Find the solution to the equation

$$u_t = 4u_{xx}$$

with IC $u(x,0) = 0$

and BCs $u(0,t) = 2$

$u(3,t) = 5$

$u_{ss}(x) = ax + b$

$u_{ss}(0) = b = 2$

$u_{ss}(3) = 3a + 2 = 5 \Rightarrow a = 1$

$u_{ss}(x) = x + 2$

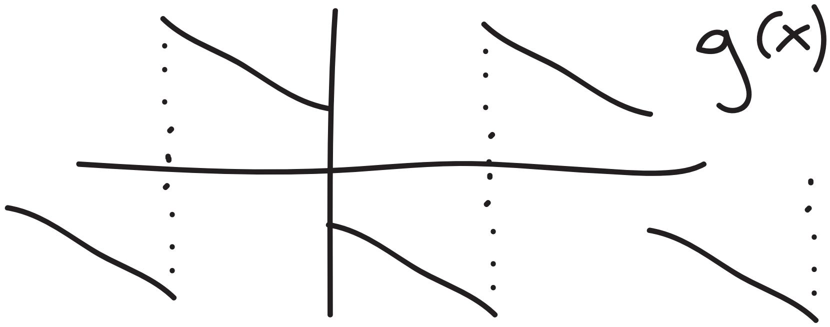
$$u(x,t) = x + 2 + \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{9} \cdot 4t} \sin \frac{n\pi x}{3}$$

$$u(x,0) = x + 2 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3} = 0$$

Choose b_n so that

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3} = -x - 2 \quad \text{on } (0,3).$$

Extend $-x-2$ as odd about $x=0$ and then periodic with period 6.



$$b_n = \frac{1}{3}\int_{-3}^{3} g(x)\sin\frac{n\pi x}{3}\,dx$$

$$= \frac{2}{3}\int_{0}^{3} (-x-2)\sin\frac{n\pi x}{3}\,dx$$

$$= \cdots = 2\left(\frac{-2+5(-1)^n}{n\pi}\right)$$

$$u(x,t) = x+2+\sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{9}\cdot 4t}\sin\frac{n\pi x}{3}$$

Check: $u_t = \sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{9}\cdot 4\right) b_n e^{-\frac{n^2\pi^2}{9}\cdot 4t}\sin\frac{n\pi x}{3}$

$$4u_{xx} = 4\sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{9}\right) b_n e^{-\frac{n^2\pi^2}{9}\cdot 4t}\sin\frac{n\pi x}{3}$$

So $u_t = 4u_{xx}$

$$u(x,0) = x+2+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3} = \underbrace{x+2+(-x-2)}_{=0}$$

$u(0,t) = 2$, $u(3,t) = 5$.