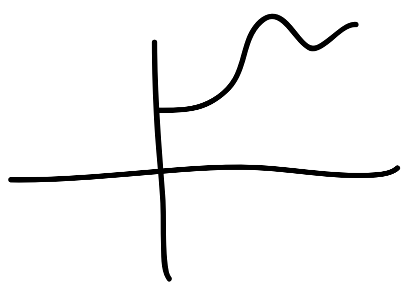
(ii) the Fourier series for $f_{ext}(x)$ has $c=0$ and $a_n = 0$ for $n=1,2,3\cdots$;

(iii) the extended function has period 6. $\left(\frac{2}{p} = \frac{1}{3}\right)$

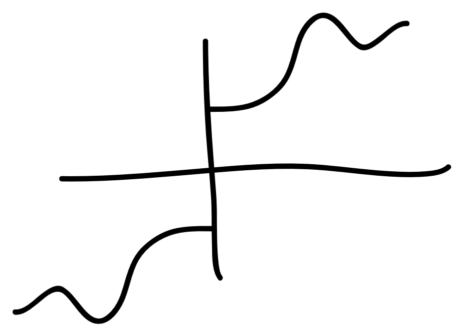
We can ensure (i) by defining $f_{ext}(x) = f(x)$ on $[0,3]$.

Next, for (ii) we define $f_{ext}(x) = -f(-x)$ for $-3 \le x < 0$ so that $f_{ext}(x)$ is odd. Finally, define $f_{ext}(x)$ beyond $[-3,3]$ by copying it over to $[3,9]$, $[-9,-3]$, $[9,15]$ etc.

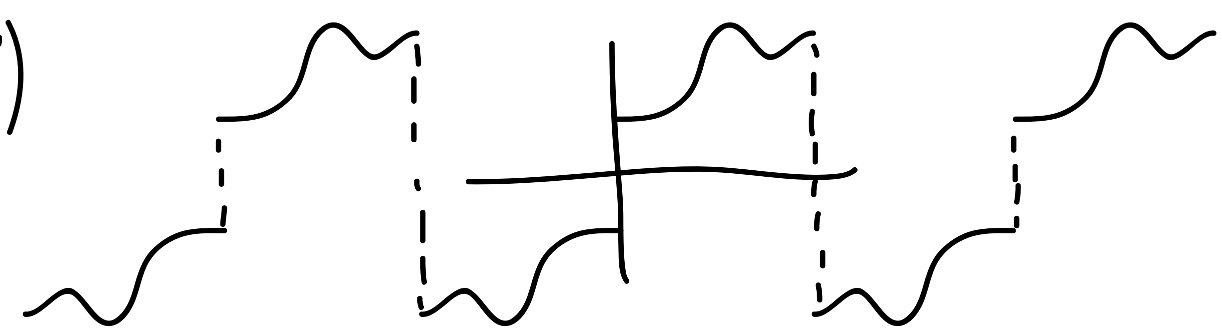
(i)



(ii)



(iii)



Now, the Fourier series of $f_{ext}(x)$ will consist of only sin terms and will have frequencies $\frac{n\pi}{3}$. It will also converge to $f(x)$ at all $x \in [0,3]$ at which $f_{ext}(x)$ is continuous.

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{9} Dt} \sin\left(\frac{n\pi x}{3}\right)$$

where $b_n = \frac{1}{3}\int_{-3}^{3} f_{ext}(x) \sin\left(\frac{n\pi x}{3}\right) dx = \frac{2}{3}\int_{0}^{3} f(x) \sin\left(\frac{n\pi x}{3}\right) dx$

this = follows from the fact that $f_{ext}(x) \cdot \sin\left(\frac{n\pi x}{3}\right)$ is an even function.

Non-homogeneous BCs

In general, solutions might include a linear term as well:

$$u(x,t) = ax + b + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2 Dt}{L^2}} \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2 Dt}{L^2}} \sin\left(\frac{n\pi x}{L}\right).$$

For zero Dirichlet BCs, $a = b = a_n = 0$ and $b_n = \frac{2}{L}\int_0^L u(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$

For zero Neumann BCs, $a = b_n = 0$ and $b = \frac{1}{L}\int_0^L u(x,0)\,dx$

$$a_n = \frac{2}{L}\int_0^L u(x,0)\cos\left(\frac{n\pi x}{L}\right)dx$$

When the BCs are not zero, we need to find a and b so that $u_{ss}(x) = ax + b$ satisfies the BCs. For Dirichlet BCs, this is simple (given two points on a line). For Neumann BCs, it's trickier. Say $u_x(0,t) = \alpha$, $u_x(0,t) = \beta$. Because flux is $-Du_x$, these BCs tell us how much mass is leaving/entering. Unless $\alpha = \beta$ the total mass changes constantly so we require $\alpha = \beta$ (otherwise, no steady state). This means $a = \alpha$. To find b,

$$\underset{\text{at } t=0}{\text{Total mass}} = \int_0^L u(x,0)\,dx = \int_0^L u_{ss}(x)\,dx = \underset{\text{as } t\to\infty}{\text{Total mass}}.$$

Solve this eq for b.

The general solution to a non-homogeneous Dirichlet problem

$$u_t = Du_{xx}, \quad u(0,t) = p, \quad u(L,t) = q \quad \text{is}$$

$$u(x,t) = p + \frac{q-p}{L}x + \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2 Dt}{L^2}} \sin\left(\frac{n\pi x}{L}\right).$$

The general solution to a non-homogeneous Neumann problem

$$u_t = Du_{xx}, \quad u_x(0,t) = a = u_x(L,t) \quad \text{is}$$

$$u(x,t) = ax + b + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2 Dt}{L^2}} \cos\left(\frac{n\pi x}{L}\right)$$

where b is found by equating total mass initially and as $t \to \infty$.

Satisfying an IC:

Suppose we have an IC $u(x,0) = f(x)$ on $0 \leq x \leq L$. For the Dirichlet case, we need to pick the b_n so that

$$u(x,0) = p + \frac{q-p}{L}x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \quad \text{on } 0 \leq x \leq L.$$

Or

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x) - \left(p + \frac{q-p}{L}x\right).$$ So choose b_n using

$$b_n = \frac{2}{L}\int_0^L \left[f(x) - \left(p + \frac{q-p}{L}x\right)\right] \sin\left(\frac{n\pi x}{L}\right) dx.$$