Today

- Implicit differentiation
  - Power rule for fractional powers
- Exponential functions

For $f(x)=x^n$, $f'(x)=nx^{n-1}$ when n is an integer. What if n is rational?

For $f(x) = x^{1/n}$ with n an integer, what is $f'(x)$?

Rewrite the equation as $f(x)^n = x$ and take derivatives:

(A) $n\, f(x)^{n-1} = 1$

(B) $n\, f(x)^{n-1} f'(x) = 1$

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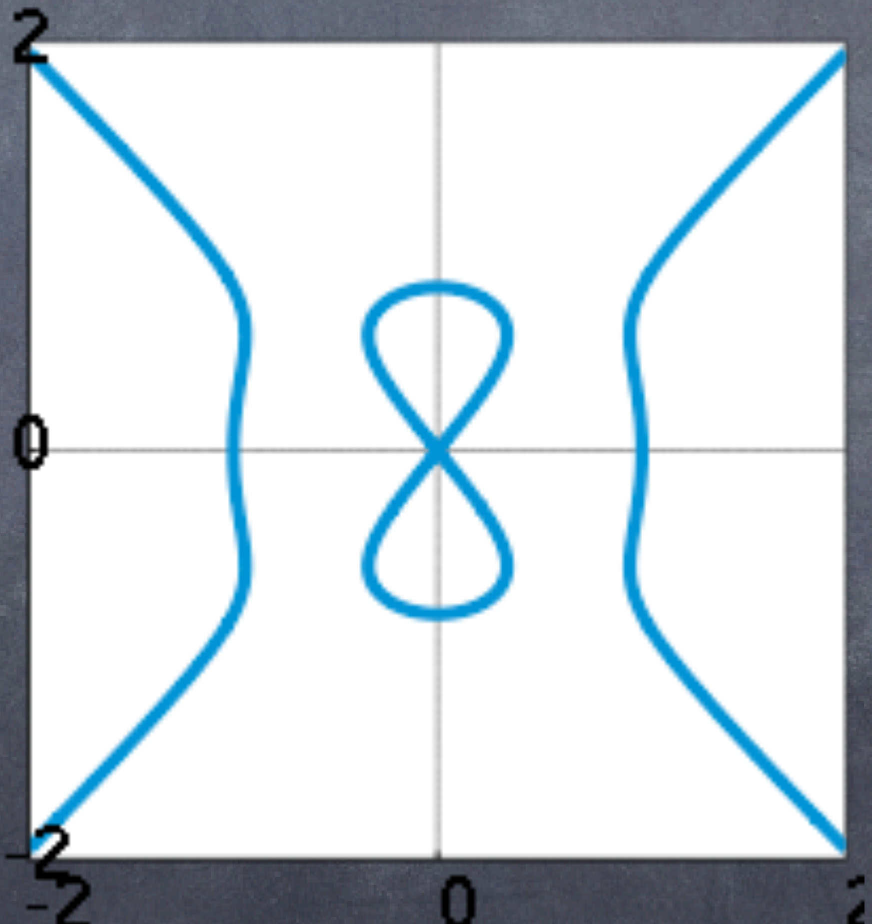
- When p=m/n and $f(x) = x^p$, $f'(x)=px^{p-1}$.
    - You do this (Problem 7.22 in LK notes).

The Devil's curve:

$y^2(y^2-a^2) = x^2(x^2-b^2)$

- Where is the Devil's curve horizontal?



- For you to think about...

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(D) If $a<b$ then $a^x<b^x$ for all $x>0$ and $a^x>b^x$ for all $x<0$.

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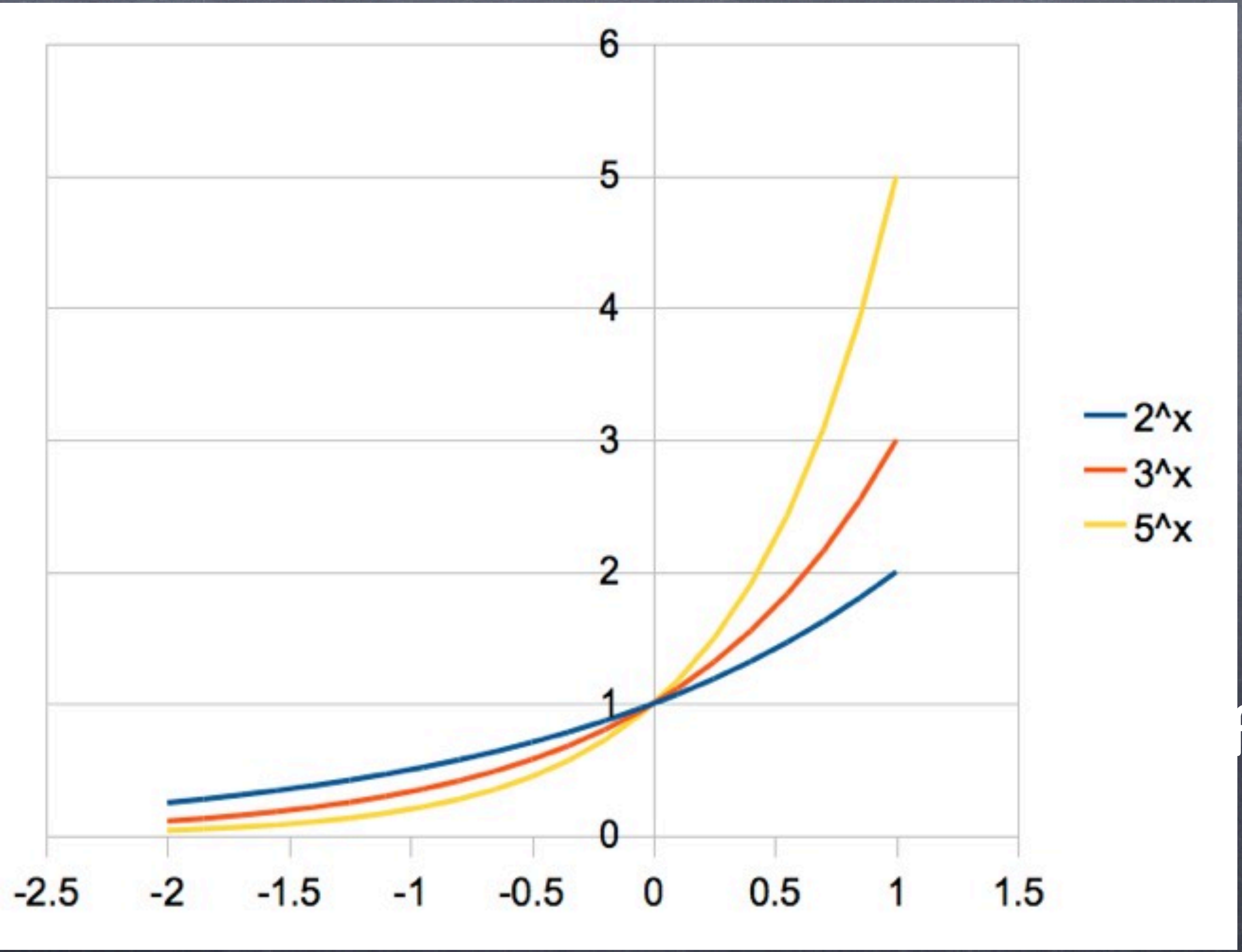

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- What function is equal to its own derivative?
  - Equivalent to asking "what f(x) satisfies $f'(x)=f(x)$?"
  - Call this a "differential equation". (DE)

DE example: Which of the following satisfies $f'(x)=f(x)$?

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