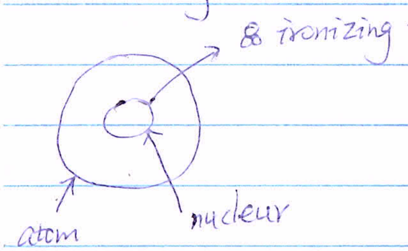
Lecture 23 (Oct. 30, 2013)

Learning Goals: ① Exponential Decay

② DE $\frac{dy}{dx} = ay + b$

• Radioactivity:



Assume some material includes $y(t)$ amount of radioactivity at time t

The amount of radioactivity that is lost per unit time is proportional to $y(t)$

$\Rightarrow \frac{dy}{dt} = -ky, \quad k > 0$

$\Rightarrow y(t) = y_0 e^{-kt}$ with $y_0 = y(t=0)$

Half life: How long it takes for half of the initial amout of radioactivity to remain

$\Rightarrow T$ that satisfies $y(T) = \frac{1}{2} y_0$

$\Rightarrow y_0 \cdot e^{-kT} = \frac{1}{2} y_0 \quad \Rightarrow -kT = \ln\frac{1}{2} = -\ln 2 \quad \Rightarrow T = \frac{\ln 2}{k}$

Example 1: 1986, chernobyl. Nuclear power plant exploded. Radioactive elements iodine-131 (I^{131}) and cesium-137 (C^{137}) were scattered. Half life of I^{131} is 8 days and half life of C^{137} is 30 years. How much of those two would remain after 25 years?

① I^{131}. $k = \frac{\ln 2}{8} \approx 0.0866$ (per day)

$y(t) = y_0 e^{-kt} \quad \Rightarrow y(25 \text{ years}) = y_0 e^{-\frac{\ln 2}{8} \cdot 365 \times 25} \approx y_0 e^{-790.25} \approx 0$

② C^{137}. $k = \frac{\ln 2}{30} \approx 0.0231$ (per year)

$y = y_0 e^{-kt} \quad \Rightarrow y(25 \text{ years}) = y_0 e^{-\frac{\ln 2}{30} \cdot 25} = y_0 \cdot e^{-\frac{5\ln 2}{6}} \approx 0.5612 y_0$

$\Rightarrow$ 56.12% of the initial amout remains

Notice: function $y(t) = y_0 e^{-kt} \to 0$ for $t \to +\infty$

• DE: $\frac{dy}{dx} = ay + b$, a, b - nonzero constants

Solution: change variable to make DE in the form of $\frac{dz}{dx} = az$

$\frac{dy}{dx} = ay + b = a(y + \frac{b}{a})$, assume $z = y + \frac{b}{a}$

Then $\left.\begin{array}{l} \frac{dz}{dx} = \frac{dy}{dx} \\ \frac{dy}{dx} = ay + b = a \cdot z \end{array}\right\} \Rightarrow \frac{dz}{dx} = a \cdot z \Rightarrow z = z_0 e^{ax}$ with $z_0 = z(x=0)$

$= y(x=0) + \frac{b}{a}$

$= y_0 + \frac{b}{a}$

$\Rightarrow y = -\frac{b}{a} + (y_0 + \frac{b}{a}) e^{ax}$ is the solution to DE $\frac{dy}{dx} = ay + b$

Then given y_0, we are able to find y for any value of x

① Given $y_0 \neq -\frac{b}{a}$, then there's no x making $y(x) = -\frac{b}{a}$

② Given $y_0 = -\frac{b}{a}$, then $y(x) = -\frac{b}{a}$ for all the value of x