# Differential Calculus: Mathematics 102

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# Contents

# List of Figures

# List of Tables

# Chapter 1

# How big can a cell be? (The power of functions)

The shapes of living cells are designed to be uniquely suited to their functions. Few cells are really spherical. Many have long appendages, cylindrical parts, or branch-like structures. But here, we will neglect all these beautiful complexities and look at a simple egg-like spherical cell. The question we want to explore is what determines the size (and shape) of a cell and why some size limitations exist. Why should animals be made of millions of tiny cells, instead of just a few hundred large ones?

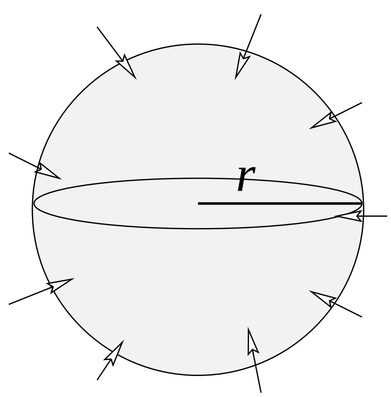



**Figure 1.1.** *A cell (assumed spherical) absorbs nutrients at a rate proportional to its surface area $k_1 S$, but consumes nutrients at a rate proportional to its volume $k_2 V$. We use the facts that the surface area and volume of a sphere of radius $r$ are given by $S = 4\pi r^2, V = \frac{4}{3}\pi r^3$*

While these questions seem extremely complicated, a relatively simple mathematical argument can go a long way in illuminating the situation. To delve into this mystery of size and shape, we will formulate a **mathematical model**. A model is just a representation of a real situation which simplifies things by representing the most important aspects, while neglecting or idealizing the other aspects. Below we follow a reasonable set of assumptions and mathematical facts to explore how nutrient balance can affect and limit cell size.

## 1.1 A simple model for nutrient balance in the cell

We base the model on the following assumptions:

1. The cell is roughly spherical (See Figure 1.1).

2. The cell absorbs oxygen and nutrients from the environment through its surface. If the surface area, $S$, of the cell is bigger, it can absorb these substances at a faster rate. We will assume that the rate at which nutrients (or oxygen) are absorbed is *proportional* to the surface area of the cell.

3. The rate at which nutrients are consumed in metabolism (i.e. used up) is proportional to the volume, $V$, of the cell; that is, the bigger the volume, the more nutrients are needed to keep the cell alive. We will assume that the rate at which nutrients (or oxygen) are consumed is *proportional* to the volume of the cell.

We define the following quantities for our model of a single cell:

$A =$ net rate of absorption of nutrients per unit time.
$C =$ net rate of consumption of nutrients per unit time.
$V =$ cell volume.
$S =$ cell surface area.
$r =$ radius of the cell.

We now rephrase the assumptions mathematically. By assumption (2), $A$ is proportional to $S$: This means that

$$A = k_1 S,$$

where $k_1$ is a constant of proportionality. Since absorption and surface area are positive quantities, in this case only positive values of the proportionality constant make sense, so that $k_1 > 0$. (The value of this constant would depend on the permeability of the cell membrane, how many pores or channels it contains, and/or any active transport mechanisms that help transfer substances across the cell surface into its interior.

Further, by assumption (3), $C$ is proportional to $V$, so that

$$C = k_2 V,$$

where $k_2 > 0$ is a second proportionality constant. The value of $k_2$ would depend on the rate of metabolism of the cell, i.e. how quickly it consumes nutrients in carrying out its activities.

Since we have assumed that the cell is spherical, by assumption (1), the surface area, $S$, and volume $V$ of the cell are:

$$S = 4\pi r^2, \quad V = \frac{4}{3}\pi r^3. \tag{1.1}$$

Putting these facts together leads to the following relationships between nutrient absorption, consumption, and cell radius:

$$A = k_1(4\pi r^2) = (4\pi k_1)r^2, \qquad C = k_2(\frac{4}{3}\pi r^3) = (\frac{4}{3}\pi k_2)r^3.$$

We note that $A, C$ are now quantities that depend on the radius of the cell. Indeed, since the terms in brackets on the right hand sides are just constant coefficients, *each of the above expressions is simply a power function*, with $r$ the independent variable. That is, each of these expressions has the form

$$y = K\, r^n$$

for some positive constant coefficient $K$ (for consumption, $K = \frac{4}{3}\pi k_2$ and for absorption $K = 4\pi k_1$). Most importantly, the powers are $n = 3$ for consumption and $n = 2$ for absorption.

In order to appreciate how the size of the cell affects each of the two processes consumption and absorption of nutrients, let us review some elementary facts about power functions.

## 1.2 Power functions

Power functions are among the most elementary and "elegant" functions. They are easy to calculate[1], are very predictable and smooth, and, from the point of view of calculus, are very easy to handle.

A power function has the form

$$y = f(x) = x^n$$

where $n$ is a positive integer. As shown in Fig. 1.2, even and odd powers lead to power functions of distinct symmetry properties. Indeed the terms **even** and **odd functions** stem directly from the symmetry properties of the power functions. (See Appendix B for a review of symmetry.) From Figure 1.2, we see that all the elementary power functions intersect at $x = 0$ and $x = 1$. Each of the even (odd) power functions also intersect one another at $x = -1$.

Figure 1.2 also demonstrates another extremely important feature of the power functions: the higher the power, the *flatter* the graph near the origin and the *steeper* the graph beyond $|x| > 1$. This can be restated in terms of the relative size of the power functions. We say that *close to the origin, the functions with lower powers dominate, while far from the origin, the higher powers dominate*.

So far, we have compared power functions whose coefficient is the constant 1. How would we compare two functions of the form

$$y_1 = ax^n, \quad \text{and} \quad y_2 = bx^m.$$

[1] We only need to use multiplication to compute the value of these functions at any point.

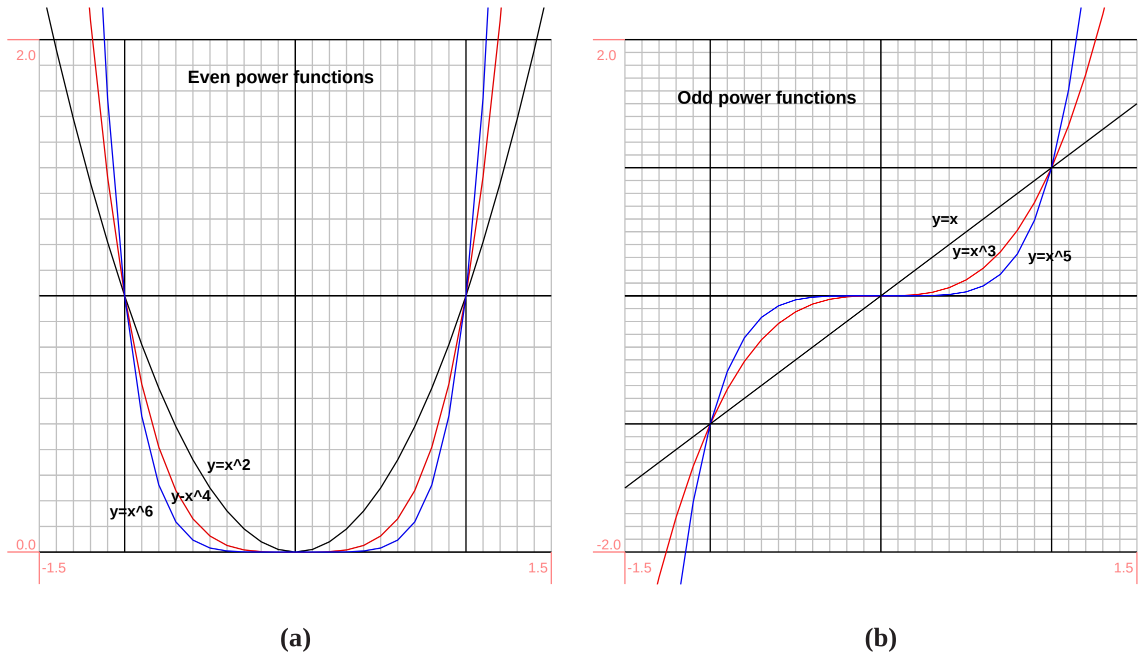



**Figure 1.2.** *Graphs of power functions (a) A few of the even ($y = x^2,\ y = x^4,\ y = x^6$ ) power functions (b) Some odd ($y = x,\ y = x^3,\ y = x^5$ ) power functions. Note symmetry properties. Also observe that as the power increases, the graphs become flatter close to the origin and steeper at large $x$ values.*

This comparison is a slight generalization of what we have seen above. First, we note that the coefficients $a$ and $b$ merely scale the vertical behaviour (i.e. stretch the graph along the $y$ axis. It is still true that the higher the power, the flatter the graph close to $x = 0$, and the steeper for large positive or negative values of $x$. However, now the points of intersection of the graphs will occur at $x = 0$ and, in the first quadrant at

$$ax^n = bx^m \quad \Rightarrow \quad x^{n-m} = (b/a) \Rightarrow \quad x = (b/a)^{1/(n-m)}$$

If $n, m$ are both even or both odd, there will also be an intersection at $x = -(b/a)^{1/(n-m)}$. As one example, for the two functions $y_1 = 3x^4$ and $y_2 = 27x^2$, intersections occur at $x = 0$ and at $\pm(27/3)^{1/(4-2)} = \pm\sqrt{9} = \pm 3$. As a second example, the two functions $y_1 = (4/3)\pi x^3, y_2 = 4\pi x^2$ intersect only at $x = 0, 3$ but not for any negative values of $x$. In many cases, the points of intersection will be irrational numbers whose decimal approximations can only be obtained by a scientific calculator or some other method (e.g. see Newton's Method in a later chapter).

## 1.3 Cell size for nutrient balance, continued

In our discussion of cell size, we found two power functions that depend on the cell radius, namely the nutrient absorption and consumption rates,

$$A(r) = (4\pi k_1)r^2, \quad \text{and} \quad C(r) = (\frac{4}{3}\pi k_2)r^3.$$

(Here we have explicitly noted that both are power functions with respect to cell radius, $r$. Further the coefficients are indicated by terms in braces, each of which is a constant.) Based on our discussion of power functions, we know that for small $r$, the power function with the lower power of $r$ (namely $A$) dominates, but for very large values of $r$, the power function with the higher power ($C$) dominates. Where does the switch take place? As before, we find this by computing the point of intersection of the two graphs

$$A = C \quad \Rightarrow \quad (\frac{4}{3}\pi k_2)r^3 = (4\pi k_1)r^2.$$

One solution to this equation (which is not too interesting here) is $r = 0$. If $r \neq 0$, then we can cancel a factor of $r^2$ from both sides to obtain:

$$r = 3\frac{k_1}{k_2}.$$

For cells of this radius, absorption and consumption are equal, it follows that for smaller cell sizes the absorption $A \approx r^2$ is the dominant process, which for large cells, the consumption $C \approx r^3$ is higher than absorption. We conclude that cells larger than the critical size $r = 3k_1/k_2$ will be unable to keep up with the nutrient demand, and will not survive.

Thus, using this simple geometric argument, we have deduced that the size of the cell has strong implications on its ability to absorb nutrients quickly enough to feed itself. The restriction on oxygen absorption is even more critical than the replenishment of other substances such as glucose. For these reasons, cells larger than some maximal size (roughly 1 mm in diameter) rarely occur. Furthermore, organisms that are bigger than this size cannot rely on simple diffusion to carry oxygen to their parts—they must develop a circulatory system to allow more rapid dispersal of such life-giving substances or else they will perish.

## 1.4 Lessons learned

To be written: Some important observations implicit in the above discussion.

- Inverse power functions and fractional powers (even and odd).
- General observations about the shapes of graphs, e.g. which have minima, which do not. Behaviour at $x \to \pm\infty$. Smoothness properties, etc.
- Other examples of even and odd functions.
- Other information we obtain from graphs.

## 1.5 Polynomials

A polynomial is a function in which the simple power functions are combined in a simple way. A typical example is

$$y = p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

This **superposition** of the basic power functions with integer powers and real coefficients $a_k$ proves to be a function with particularly convenient features in terms of computations: evaluating $p$ at any point $x$ reduces to simple arithmetical operations of multiplications and addition (something that computers are well equipped for). Furthermore, as we shall see, these functions are easy to treat using basic calculus operations that we will describe in the following chapters. The highest power $n$ is called the **degree** of the polynomial.

In Appendix B, we present some of the special features of polynomials. Here we can briefly mention that a polynomial of degree $n$ can have up to $n-1$ "wiggles" (by which we mean **maxima** and **minima**). Every polynomial is unbounded as $x \to \infty$ and as $x \to -\infty$. In fact, for large enough values of $x$, the power function $y = f(x) = a_n x^n$ with the largest power, $n$, dominates so that

$$p(x) \approx a_n x^n \quad \text{for large } x.$$

Similarly, for small $x$, close to the origin, the smallest powers dominates so that

$$p(x) \approx a_1 x + a_0 \quad \text{for small } x.$$

**Example 1.1** Sketch the polynomial

$$y = p(x) = x^3 + ax.$$

How would the sketch change if the constant $a$ changes from positive or negative? ■

**Solution:** The polynomial has two terms, and we will consider their effects individually. Near the origin, for $x \approx 0$ the term $ax$ dominates so that, close to $x = 0$, the function behaves as

$$y \approx ax.$$

This is a straight line with slope $a$. If $a > 0$ we should see a line with positive slope here, whereas if $a < 0$ the slope of the line should be negative. Far away from the origin, the cubic term dominates, so

$$y \approx x^3.$$

That means that we would see a nearly cubic curve when we look at large (positive or negative) $x$ values. Figure 1.3 illustrates these ideas. In column (a) we see the behaviour of $y = p(x) = x^3 + ax$ for large $x$, in (b) for small $x$. Column (c) shown the graph for an intermediate range. We might notice that for $a < 0$, the graph has a local minimum as well as a local maximum.

The zeros of the polynomial can be found by setting

$$y = p(x) = 0 \quad \Rightarrow \quad x^3 + ax = 0 \quad \Rightarrow \quad x^3 = -ax$$

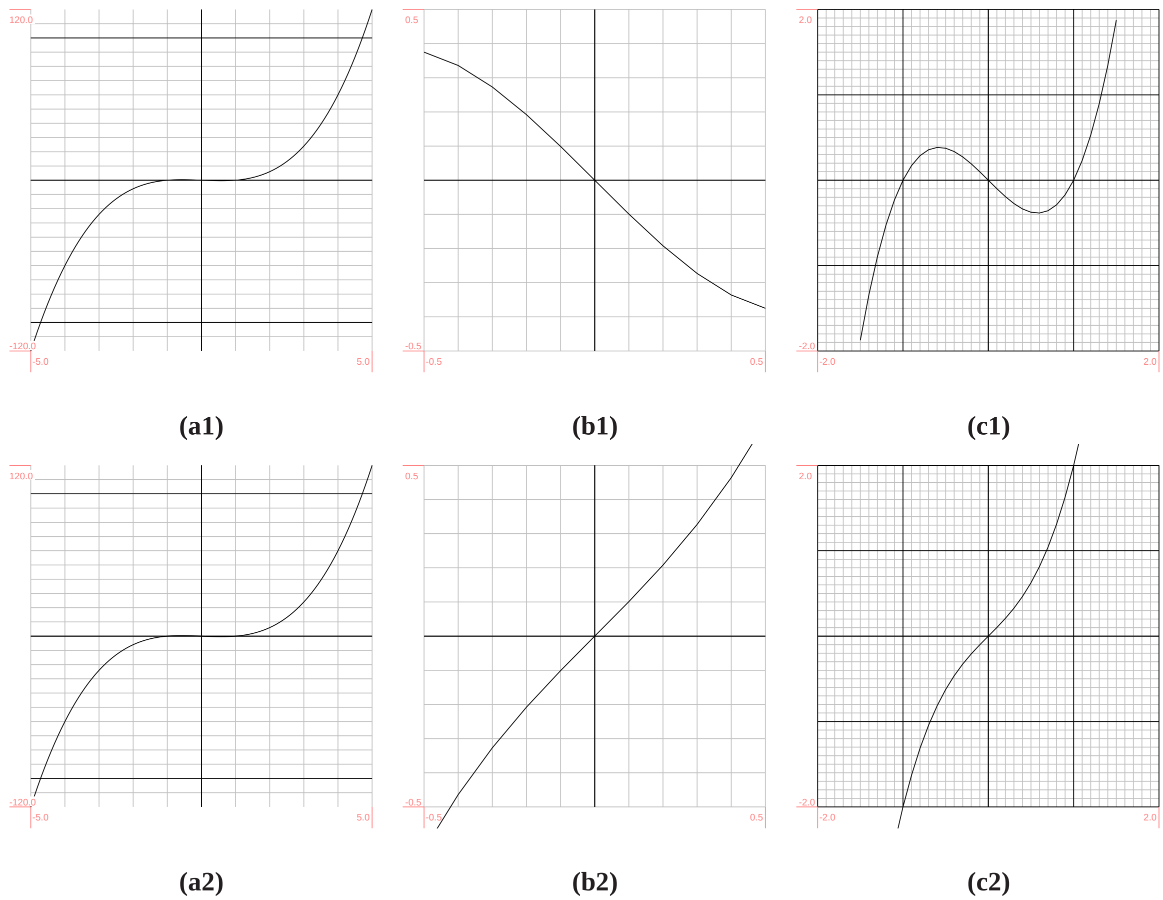



**Figure 1.3.** *The behaviour of $y = p(x) = x^3 + ax$ is shown here for $a < 0$ (top row) and $a > 0$ (bottom row). (a): Zooming out to the range $-5 < x < 5$, we see that for very large $x$ the graph looks a lot like the cubic curve $y = x^3$. (b): Zooming in to the range $-0.5 < x < 0.5$, i.e. at small $x$, the graph looks almost like the straight line $y = ax$, whose slope is $a$. (c): If we plot the curve for intermediate $x$ range, $-2 < x < 2$, we see the behaviour for both small and large $x$ values.*

The above equation always has a solution $x = 0$, but if $x \neq 0$, we can cancel and obtain

$$x^2 = -a.$$

This would have no solutions if $a$ is a positive number, so that in that case, the graph crosses the $x$ axis only once, at $x = 0$, as shown in Figure 1.3 (c2). If $a$ is negative, then the negatives cancel, so the equation can be written in the form

$$x^2 = |a|$$

and we would have two new zeros at

$$x = \pm\sqrt{|a|}.$$

For example, if $a = -1$ then the function $y = x^3 - x$ has zeros at $x = 0, 1, -1$.

# 1.6 Rate of an enzyme-catalyzed reaction

## 1.6.1 Saturation and Michaelis-Menten kinetics

Biochemical reactions are often based on the action of proteins known as enzymes that catalyze many reactions in living cells. Shown in Fig. 1.4 is a typical scheme. The enzyme E binds to its substrate S to form a complex C. The coplex then breaks apart into a product, P, and an enzyme molecule that can repeat its action again. Generally, the substrate is much more plentiful than the enzyme.

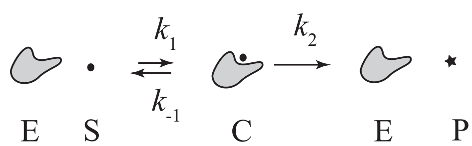



**Figure 1.4.** *An enzyme (catalytic protein) is shown binding to a substrate molecule (circular dot) and then processing it into a product (star shaped molecule).*

Suppose we let $x$ represent the concentration of substrate in the reaction mixture. The speed of the reaction, $v$, (namely the rate at which product is formed) depends on $x$. But the relationship is not linear, as shown in Fig. 1.5. In fact, this relationships, known as **Michaelis Menten** kinetics, has the form

$$v = \frac{Kx}{k_n + x}, \tag{1.2}$$

where $K, k_n$ are positive constants that are specific to the enzyme and the experimental conditions.

Equation (1.2) is a **rational function**, that is, a ratio of two polynomials. We can use a graphics calculator or graphing software to plot a graph of this function, as done in Fig 1.5, or else we can put our understanding of polynomials to use in sketching this function.

Since $x$ is a concentration, it must be a positive quantity, so we restrict attention to $x \geq 0$. The following observations can be made

1. The graph of (1.2) goes through the origin. Indeed, when $x = 0$ we have $v = 0$.

2. Close to the origin, the graph "looks like" a straight line. We can see this by considering values of $x$ that are much smaller than $k_n$. Then the denominator $(k_n + x)$ is well approximated by the constant $k_n$. Thus, for small $x$, $v \approx (K/k_n)x$. Thus for small $x$ the graph resembles a straight line with slope $(K/k_n)$.

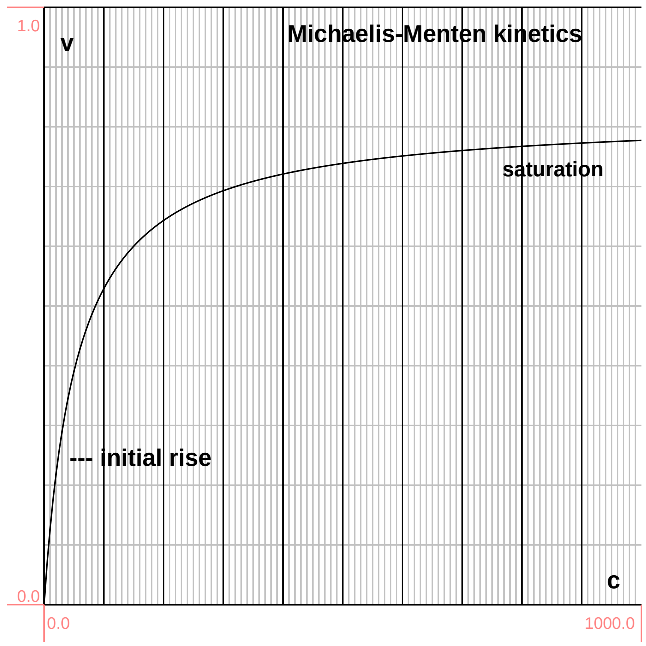



**Figure 1.5.** *The graph of reaction speed, $v$, versus substrate concentration, $c$ in an enzyme-catalyzed reaction. This behaviour is called Michaelis-Menten kinetics. Note that the graph at first rises almost like a straight line, but then it curves over and approaches a horizontal asymptote. We refer to this as "saturation". This graph tells us that the speed of the enzyme cannot exceed some maximal level, i.e. it cannot be faster than $K$. See Eqn. 1.2.*

3. For large $x$, there is a horizontal asymptote. The reader can use a similar argument for $x \gg k_n$, to show that $v$ is approximately constant.

Michaelis-Menten kinetics thus represents one type of relationship in which the phenomenon of **saturation** occurs: the speed of the reaction increases for small increases in the level of substrate, but it cannot increase indefinitely, i.e. the enzymes saturate and operate at their fixed constant speed when the substrate concentration is very high.

It is worth pointing out the units of terms in (1.2). $x$ carries units of concentration (e.g. nano Molar written nM, which means $10^{-9}$ Moles per litre) $v$ carries units of concentration over time (e.g. nM min$^{-1}$). $k_n$ *must* have same units as $x$ (only quantities with identical units can be added or compared !). The units on the two sides of the relationship (1.2) have to balance too, meaning that $K$ must have the same units as the speed of the reaction, $v$.

### 1.6.2 Hill functions

The Michaelis-Menten kinetics we discussed above fit into a broader class of **Hill functions**, which are rational functions of the form

$$y = \frac{Ax^n}{a^n + x^n}. \tag{1.3}$$

Here $A, a > 0$ is a constant and $n$ is some power. This function is often referred to as *a Hill function with coefficient* $n$, (although the "coefficient" is actually a power in terms of the terminology used in this chapter). Hill functions occur in biology in situations where the rate of some enzyme-catalyzed reaction is affected by cooperative behaviour of a number of subunits, or by a chain of steps.

We see that Michaelis Menten kinetics corresponds to a Hill function with $n = 1$. In biochemistry, expressions of the form (1.3) with $n > 1$ are often denoted "sigmoidal" kinetics, and a few such functions are plotted in Fig 1.6. Using arguments similar to those of Section 1.6.1, we can infer the shapes of these functions as follows:

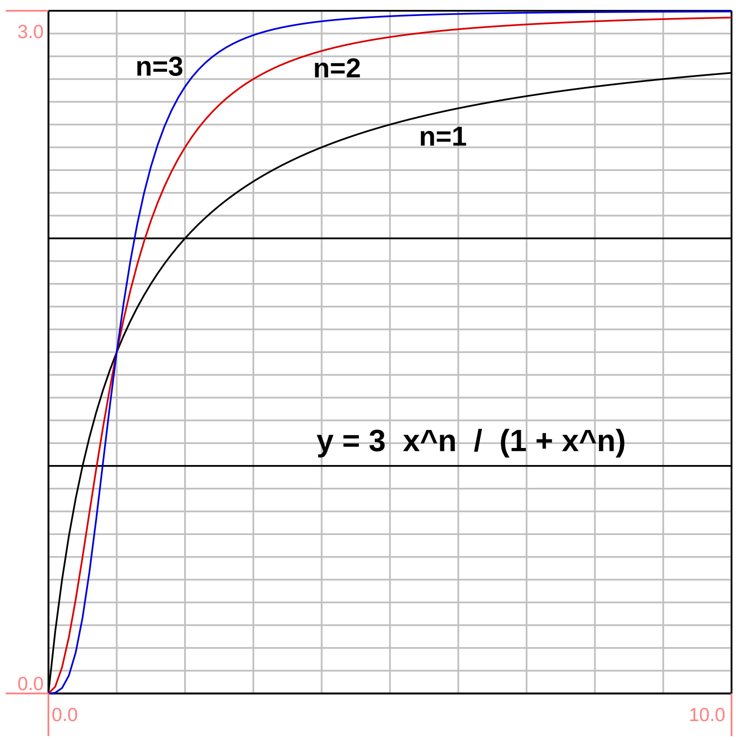



**Figure 1.6.** *Three Hill functions with* $A = 3, a = 1$ *and coefficient* $n = 1, 2, 3$ *are compared on this graph. As the Hill coefficient increases, the graph becomes flatter close to the origin, and steeper in its rise to the asymptote at* $y = A$.

- The graph of the Hill function (Figure 1.6) goes through the origin. (At $x = 0$, we see that $y = 0$.)

- For very small $x$, (i.e., $x << a$) we can make the approximation $a^n + x^n \approx a^n$ so that
$$y = \frac{Ax^n}{a^n + x^n} \approx \frac{Ax^n}{a^n} = \left(\frac{A}{a^n}\right) x^n \quad \text{for small x.}$$
This means that near the origin, the graph looks like a power function, $Cx^n$ (where $C = A/a^n$).

- For large $x$, i.e. $x >> a$, it is approximately true that $a^n + x^n \approx x^n$ so that
$$y = \frac{Ax^n}{a^n + x^n} \approx \frac{Ax^n}{x^n} = A \quad \text{for large x.}$$
This reveals that the graph has a horizontal asymptote $y = A$ at large values of $x$. This means that the largest ("maximal") value that $y$ approaches is $y = A$. If $y$ represents the speed of a chemical reaction (analogous to the variable we labeled $v$ in chapter 1), then $A$ is the "maximal rate" or "maximal speed".

- Since the Hill function behaves like a simple power function close to the origin, we conclude directly that the higher the value of $n$, the flatter is its graph near 0. Further, large $n$ means sharper rise to the eventual asymptote. Hill functions with large $n$ are often used to represent "switch-like" behaviour in genetic networks or biochemical signal transduction pathways.

- The constant $a$ is sometimes called the "half-maximal activation level" for the following reason: When $x = a$ then
$$y = \frac{Aa^n}{a^n + a^n} = \frac{Aa^2}{2a^2} = \frac{A}{2}.$$
This shows that the level $x = a$ leads to the half-maximal level of $y$.

## 1.7 For further study: Spacing of fish in a school

Many animals live or function best when they are in a group. Social groups include herds of wildebeest, flocks of birds, and schools of fish, as well as swarms of insects. Life in a group can affect the way that individuals forage (search for food), their success at detecting or avoiding being eaten by a predator, and other functions such as mating, protection of the young, etc. Biologists are interested in the ecological implications of groups on their own members or on other species with whom they interact, and how individual behaviour, combined with environmental factors and random effects affect the shape of the groups, the spacing, and the function.

In many social groups, the spacing between individuals is relatively constant from one part of the formation to another, because animals that get too close start to move away from one another, whereas those that get too far apart are attracted back. These spacing distances can be observed in a variety of groups, and were described in many biological publications. For example, Emlen [**?**] found that in flocks, gulls are spaced at about one body length apart, whereas Conder [**?**] observed a 2-3 body lengths spacing distance in tufted ducks. Miller [**?**] observed that sandhill cranes try to keep about 5.8 ft apart in the flock he observed.

To try to explain why certain spacing is maintained in a group of animals, it was proposed that there are mutual attraction and repulsion interactions, (effectively acting like simple forces) between individuals. Breder [**?**] followed a number of species of fish that school, and measured the individual spacing in units of the fish body length, showing that individuals are separated by 0.16-0.25 body length units. He suggested that the effective forces between individuals were similar to inverse power laws for repulsion and attraction. Breder considered a quantity he called *cohesiveness*, defined as:

$$c = \frac{A}{x^m} - \frac{R}{x^n}, \tag{1.4}$$

where $A, R$ are magnitudes of attraction and repulsion, $x$ is the distance between individuals, and $m, n$ are integer powers that govern how quickly the interactions fall off with distance. We could re-express (1.4) as

$$c = Ax^{-m} - Rx^{-n}$$

Thus, the function shown in Breder's cohesiveness formula is related to our power functions, but the powers are negative integers. A specific case considered by Breder was $m = 0, n = 2$, i.e. constant attraction and inverse square law repulsion,

$$c = A - (R/x^2)$$

Breder specifically considered the "point of neutrality", where $c = 0$. The distance at which this occurs is:

$$x = (R/A)^{1/2}$$

where attraction and repulsion are balanced. This is the distance at which two fish would be most comfortable: neither tending to move apart, nor get closer together.

Other ecologists studying a similar problem have used a variety of assumptions about forces that cause group members to attract or repel one another.

## 1.8 For further study: Transforming Michaelis-Menten kinetics to a linear relationship

Michaelis-Menten kinetics that we explored in (1.2) is a nonlinear saturating function in which the concentration $x$ is the independent variable on which the reaction velocity, $v$ depends. As discussed in Section 1.6.1, the constants $K$ and $k_n$ depend on the enzyme and are often quantified in a biochemical assay of enzyme action. In older times, a convenient way to estimate the values of $K$ and $k_n$ was to measure $v$ for many different values of the initial substrate concentration. Before nonlinear fitting software was widely available, the expression (1.2) was transformed (meaning that it was rewritten as a linear relationship.

We can do so with the following algebraic steps:

$$v = \frac{Kx}{k_n + x}$$

so, taking reciprocals and expanding leads to

$$\begin{aligned}\frac{1}{v} &= \frac{k_n + x}{Kx},\\ &= \frac{k_n}{Kx} + \frac{x}{Kx}\\ &= \left(\frac{k_n}{K}\right)\frac{1}{x} + \left(\frac{1}{K}\right)\end{aligned}$$

This suggests defining the two constants:

$$m = \frac{k_n}{K}, \quad b = \frac{1}{K}.$$

In which case, the relationship between $1/v$ and $1/x$ becomes linear:

$$\left[\frac{1}{v}\right] = m\left[\frac{1}{x}\right] + b. \tag{1.5}$$

Both the slope, $m$ and intercept $b$ of the straight line provide information about the parameters. The relationship (1.5), which is a disguised variant of Michaelian kinetics is called the Linweaver-Burke relationship. Later, we will see how this can be used to estimate the values of $K$ and $k_n$ from biochemical data about an enzyme.

## Exercises

1.1. **Simple transformations:** Consider the graphs of the simple functions $y = x$, $y = x^2$, and $y = x^3$. What happens to each of these graphs when the functions are *transformed* as follows:

(a) $y = Ax$, $y = Ax^2$, and $y = Ax^3$ where $A > 1$ is some constant.

(b) $y = x + a$, $y = x^2 + a$, and $y = x^3 + a$ where $a > 0$ is some constant.

(c) $y = (x - b)^2$, and $y = (x - b)^3$ where $b > 0$ is some constant.

1.2. **Simple sketches:** Sketch the graphs of the following functions:

(a) $y = x^2$,

(b) $y = (x + 4)^2$

(c) $y = a(x - b)^2 + c$ for the case $a > 0$, $b > 0$, $c > 0$.

(d) Comment on the effects of the constants $a$, $b$, $c$ on the properties of the graph of $y = a(x - b)^2 + c$.

1.3. **Power functions:** Consider the functions $y = x^n$, $y = x^{1/n}$, $y = x^{-n}$, where $n$ is an integer ($n = 1, 2..$) Which of these functions increases most steeply for values of $x$ greater than 1? Which decreases for large values of $x$? Which functions are not defined for negative $x$ values? Compare the values of these functions for $0 < x < 1$. Which of these functions are not defined at $x = 0$?

1.4. **Finding points of intersection(I):**

(a) Consider the two functions $f(x) = 3x^2$ and $g(x) = 2x^5$. Find all points of intersection of these functions.

(b) Repeat the calculation for the two functions $f(x) = x^3$ and $g(x) = 4x^5$.

1.5. **Finding points of intersection(II):** Consider the two functions $f(x) = Ax^n$ and $g(x) = Bx^m$. Suppose $m > n > 1$ are integers, and $A, B > 0$. Determine the values of $x$ at which the values of the functions are the same. Are there two places of intersection or three? How does this depend on the integer $m - n$? (Remark: The point (0,0) is always an intersection point. Thus, we are asking when there is only *one* more and when there are *two* more intersection points. See Problem 4 for a simple example of both types.)

1.6. **More intersection points:** Find the intersection of each pair of functions.

(a) $y = \sqrt{x}$, $y = x^2$

(b) $y = -\sqrt{x}$, $y = x^2$

(c) $y = x^2 - 1$, $\frac{x^2}{4} + y^2 = 1$

1.7. **Roots of a quadratic:** Find the range of $m$ such that the equation $x^2 - 2x - m = 0$ has two unequal roots.

1.8. **Power functions with negative powers:** Consider the function

$$f(x) = \frac{A}{x^a}$$

where $A > 0, a > 1$, with $a$ an integer. This is the same as the function $f(x) = Ax^{-a}$, which is a power function with a negative power.

(a) Sketch a rough graph of this function for $x > 0$.

(b) How does the function change if $A$ is increased?

(c) How does the function change if $a$ is increased?

1.9. **Intersections of functions with negative powers:** Consider two functions of the form

$$f(x) = \frac{A}{x^a}, \quad g(x) = \frac{B}{x^b}.$$

Suppose that $A, B > 0,\ a, b > 1$ and that $A > B$. Determine where these functions intersect for positive $x$ values.

1.10. **Zeros of polynomials:** Find all real zeros of the following polynomials:

(a) $x^3 - 2x^2 - 3x$

(b) $x^5 - 1$

(c) $3x^2 + 5x - 2$.

(d) Find the points of intersection of the functions $y = x^3 + x^2 - 2x + 1$ and $y = x^3$.

1.11. **Qualitative sketching skills:**

(a) Sketch the graph of the function $y = ax - x^5$ for positive and negative values of the constant $a$. Comment on behaviour close to zero and far away from zero.

(b) What are the zeros of this function and how does this depend on $a$ ?

(c) For what values of $a$ would you expect that this function would have a local maximum ("peak") and a local minimum ("valley")?

1.12. **Inverse functions:** The functions $y = x^3$ and $y = x^{1/3}$ are *inverse functions*.

(a) Sketch both functions on the same graph for $-2 < x < 2$ showing clearly where they intersect.

(b) The tangent line to the curve $y = x^3$ at the point (1,1) has slope $m = 3$, whereas the tangent line to $y = x^{1/3}$ at the point (1,1) has slope $m = 1/3$. Explain the relationship of the two slopes.

1.13. **Properties of a cube:** The volume $V$ and surface area $S$ of a cube whose sides have length $a$ are given by the formulae

$$V = a^3, \quad S = 6a^2.$$

Note that these relationships are expressed in terms of power functions. The independent variable is $a$, not $x$. We say that "$V$ is a function of $a$" (and also "$S$ is a function of $a$").

(a) Sketch $V$ as a function of $a$ and $S$ as a function of $a$ on the same set of axes. Which one grows faster as $a$ increases?

(b) What is the ratio of the volume to the surface area; that is, what is $\frac{V}{S}$ in terms of $a$? Sketch a graph of $\frac{V}{S}$ as a function of $a$.

(c) The formulae above tell us the volume and the area of a cube of a given side length. But suppose we are given either the volume or the surface area and asked to find the side. Find the length of the side as a function of the volume (i.e. express $a$ in terms of $V$). Find the side as a function of the surface area. Use your results to find the side of a cubic tank whose volume is 1 litre (1 litre = $10^3$ $\text{cm}^3$). Find the side of a cubic tank whose surface area is $10$ $\text{cm}^2$.

1.14. **Properties of a sphere:** The volume $V$ and surface area $S$ of a sphere of radius $r$ are given by the formulae

$$V = \frac{4\pi}{3}r^3, \quad S = 4\pi r^2.$$

Note that these relationships are expressed in terms of power functions with constant multiples such as $4\pi$. The independent variable is $r$, not $x$. We say that "$V$ is a function of $r$" (and also "$S$ is a function of $r$").

(a) Sketch $V$ as a function of $r$ and $S$ as a function of $r$ on the same set of axes. Which one grows faster as $r$ increases?

(b) What is the ratio of the volume to the surface area; that is, what is $\frac{V}{S}$ in terms of $r$? Sketch a graph of $\frac{V}{S}$ as a function of $r$.

(c) The formulae above tell us the volume and the area of a sphere of a given radius. But suppose we are given either the volume or the surface area and asked to find the radius. Find the radius as a function of the volume (i.e. express $r$ in terms of $V$). Find the radius as a function of the surface area. Use your results to find the radius of a balloon whose volume is 1 litre. (1 litre = $10^3$ $\text{cm}^3$). Find the radius of a balloon whose surface area is $10$ $\text{cm}^2$

1.15. **The size of cell:** Consider a cell in the shape of a thin cylinder (length $L$ and radius $r$). Assume that the cell absorbs nutrient through its surface at rate $k_1 S$ and consumes nutrients at rate $k_2 V$ where $S, V$ are the surface area and volume of the cylinder. Here we assume that $k_1 = 12\mu$M $\mu$m$^{-2}$ per min and $k_2 = 2\mu$M $\mu$m$^{-3}$ per min. (Note: $\mu$M is $10^{-6}$ moles. $\mu$m is $10^{-6}$meters.) Use the fact that a cylinder (without end-caps) has surface area $S = 2\pi rL$ and volume $V = \pi r^2 L$ to determine the cell radius such that the rate of consumption exactly balances the rate of absorption. What do you expect happens to cells with a bigger or smaller radius? How does the length of the cylinder affect this nutrient balance?

1.16. **Allometric relationship:** Properties of animals are often related to their physical size or mass. For example, the metabolic rate of the animal ($R$), and its pulse rate ($P$) may be related to its body mass $m$ by the approximate formulae $R = Am^b$ and $P = Cm^d$, where $A, C, b, d$ are positive constants. Such relationships are known as *allometric* relationships.

(a) Use these formulae to derive a relationship between the metabolic rate and the pulse rate (Hint: eliminate $m$).

(b) A similar process can be used to relate the Volume $V = (4/3)\pi r^3$ and surface area $S = 4\pi r^2$ of a sphere to one another. Eliminate $r$ to find the corresponding relationship between volume and surface area for a sphere.

1.17. **Rate of a very simple chemical reaction:** Here we consider a chemical reaction that does not saturate, and consider the simple linear relationship between reaction speed and reactant concentration. A chemical is being added to a mixture and is used up by a reaction that occurs in that mixture. The rate of change of the chemical, (also called "the rate of the reaction") $v$ (in units of M /sec where M stands for Molar, which is the number of moles per litre) is observed to follow a relationship $v = a - bc$ where $c$ is the reactant concentration (in units of M) and $a, b$ are positive constants. (Note that here $v$ is considered to be a function of $c$, and moreover, the relationship between $v$ and $c$ is assumed to be linear.)

(a) What units should $a$ and $b$ have to make this equation consistent? (Remember: in an equation such as $v = a - bc$, each of the three terms *must have* the same units. Otherwise, the equation would not make sense.)

(b) Use the information in the graph shown in Figure 1.7 to find the values of $a$ and $b$. (To do so, you should find the equation of the line in the figure, and compare it to the relationship $v = a - bc$.)

(c) What is the rate of the reaction when $c = 0.005$ M?

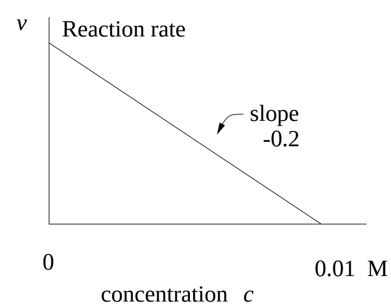



**Figure 1.7.** *Figure for problem 17*

1.18. **Michaelis-Menten kinetics:** Consider the Michaelis-Menten kinetics where the speed of an enzyme-catalyzed reaction is given by $v = Kx/(k_n + x)$.

(a) Explain the statement that "when $x$ is large there is a horizontal asymptote" and find the value of $v$ to which that asymptote approaches.

(b) Determine the reaction speed when $x = k_n$ and explain why the constant $k_n$ is sometimes called the "half-max" concentration.

1.19. **A polymerization reaction:** Consider the speed of a polymerization reaction shown in Figure 1.8. Here the rate of the reaction is plotted as a function of the substrate concentration. (The experiment concerned the polymerization of actin, an important structural component of cells; data from Rohatgi et al (2001) J Biol Chem

276(28):26448-26452.) The experimental points are shown as dots, and a Michaelis-Menten curve has been drawn to best fit these points. Use the data in the figure to determine approximate values of $K$ and $k_n$ in the two treatments shown.

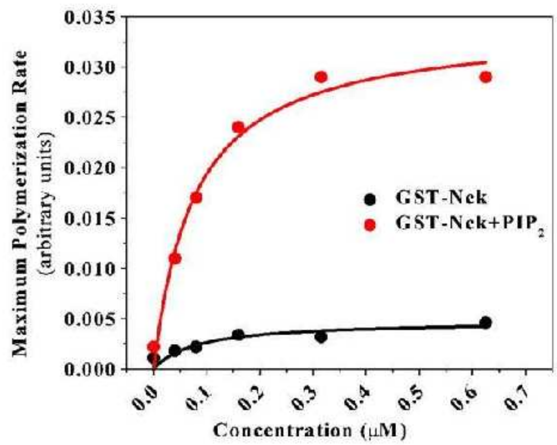



**Figure 1.8.** *Figure for problem 19*

1.20. **Hill functions:** Hill functions are sometimes used to represent a biochemical "switch", that is a rapid transition from one state to another. Consider the Hill functions

$$y_1 = \frac{x^2}{1+x^2}, \quad y_2 = \frac{x^5}{1+x^5},$$

(a) Where do these functions intersect?

(b) What are the asymptotes of these functions?

(c) Which of these functions increases fastest near the origin?

(d) Which is the sharpest "switch" and why?

1.21. **Transforming a Hill function to a linear reationship:** A Hill function is a nonlinear function. But if we redefine variables, we can transform it into a linear relationship. The process is analogous to transforming Michaelis-Menten kinetics into a Linweaver-Burke plot. Determine how to define appropriate variables $X$ and $Y$ (in terms of the original variables $x$ and $y$) so that the Hill function $y = Ax^3/(a^3 + x^3)$ is turned into a linear relationship between $X$ and $Y$. Then indicate how the slope and intercept of the line are related to the original constants $A, a$ in the Hill function.

1.22. **Hill function and sigmoidal chemical kinetics:** It is known that the *rate* $v$ at which a certain chemical reaction proceeds depends on the *concentration* of the reactant $c$ according to the formula

$$v = \frac{Kc^2}{a^2 + c^2}$$

where $K$, $a$ are some constants. When the chemist plots the values of the quantity $1/v$ (on the "$y$" axis) versus the values of $1/c^2$ (on the "$x$ axis"), she finds that the points are best described by a straight line with $y$-intercept $2$ and slope $8$. Use this result to find the values of the constants $K$ and $a$.

1.23. **Linweaver-Burke plots:** Shown in the Figure (a) and (b) are two Linweaver Burke plots. By noting properties of these figures comment on the comparison between the following two enzymes:

(a) Enzyme (1) and (2).

(b) Enzyme (1) and (3).

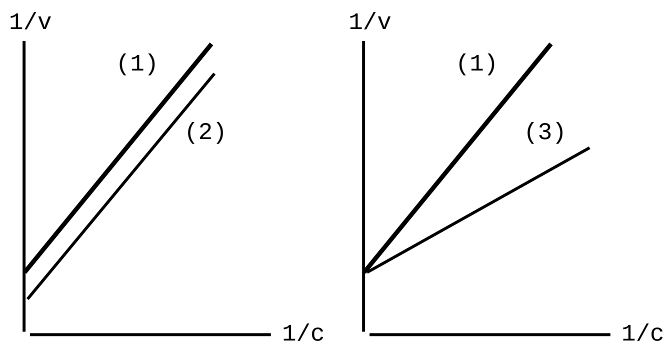



**Figure 1.9.** *Figure for problem 23*

1.24. **Michaelis Menten Enzyme kinetics:** The rate of an enzymatic reaction according to the *Michaelis Menten Kinetics* assumption is

$$v = \frac{Kc}{k_n + c},$$

where $c$ is concentration of substrate (shown on the $x$ axis) and $v$ is the reaction speed (given on the $y$ axis). Consider the data points given in the table below:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Substrate conc | $nM$ | c | 5. | 10. | 20. | 40. | 50. | 100. |
| Reaction speed | $nM/min$ | v | 0.068 | 0.126 | 0.218 | 0.345 | 0.39 | 0.529 |

Convert this data to a Linweaver-Burke (linear) relationship. Plot the transformed data values on a graph or spreadsheet, and estimate the slope and $y$-intercept of the line you get. Use these results to find the best estimates for $K$ and $k_n$.

1.25. **Spacing in a school of fish:** According to the biologist Breder (1950), two fish in a school prefer to stay some specific distance apart. Breder suggested that the fish that are a distance $x$ apart are attracted to one another by a force $F_A(x) = A/x^a$ and repelled by a second force $F_R(x) = R/x^r$, to keep from getting too close. He found the preferred spacing distance (also called the *individual distance*) by determining the value of $x$ at which the repulsion and the attraction exactly balance. Find the *individual distance* in terms of the quantities $A, R, a, r$ (all assumed to be positive constants.)

# Chapter 2

# Average rates of change, average velocity and the secant line

In this chapter, we introduce the idea of an average rate of change. To motivate ideas, we examine data for two common processes, changes in temperature, and motion of a falling object. Simple experiments are described in each case, and some features of the data are discussed. Based on each example, we define and calculate net change over some time interval and so define the average rate of change.This concept generalizes to functions of any variable (not only time). We interpret this idea geometrically, in terms of the slope a secant line.

In both cases, we then ask how to use the idea of an average rate of change (over a given interval) to find better and better approximations of the rate of change at a single instant, (i.e. at a point). We will see that one way to arrive at this abstract concept entails refining the dataset - collecting data at closer and closer time points. A second, more abstract way, is to use the idea of a limit. Eventually, this procedure will allow us to arrive at the definition of the derivative, which is the instantaneous rate of change.

## 2.1 Milk temperature in a recipe for yoghurt

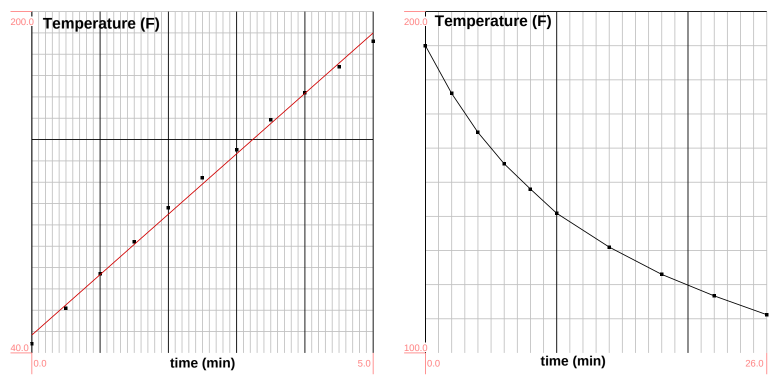
Making yoghurt calls for heating milk to 190°F to kill off undesirable bacteria, and then cooling to 110°F. Some pre-made yoghurt with "live culture" is added, and the mixture kept at 110°F for 7-8 hours. This is the ideal temperature for growth of *Lactobacillus*,, a useful micro-organism turns milk into yoghurt as a byproduct of its growth[2].

Experienced yoghurt-makers follow the temperature of the milk with a thermometer to avoid scalding the milk or missing the desired final temperature. A set of temperature measurements[3] is shown in Table 2.1.

To visualize the trend of the data we plot the temperature versus time in Fig. 2.1(a,b), where (a) is the heating phase and (b) the cooling phase of the process. This makes a number of features stand out.

[2] The initial heating also denatures milk proteins, which prevents the milk from turning into curds. Adding some ready-made yoghurt with live culture and keeping the mixture warm for 8-10 hrs will then result in fresh yoghurt.

[3] The data was collected by your instructor in her kitchen.




**(a)** **(b)**

**Figure 2.1.** *Plots of the data shown in Table 2.1 (a) Heating and (b) cooling milk in yoghurt production.*

(a) Heating

| time (min) | Temperature (F) |
|---|---|
| 0.0 | 44.3 |
| 0.5 | 61 |
| 1.0 | 77 |
| 1.5 | 92. |
| 2.0 | 108 |
| 2.5 | 122 |
| 3.0 | 135.3 |
| 3.5 | 149.2 |
| 4.0 | 161.9 |
| 4.5 | 174.2 |
| 5.0 | 186 |

(b) Cooling

| time (min) | Temperature (F) |
|---|---|
| 0 | 190 |
| 2 | 176 |
| 4 | 164.6 |
| 6 | 155.4 |
| 8 | 148 |
| 10 | 140.9 |
| 14 | 131 |
| 18 | 123 |
| 22 | 116 |
| 26 | 111.2 |

**Table 2.1.** *Temperature of the milk as it is (a) Heated and (b) Cooled before adding live yoghurt bacteria.*

- In Fig. 2.1(a) the temperature increases and in Fig. 2.1(b) it decreases.
- The measurements are discrete, that is, we only have a finite number of points at which the temperature was recorded.
- In (a) the increasing phase “looks like” a straight line, whereas the cooling phase in (b) is clearlynot linear. That is (a) appears to be close to *linear* whereas (b) displays an obvious *nonlinear* relationship.

- In Fig. 2.1(a), we have drawn a straight line that appears to capture the data trend fairly well[4]. In Fig. 2.1(b), simply connecting the data points leads to the resulting black curve.

To discuss the trends we have observed, it will be beneficial to define convenient notation. We will let

$$\begin{aligned} t &= \text{time}, \\ T_0 &= \text{initial temperature of the milk}, \\ T(t) &= \text{temperature of the milk at time } t, \\ \Delta t &= \text{an interval of time}, \\ \Delta T &= \text{a change in the temperature of the milk} \end{aligned}$$

These observations lead to several questions.

1. How "fast" is the temperature $T(t)$ increasing in (a)?

2. How fast is it decreasing in (b)?

The notion of a rate of change will be useful in addressing these questions. We define this concept shortly, but first we consider another common example.

## 2.2 A moving object

We next consider an example that will motivate the rate of change of position of a moving object, for which the term **velocity** is commonly used. **Uniform motion** is defined as motion in which a constant distance is covered in constant time intervals. For particles moving uniformly, velocity is constant, and is simply the distance travelled per unit time, or simply displacement divided by time taken. In uniform motion, velocity does not change over time.

Most types of motion that occur in natural systems are not that simple: an example in Figure 2.2 shows successive heights of an object falling under the effect of gravity. As shown, a falling object covers *increasing* distances as time progresses so that the velocity changes with time. In this situation, we have to rethink how to define the notion of velocity at a given time, and we have to formulate more precisely how we will calculate it. Such questions lead us to the central idea in this chapter: the definition of **average** and **instantaneous** velocity.

Figure 2.2 displays a set of three stroboscopic images combined (for visualization purposes) on a single graph. Each set of dots shows successive vertical positions of an object falling from a height of 20 meters over a 2 second time period. In (a) the location of the ball is given at times $t = 0, 0.5, 1, 1.5$, and $2.0$ seconds, i.e. at intervals of $\Delta t = 0.5$ seconds. (A strobe flashing five times, once every $\Delta t = 0.5$ would produce this data.)

We might wonder where the ball is located at times between these successive measurements. Did it vanish? Did it continue in a straight or a looped path? To find out what happened during the intervals between data points, we increase the strobe frequency, and

[4] How to pick the best such line will be the subject of future discussion.

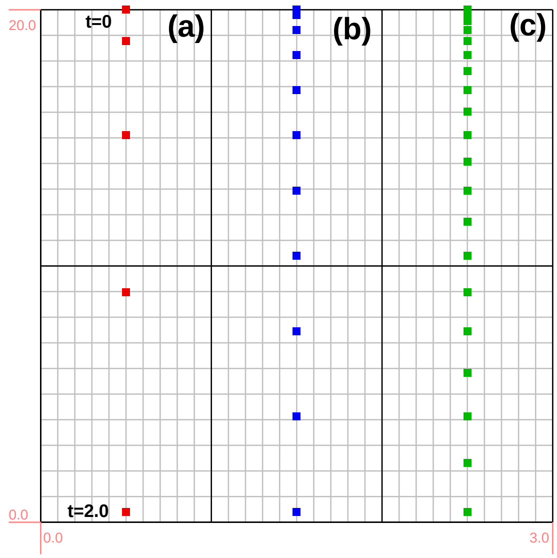


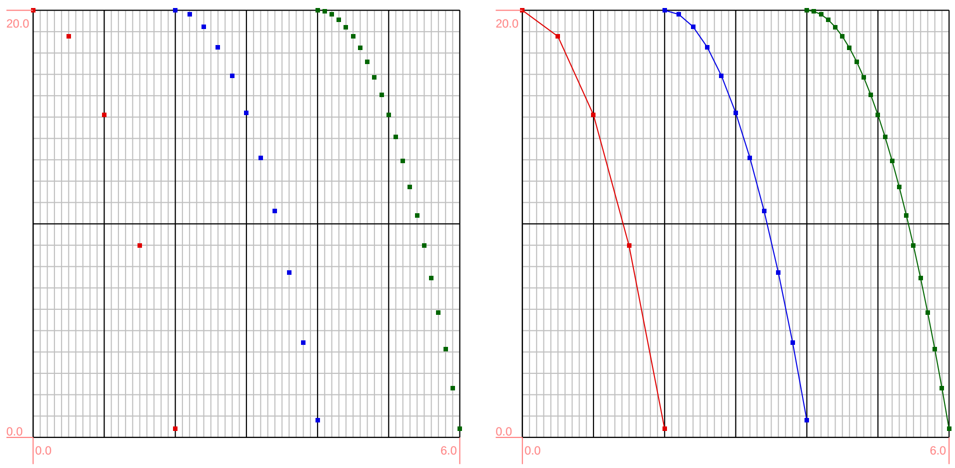

**Figure 2.2.** *The height of an object falling under the effect of gravity is shown (from $t = 0$ top, to $t = 2$ bottom). The time interval $\Delta t$ at which data is collected has been refined (left to right) to get more and more accurate tracking of the object. (a) $\Delta t = 0.5$, (b) $\Delta t = 0.2$, (c) $\Delta t = 0.1$.*

record measurements more often: for example, in Figure 2.2(b) measurements were made for $t = 0, 0.2, 0.4, \ldots, 2.0$ seconds, i.e. at intervals $\Delta t = 0.2$s. An even closer set of points appears in (c), where the time interval between strobe flashes was decreased to $\Delta t = 0.1$s. By determining the position of the ball at closer time points, we can determine the trajectory of the ball with greater accuracy. The idea of making measurements at finer and finer time increments is important in this example. We will return to it often in our goal of understanding rates of change of natural processes.

We represent the data for the motion of the object in a graph in Figure 2.3. Here we have added a time axis for each of the sequences, so that the position(on the vertical axis) and time horizontal axes) are plotted together. Again for ease of visualization, we have combined three possible experiments on the same grid.

We will use the following notation:

$$\begin{aligned}
t &= \text{time},\\
Y_0 &= \text{initial height of the object},\\
Y(t) &= \text{height of the object at time } t,\\
\Delta t &= \text{an interval of time},\\
\Delta Y &= \text{a change in the vertical position of the object}\\
&= \text{the displacement of the object.}
\end{aligned}$$




**(A)** **(B)**

**Figure 2.3.** *(A) The positions of the object are plotted versus time in each of three experiments. We have decreased the time between strobe flashes:* $\Delta t = 0.5, 0.2, 0.1$ *for the trials (from left to right). (B) The same plot, with straight line segments connecting the data points. The slopes of these straight lines (secant lines) are defined as average velocity over the given time intervals.*

Note that we have indicated above that we consider $Y$ to be a function of time by writing $Y(t)$. Also worth noting is that we have defined the change in height or in position $\Delta Y$ as the **displacement** of the object. We later study the motion of a falling object using a variable $y(t)$ that represents the distance fallen. Problem 3 explores the connection between these.

## 2.3 Average rate of change

In the examples discussed in this chapter, the independent variable is time, and we have been considering variables such as temperature and position that depend on time. In this case, we could represent these relationships by a function $f(t)$. Then for a given time interval, $a \leq t \leq b$, we define the **average rate of change** of $f$ over the given interval to be:

$$\bar{r} = \frac{\text{Change in } f}{\text{Change in } t} = \frac{\Delta f}{\Delta t} = \frac{f(b) - f(a)}{b - a}.$$

We use this definition to compute the average rate of change for each of the examples presented earlier.

**Example 2.1 (Average velocity of a falling object)** Find the average velocity of the falling object over the time interval $0 \leq t \leq 0.5$

**Solution:** Since $Y(t)$ is the height of the object, the average rate of change of $Y$ with respect to time is an average velocity. For a given time interval, the average velocity $\bar{v}$, is simply defined as

$$\bar{v} = \frac{\text{Change in height}}{\text{Time taken}} = \frac{\Delta Y}{\Delta t} = \frac{Y(b) - Y(a)}{b - a}.$$

For example, in the data shown in Fig 2.3, it turns out that over the time interval $0 \leq t \leq 0.5$ the ball fell from $Y(0) = 20$ to $Y(0.5) = 18.775$m. This leads to an average velocity over the given time interval of

$$\bar{v} = \frac{Y(0.5) - Y(0)}{0.5} = \frac{18.775 - 20}{0.5} = -2.45\text{m/s.}$$

The average velocity is negative since the height decreases over the given interval.

The average velocity can be computed between any two data points. As we have already seen, this quantity depends on the time interval over which it is computed. As a second computation, we could compute average velocity for $1.5 \leq t \leq 2$. Over that time, we find that the height changed from $Y(1.5) = 8.9750$ to $Y(2) = 0.4$m. Computing the average velocity over this time interval leads to -17.15 m/s. Clearly the object is falling at a faster average rate.

We can put a geometric meaning to this formula by making the following observation. On our graphs of the position of a falling object versus time, let us connect successive points by straight lines, as shown in Figure 2.3(B). Then we say that: *The average rate of change of the height $Y$ between any two time points is just the slope of the straight line connecting the corresponding points on the graph of* $Y(t)$. We denote that line by the term **secant line**.

**Example 2.2 (Average rate of change of temperature)** Find the average rates of change of the temperature over the time interval $2 < t < 4$ for both the heating and the cooling milk. ■

**Solution:** The data shown in Table 2.1 tabulated the temperature $T(t)$ versus time $t$ in minutes. Over a given time interval, the average rate of change of the temperature is

$$\frac{\text{Change in temperature}}{\text{Time taken}} = \frac{\Delta T}{\Delta t}.$$

For example, the average rate of change of temperature as the milk cools over the interval $2 < t < 4$ min is

$$\frac{(164.6 - 176)}{(4 - 2)} = -5.7^\circ/\text{min.}$$

Over a similar time interval for the heating milk, the average rate of change of the temperature is

$$\frac{(161.9 - 108)}{(4 - 2)} = 26.95^\circ/\text{min.}$$

Were we to connect two points $(2, T(2))$ and $(4, T(4)$ on one of the graphs in Fig. 2.1, we would find a line segment whose slope matches the average rate of change we have computed here. As before, this is the secant line.

We can write down the **equation of the secant line** (if desired) by using the fact that it goes through the given points (or, alternately, by using one point and the computed slope). Since the latter secant line goes through the point $(t, T) = (2, 108)$ and has slope 26.95, we find that

$$\frac{(y_T - 108)}{t-2} = 26.95 \quad \Rightarrow \quad y_T = 108 + 26.95(t-2) \quad \Rightarrow \quad y_T = 26.5t + 54.1,$$

where we have used $y_T$ as the height of the secant line, to avoid confusion with $T(t)$ which is the actual temperature as a function of the time.

We can extend the definition of the average rate of change to any function.

**Definition 2.3 (Average rate of change of a function:).** *Suppose $y = f(x)$ is a function of some arbitrary variable $x$. The the* average rate of change *of $f$ between two points $x_0$ and $x_0 + h$ is given by*

$$\frac{\textit{change in y}}{\textit{change in t}} = \frac{\Delta y}{\Delta x} = \frac{[f(x_0+h) - f(x_0)]}{(x_0+h) - x_0} = \frac{[f(x_0+h) - f(x_0)]}{h}$$

*Here $h$ is the difference of the $x$ coordinates and the ratio we have just computed is the slope of the secant line shown in Figure 2.4(b).*

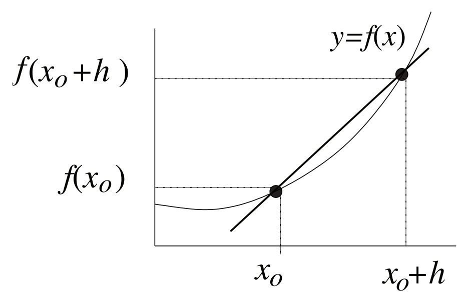



**Figure 2.4.** *The graph of some arbitrary function $f(x)$ is shown. Two points on this graph, $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ are identified. and the line connecting these is the secant line.The slope of this line is the average rate of change of $f$ over the interval $x_0 < x < x_0 + h$.*

We caution that the word "average" sometimes causes confusion. One often speaks in a different context of the average value of a set of numbers (e.g. the average of $\{7, 1, 3, 5\}$ is $(7 + 1 + 3 + 5)/4 = 4$.) However the average rate of change is always defined in terms of a pair of points. It is not the average of some arbitrary set of values.

**Example 2.4** Prof Molly Lutcavage studied the swimming behaviour of Atlantic bluefin tuna (*Thunnus thynnus L.*) in the Gulf of Maine. She recorded their position over a period of 1-2 days. Here we consider the length of the tuna tracks. Two approximate data sets are shown in Fig 2.5. Determine the average velocity of each of these two fish over the 35h shown in the figure. What is the fastest average velocity shown in this figure, and over what time interval and which fish did it occur? ■

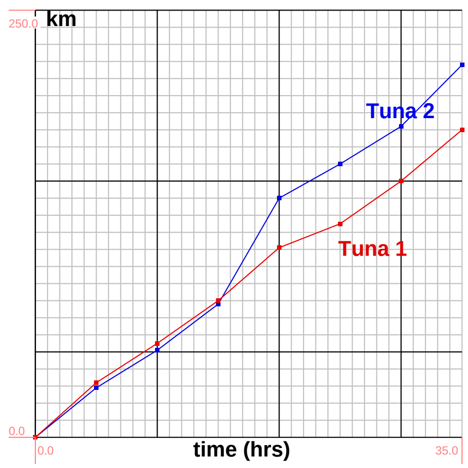



**Figure 2.5.** *Distance travelled by two bluefin tuna over 35 hrs*

**Solution:** We find that Tuna 1 swam 180 km over the course of 35 hr, whereas Tuna 2 swam 218 km during the same time period. Thus the average velocity of Tuna 1 was $\bar{v} = 180/35 \approx 5.14$ km/h, whereas a similar calculation for Tuna 2 yields 6.23 km/h. The fastest average velocity would correspond to the segent of the graph that has the largest slope. We see that the blue curve (Tuna 2) has the greatest slope during the time interval $15 < t < 20$. Indeed, we find that the tuna covered a distance from the distance covered over that 5 hr interval was from 78 to 140 km over that time, a displacement of 140-78=62km. Its average velocity over that tie interval was thus $62/5 = 12.4$km/h.

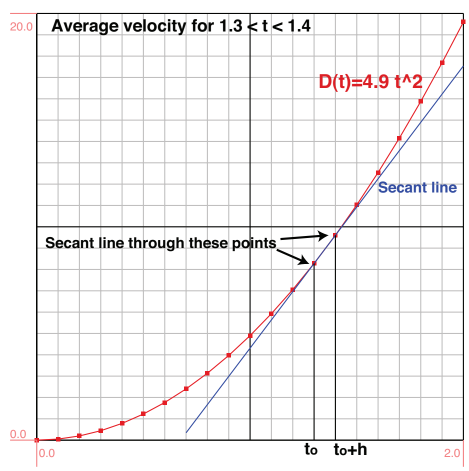
## 2.4 Gallileo's remarkable finding

Observations recording the position of a falling object were made long ago by Galileo. He devised some ingenious experiments in which he was able to uncover an interesting relationship between the total distance that an object (falling under the force of gravity) moves during a given total time. Here we will define the variable $y(t)$ to be the distance fallen at time $t$. There is a simple relationship between the height $Y(t)$ and the distance fallen $y(t)$ that the reader should note. (See Exercise 3.)

Galileo realized that a simple relationship exists between the distance fallen by an object under the effect of gravity and the time taken. Galileo discovered that the distance fallen under the effect of gravity was proportional to the square of the time, i.e., that

$$y(t) = ct^2, \tag{2.1}$$

where $c$ is a constant. We recognize this quadratic relationship as a simple power function with a constant coefficient. (Later in this course, we will see that this follows directly from the fact that gravity causes constant acceleration - but Galileo, did not realize this fact, nor did he have a clear idea about what acceleration meant.) When precise measurements are carried out, with units of meters (m) for the distance, and seconds (s) for the time, then it is found that $c = 4.9\mathrm{m/s}^2$. Although Galileo did not have formulae or graph-paper in his day, (and was thus forced to express this relationship in a cumbersome verbal way), what he had discovered was quite remarkable.




| $t$ | $y(t) = 4.9t^2$ |
|---|---|
| 0.0 | 0.0 |
| 0.1 | 0.0490 |
| 0.2 | 0.1960 |
| $\vdots$ | $\vdots$ |
| 1. 0 | 4.9 |
| 1.1 | 5.9290 |
| 1.2 | 7.0560 |
| 1.3 | 8.2810 |
| 1.4 | 9.6040 |
| 1.5 | 11.0250 |
| $\vdots$ | $\vdots$ |
| 1.9 | 17.6890 |
| 2. 0 | 19.6 |

**Table 2.2.** *The average velocity between time $t_0 + h$ and $t_0$ is the slope of the secant line shown on this graph. Some values of $(t, y)$ for the function $y = 4.9t^2$ are given in the table.*

We show a graph of the relationship (2.1) for $c = 4.9$ together with some points on that graph in Table 2.2. On our sketch, we have superimposed the secant line connecting the points at $t_0$ and $t_0 + h$. We now compute the average velocity in this more general setting.

**Example 2.5 (Average velocity using Galileo's formula)** Consider a falling object. Suppose that the total distance fallen at time $t$ is given by Eqn. (2.1). Find the average velocity $\bar{v}$, of the object over the time interval $t_0 \le t \le t_0 + h$. ■

**Solution:**

$$\begin{aligned}
\bar{v} &= \frac{y(t_0+h)-y(t_0)}{h} \\
&= \frac{c(t_0+h)^2 - c(t_0)^2}{h} \\
&= c\left(\frac{(t_0^2+2ht_0+h^2)-(t_0^2)}{h}\right) \\
&= c\left(\frac{2ht_0+h^2}{h}\right) \\
&= c(2t_0+h)
\end{aligned}$$

Thus the average velocity over the time interval $t_0 < t < xt_0 + h$ is $\bar{v} = c(2t_0 + h)$.

In an upcoming section, we will use this result to ask what would be a reasonable definition of the **instantaneous velocity** at some time. The essential idea will be to compute the average velocity over a smaller and smaller time interval, which is equivalent to letting the value $h$ become smaller and smaller. This notion has already been encountered in the idea of refining the measurements. For the falling object, collecting data over smaller time intervals corresponds to the letting the time between strobe flashes get smaller and smaller.

## 2.5 Refining the data

### 2.5.1 Refined temperature data

In Fig 2.6 we show a similar process of refining the data for temperature versus time. We not that this process of refinement will allow us to define a better and better concept of the rate of change close to a given time.

### 2.5.2 Instantaneous velocity

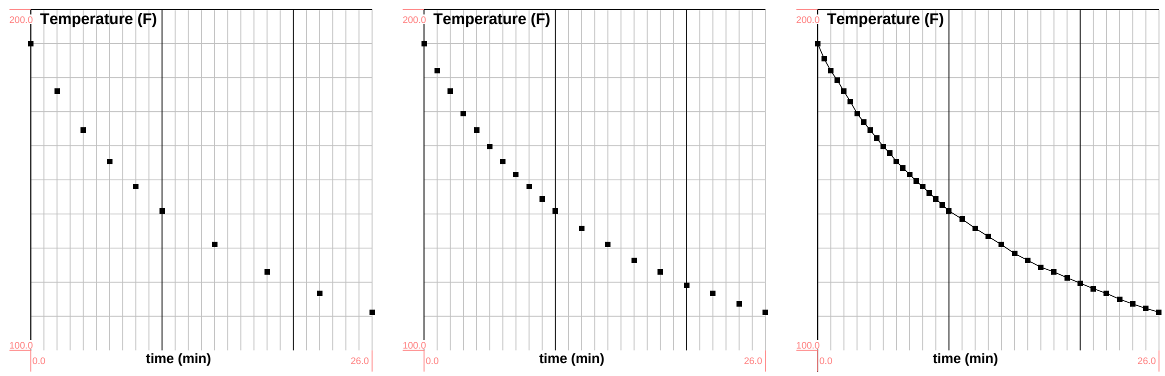
To arrive at a notion of an instantaneous velocity at some time $t_0$, we will consider defining average velocities over time intervals $t_0 \le t \le t_0 + h$, that get smaller and smaller: For example, we might make the strobe flash faster, so that the time between flashes, $\Delta t = h$ decreases. (We use the notation $h \to 0$ to denote the fact that we are interested in shrinking the time interval.)

At each stage, we calculate an average velocity, $\bar{v}$ over the time interval $t_0 \le t \le t_0 + h$. As the interval between measurements gets smaller, i.e the process of refining our measurements continues, we arrive at a number that we will call *the instantaneous velocity*. This number represents ”the velocity of the ball at the very instant $t = t_0$”.

More precisely,

$$\text{Instantaneous velocity} = v(t_0) = \lim_{h\to 0} \bar{v} = \lim_{h \to 0} \frac{y(t_0+h)-y(t_0)}{h}.$$

**Example 2.6** Find the instantaneous velocity of the same falling object at time $t_0$. ■




**(a)** **(b)** **(c)**

**Figure 2.6.** *Three graphs of the temperature of cooling milk showing (a) a coarse data set (measurements every $\Delta t = 2$ min.), (b) a more refined data set, (measurements every $\Delta t = 1$ min) (c) an even more refined dataset (measurements every $\Delta t = 0.5$ min.)*

**Solution:** According to our definition, we must determine

$$v(t_0) = \lim_{h \to 0} \frac{y(t_0 + h) - y(t_0)}{h}$$

Our calculation would be nearly identical for $v(t_0)$ as for $\bar{v}$, but for a final step of taking the limit as $h$, the interval between time-points shrinks to zero:

$$\begin{aligned}
v(t_0) &= \lim_{h \to 0} \frac{y(t_0 + h) - y(t_0)}{h} \\
&= \lim_{h \to 0} \frac{c(t_0 + h)^2 - c(t_0)^2}{h} \\
&= \lim_{h \to 0} c\left(\frac{(t_0^2 + 2ht_0 + h^2) - (t_0^2)}{h}\right) \\
&= \lim_{h \to 0} c\left(\frac{2ht_0 + h^2}{h}\right) \\
&= \lim_{h \to 0} c(2t_0 + h) \\
&= c(2t_0) = 2ct_0
\end{aligned} \tag{2.2}$$

Remarks: In our final steps, we have allowed $h$ to shrink. In the limit, and $h \to 0$, we obtain the instantaneous velocity, i.e. $v(t_0) = 2ct_0$.

## 2.6 Introduction to the derivative

With the concepts introduced in this chapter, we are ready for the the definition of the derivative.

**Definition 2.7 (The derivative:).**
*denoted $f'(x_0)$ and defined as*

$$f'(x_0) = \lim_{h \to 0} \frac{[f(x_0+h) - f(x_0)]}{h}$$

**Example 2.8 (Calculating the derivative)** Compute the derivative of the function $f(x) = Cx^2$ at some point $x = x_0$. $\blacksquare$

**Solution:** In the previous section, we used the function $y = f(t) = ct^2$ to calculate an instantaneous velocity. We now recognize that our result in that problem, namely $2ct_0$ is the same as the derivative of the function evaluated at $t_0$. Thus, in a sense, we have already solved this problem. By switching notation ($t_0 \to x_0$ and $c \to C$) we can write down the answer, $2cx_0$ at once. However, as practice, we can rewrite the steps in the case of the general point $x$

For $y = f(x) = Cx^2$ we have

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{C(x+h)^2 - Cx^2}{h} \\
&= \lim_{h \to 0} C \frac{(x^2 + 2xh + h^2) - x^2}{h} \\
&= \lim_{h \to 0} C \frac{(2xh + h^2)}{h} \\
&= \lim_{h \to 0} C(2x + h) \\
&= C(2x) = 2Cx
\end{aligned}$$

Evaluating this result for $x = x_0$ we obtain the answer $2Cx_0$.

We recognize from this definition that the derivative is obtained by starting with the slope of a secant line (average rate of change of $f$ over the interval $x_0 < x < x_0 + h$) and proceeds to shrink the interval ($\lim_{h \to 0}$) so that it approaches a single point ($x_0$). The resultant line will be denoted the **tangent line**. , and the value obtained will be identified as the the **instantaneous rate of change** of the function with respect to the variable $x$ at the point of interest, $x_0$. Another notation used for the derivative is

$$\left.\frac{df}{dx}\right|_{x_0}.$$

We will explore properties and meanings of this concept in the next chapter.

## Exercises

2.1. **Heating milk:** Consider the data gathered for heating milk in Table 2.1 and Fig. 2.1(a).

(a) Estimate the slope and the intercept of the straight line shown in the figure and use this to write down the equation of this line. According to this approximate straight line relationship, what is the average rate of change of the temperature over the 5 min interval shown?

(b) Find a pair of points such that the average rate of change of the temperature is *smaller* than your result in part (a).

(c) Find a pair of points such that the average rate of change of the temperature is *greater* than your result in part (a).

(d) Milk boils at 212○F, and the recipe for yoghurt calls for avoiding a temperature this high. Use your common knowledge to explain why the data for heating milk is not actually linear.

2.2. **Refining the data:** Table 2.3 shows some of the data for cooling milk that was collected and plotted in Fig. 2.6. Answer the following questions.

| time | Temp | time | Temp | time | Temp |
|---|---|---|---|---|---|
| 0 | 190 | 0 | 190 | 0 | 190 |
| 2 | 176 | 1 | 182 | 0.5 | 185.5 |
| 4 | 164.6 | 2 | 176 | 1 | 182 |
| 6 | 155.4 | 3 | 169.5 | 1.5 | 179.2 |
| 8 | 148 | 4 | 164.6 | 2 | 176 |
| 10 | 140.9 | 5 | 159.8 | 2.5 | 172.9 |

**Table 2.3.** *Partial data for temperature in degrees Farenheit for the three graphs shown in Fig. 2.6. The pairs of columns indicate that the data has been collected at more and more frequent intervals.*

(a) Use the above table to determine the average rate of change of the temperature over the first 10 min.

(b) Compute the average rate of change of the temperature over the intervals $0 < t < 2$, $0 < t < 1$ and $0 < t < 0.5$.

(c) Which of your results in (b) would be closest to the "instantaneous" rate of change of the temperature at $t = 0$?

2.3. **Height and distance dropped:** We have defined the variable $Y(t)$ =height of the object at time $t$and the variable $y(t)$ as the distance dropped by time $t$. State the connection between these two variables for a ball whose initial height is $Y_0$. How is the displacement over some time interval $a < t < b$ related between these two ways of describing the motion? (Assume that the ball is in the air throughout this time interval).

2.4. **Height of a ball:** The vertical height of a ball, $Y$ (in meters) at time $t$ (seconds) after it was thrown upwards was found to satisfy $Y(t) = 14.7t - (1/2)gt^2$ where $g = 9.8$ m/s$^2$ for the first 3 seconds of its motion.

(a) What happens after 3 seconds?

(b) What is the average velocity of the ball between the times $t = 0$ and $t = 1$ second?

2.5. **Falling ball:** A ball is dropped from height $Y_0 = 490$ meters above the ground. Its height, $Y$, at time $t$ is known to follow the relationship $Y(t) = Y_0 - \frac{1}{2}gt^2$ where $g = 9.8$ m /s$^2$.

(a) Find the average velocity of the falling ball between $t = 1$ and $t = 2$ seconds.

(b) Find the average velocity between $t$ sec and $t + \epsilon$ where $0 < \epsilon < 1$ is some small time increment. (Assume that the ball is in the air during this time interval.)

(c) Determine the time at which the ball hits the ground.

2.6. **Average velocity at time t:** A ball is thrown from the top of a building of height $Y_0$. The height of the ball at time $t$ is given by

$$Y(t) = Y_0 + v_0 t - \frac{1}{2}gt^2$$

where $h_0, v_0, g$ are positive constants. Find the average velocity of the ball for the time interval $0 \le t \le 1$ assuming that it is in the air during this whole time interval. Express your answer in terms of the constants given in the problem.

2.7. **Average velocity and secant line:** The two points on Figure 2.2 through which the secant line is drawn are $(1.3, 8.2810)$ and $(1.4, 9.6040)$. Find the average velocity over this time interval and then write down the equation of the secant line.

2.8. **Average rate of change:** A certain function takes values given in the table below.

| $t$ | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| $f(t)$ | 0 | 1 | 0 | -1 | 0 |

Find the average rate of change of the function over the intervals

(a) $0 < t < 0.5$,

(b) $0 < t < 1.0$,

(c) $0.5 < t < 1.5$,

(d) $1.0 < t < 2.0$.

2.9. Consider the functions $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = x^3$.
Find the average rate of change of these functions over each of the following intervals.

(a) Over $0 \le x \le 1$.

(b) Over $-1 \le x \le 1$.

(c) Over $0 \le x \le 2$.

2.10. Find the average rate of change for each of the following functions over the given interval.

(a) $y = f(x) = 3x - 2$ from $x = 3.3$ to $x = 3.5$.

(b) $y = f(x) = x^2 + 4x$ over $[0.7, 0.85]$.

(c) $y = -\frac{4}{x}$ and $x$ changes from $0.75$ to $0.5$.

2.11. **Trig Minireview:** Consider the following table of values of the trigonometric functions $\sin(x)$ and $\cos(x)$:

| $x$ | $\sin(x)$ | $\cos(x)$ |
|---|---|---|
| $0$ | $0$ | $1$ |
| $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ |
| $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ |
| $\frac{\pi}{2}$ | $1$ | $0$ |

Find the average rates of change of the given function over the given interval. Express your answer in terms of square roots and $\pi$. Do not compute decimal expressions.

(a) Find the average rate of change of $\sin(x)$ over $0 \leq x \leq \pi/4$.

(b) Find the average rate of change of $\cos(x)$ over $\pi/4 \leq x \leq \pi/3$.

(c) Is there an interval over which the functions $\sin(x)$ and $\cos(x)$ have the same average rate of change? (Hint: consider the graphs of these functions over one whole cycle, e.g. for $0 \leq x \leq 2\pi$. Where do they intersect?)

2.12. (a) Consider the function $y = f(x) = 1 + x^2$. Consider the point $(1, 2)$ on its graph and some point nearby, for example $(1+h, 1+(1+h)^2)$. Find the slope of a secant line connecting these two points.

(b) Use this slope to figure out what the slope of the tangent line to the curve at $(1, 2)$ would be.

(c) Find the equation of the tangent line through the point $(1, 2)$.

2.13. Given the function $y = f(x) = 2x^3 + x^2 - 4$, find the slope of the secant line joining the points $(4, f(4))$ and $(4 + h, f(4 + h))$ on its graph, where $h$ is a small positive number. Then find the slope of the tangent line to the curve at $(4, f(4))$.

2.14. **Average rate of change:** Consider the function $f(x) = x^2 - 4x$ and the point $x_0 = 1$.

(a) Sketch the graph of the function.

(b) Find the average rate of change over the intervals $[1, 3]$, $[-1, 1]$, $[1, 1.1]$, $[0.9, 1]$ and $[1 - h, 1]$, where $h$ is some small positive number.

(c) Find $f'(1)$.

2.15. Given $y = f(x) = x^2 - 2x + 3$.

(a) Find the average rate of change over the interval $[2, 2 + h]$.

(b) Find $f'(2)$.

(c) Using only the information from (a), (b) and $f(2) = 3$, approximate the value of $y$ when $x = 1.99$, without substituting $x = 1.99$ into $f(x)$.

2.16. Find the average rates of the given function over the given interval. Express your answer in terms of square roots and $\pi$. Do not compute the decimal expressions.

(a) Find the average rate of change of $\tan(x)$ over $0 \leq x \leq \frac{\pi}{4}$ (Hint: $\tan(x) = \frac{\sin(x)}{\cos(x)}$).

(b) Find the average rate of change of $\cot(x)$ over $\frac{\pi}{4} \leq x \leq \frac{\pi}{3}$ (Hint: $\cot(x) = \frac{\cos(x)}{\sin(x)}$).

2.17. (a) Find the slope of the secant line to the graph of $y = 2/x$ between the points $x = 1$ and $x = 2$.

(b) Find the average rate of change of $y$ between $x = 1$ and $x = 1+\epsilon$ where $\epsilon > 0$ is some positive constant.

(c) What happens to this slope as $\epsilon \to 0$ ?

(d) Find the equation of the tangent line to the curve $y = 2/x$ at the point $x = 1$.

2.18. For each of the following motions where $s$ is measured in meters and $t$ is measured in seconds, find the velocity at time $t = 2$ and the average velocity over the given interval.

(a) $s = 3t^2 + 5$ and $t$ changes from 2 to $3s$.

(b) $s = t^3 - 3t^2$ from $t = 3s$ to $t = 5s$.

(c) $s = 2t^2 + 5t - 3$ on $[1, 2]$.

2.19. The velocity $v$ of an object attached to a spring is given by $v = -A\omega \sin(\omega t + \delta)$, where $A$, $\omega$ and $\delta$ are constants. Find the average change in velocity ("acceleration") of the object for the time interval $0 \leq t \leq \frac{2\pi}{\omega}$.

2.20. Use the definition of derivative to calculate the derivative of the function

$$f(x) = \frac{1}{x+1}$$

(intermediate steps required).

## Chapter 3

# Zooming into the graph of a function: tangent lines and derivatives

In Chapter 2 we used the concept of average rate of change (slope of secant line) to motivate and then define the notion of an instantaneous rate of change (the derivative). We arrived at a "recipe" for calculating the derivative algebraically. In this chapter we take a more geometric approach and connect the same idea to the local shape of the graph of a function.

## 3.1 Tangent lines: zooming in on the graph of a function

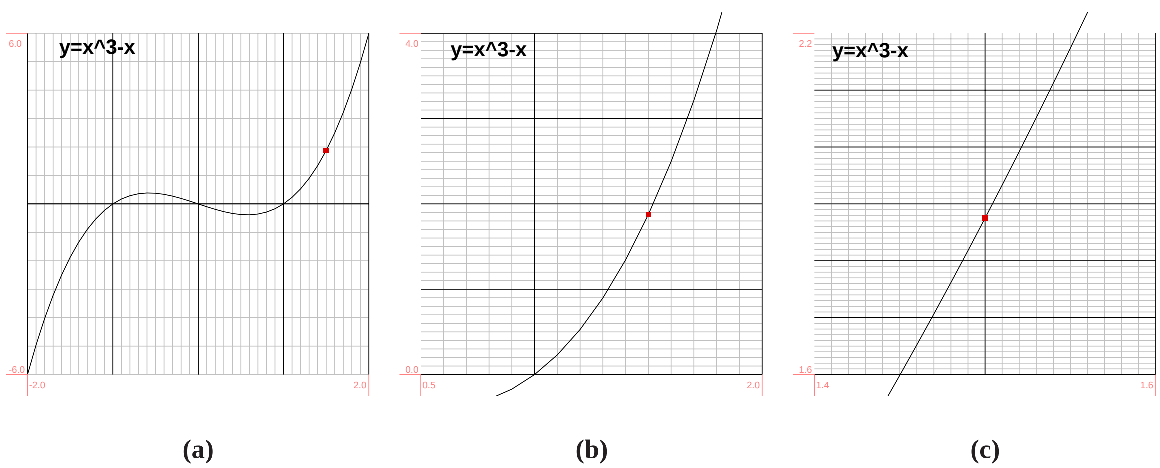



**Figure 3.1.** *Zooming in on the point* $x = 1.5$ *on the graph of the function* $y = f(x) = x^3 - x$*. As we zoom in, we see that locally, the graph "looks like" a straight line. We refer to this line as the tangent line, and its slope is the derivative of the function at that point.*

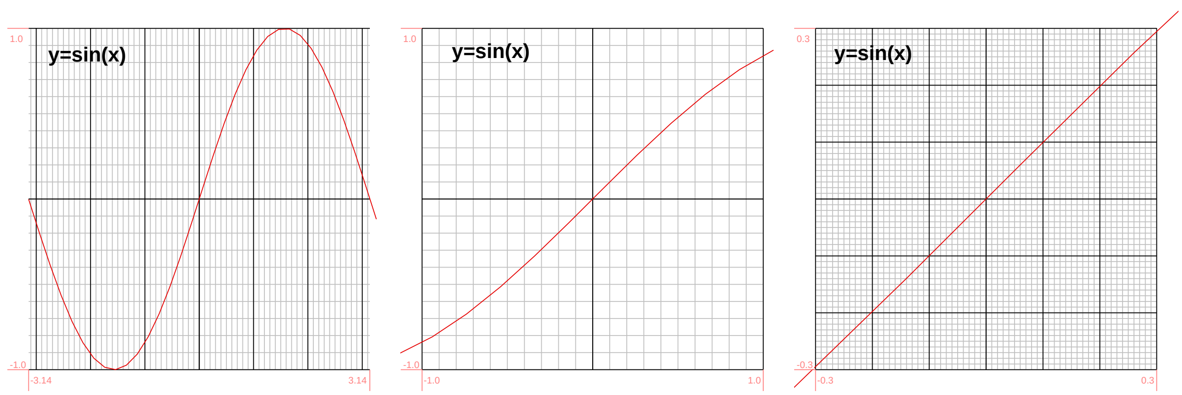
Another approach to the idea of the derivative is based on the following geometric idea. Consider the graph of some function, and pick some point on that graph. In the example in Figure 3.1 we have shown a graph of the function $y = f(x) = x^3 - x$ and a point shown by a heavy (red) dot.

Now zoom into the selected point, looking at ever higher magnification. (This is shown in the sequence of zooms in Figure 3.1). Eventually, as we get closer to the point of interest, the hills and valleys on the graph disappear off screen, and we start to feel that we live in a much flatter world. In fact, locally, the graph looks more and more like a straight line. We will refer to this straight line as the **tangent line** to the graph of the function at the given point. The slope of this tangent line will be what we refer to as the **derivative** of the function, at the given point.

Clearly the configuration of the tangent line will depend on the point we chose to zoom into. Its slope will also vary from place to place. For this reason, the derivative, denoted $f'(x)$ is, itself, a function.

In Figure 3.2 we show a zoom into the origin on the graph of the function

$$y = \sin(x).$$




**(a)** **(b)** **(c)**

**Figure 3.2.** *Zooming in on the point $x = 0$ on the graph of the function $y = f(x) = \sin(x)$. Eventually, the graph resembles a line of slope 1. This is the tangent line at $x = 0$ and its slope is the derivative of $y = \sin(x)$ at $x = 0$.*

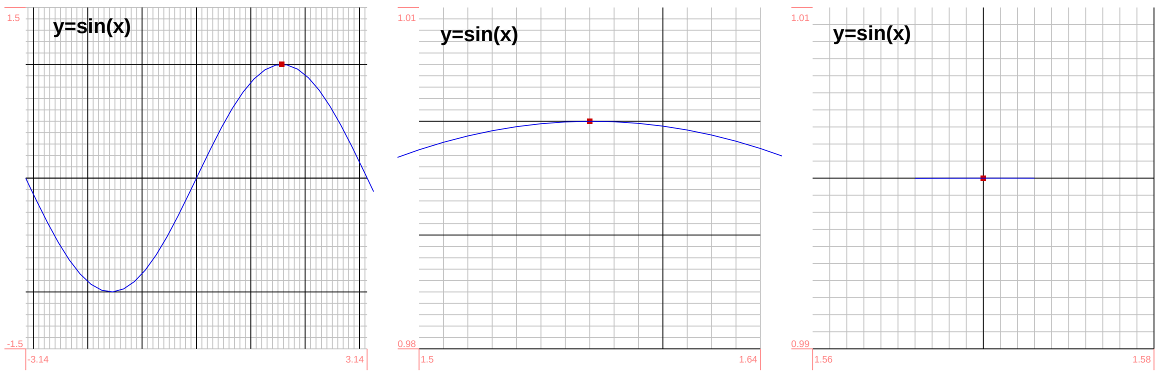
We see from this graph that the slope of the line that we obtain as we zoom into $x = 0$ is $m = 1$. We say that the derivative of the function $y = f(x) = \sin(x)$ at $x = 0$ is 1. We also observe that the line shown in the final image in the sequence of Figure 3.2 is the tangent line to the curve at $x = 0$. The equation of this line is simply $y = x$. (This follows from the fact that the line has slope 1 and goes through the point $(0, 0)$.) We can also say that close to $x = 0$ the graph of $y = \sin(x)$ looks a lot like the line $y = x$. This is equivalent to saying that

$$\sin(x) \approx x, \quad \text{or} \quad \frac{\sin(x)}{x} \approx 1$$

for small $x$, or, more formally, that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

We will find this limit useful in later calculations.




**(a)** **(b)** **(c)**

**Figure 3.3.** *Here we zoom in on the point* $x = \pi/2$ *on the graph of* $y = f(x) = \sin(x)$. *This point is a "local maximum". Eventually, we see the tangent line whose slope is 0 (it is is horizontal). Thus the derivative of* $y = \sin(x)$ *at* $x = \pi/2$ *is zero.*

**Example 3.1 (Derivative of $y = C$)** Use a geometric argument to determine the derivative of the function $y = f(x) = C$ at any point $x_0$ on its graph. ■

**Solution:** This function is a horizontal straight line, whose slope is zero everywhere. Thus "zooming in" at any point $x$, leads to the same result, so the derivative is 0 everywhere.

**Example 3.2 (Derivative of $y = Bx$)** ■

**Solution:** The function $y = Bx$ is a straight line of slope $B$. At any point on its graph, it has the same slope, $B$. Thus the derivative is equal to $B$ at any point on the graph of this function.

The reader will notice that in the above two examples, we have thereby found the derivative for the two power functions, $y = x^0$ and $y = x^1$.

## 3.2 Equation of the tangent line

The examples above allow us to visualize the tangent line by examining the local ("zoomed in") shape of the function. In two examples, we found the slope of that tangent line, which is identified by the derivative of the function at the given point. Using such information, we can write down the equation of that tangent line.

**Example 3.3** Find the equation of the tangent line to the graph of $y = \sin(x)$ at the points $x = 0$ and $x = \pi/2$ ■

**Solution:** In Fig. 3.2 we found that the tangent line to $y = \sin(x)$ at $x = 0$ has slope $m = 1$. We also know that this line goes through the point $(0, \sin(0)) = (0, 0)$. Thus its y-intercept is 0 and we can immediately write down its equation in the form $y = mx + b$. We obtain $y = x$ as the equation of this line.

In Fig. 3.3, we found that the tangent line to the same function at the point $x = \pi/2$ has slope 0. We must also use the fact that the line goes through the point $(\pi/2, \sin(\pi/2)) = (\pi/2, 1)$. Since the tangent line is horizontal (equivalent to saying it has slope 0), the $y$ value is constant everywhere, so that the equation of this line is simply $y = 1$.

**Example 3.4** Find the equation of the tangent line to the graph of $y = f(x) = x^3 - x$ at the point $x = 1.5$ shown in Fig. 3.1. ■

**Solution:** The point of interest has coordinates $(x, f(x)) = (1.5, 1.875)$. From the graph shown in Fig. 3.1 (c) we see that the "tangent line" approximately goes through $(1.7, 1.47)$ as well as the red point $(1.5, 1.875)$. We can compute its slope and find to be approximately $\Delta y/\Delta x \approx 5.8$. To find the equation of the tangent line, we use its slope and a point on that line to write

$$\frac{y - 1.875}{x - 1.5} = 5.8 \quad \Rightarrow \quad y = 1.875 + 5.8(x - 1.5) \quad \Rightarrow \quad y = 5.8x - 6.825$$

Later, we will use the definition of the derivative and an exact algebraic calculation to improve on this result.

## 3.3 Sketching the graph of the derivative

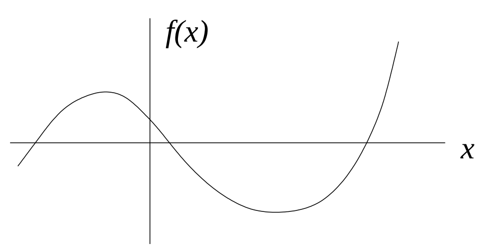



**Figure 3.4.** *The graph of a function. We will sketch its derivative*

In Figure 3.4 we show the graph of some function, $f(x)$. We would like to sketch the derivative, $f'(x)$ corresponding to this function. (Recall that the derivative is also a function.) Keep in mind that this sketch will be approximate, but will contain some important elements.

In Figure 3.5 we start by sketching in a number of tangent lines on the graph of $f(x)$. We will pay special attention to the slopes (rather than height, length, or any other property) of these dashes. Copying these lines in a row along the direction of the $x$ axis, we estimate their slopes with rather approximate numerical values.

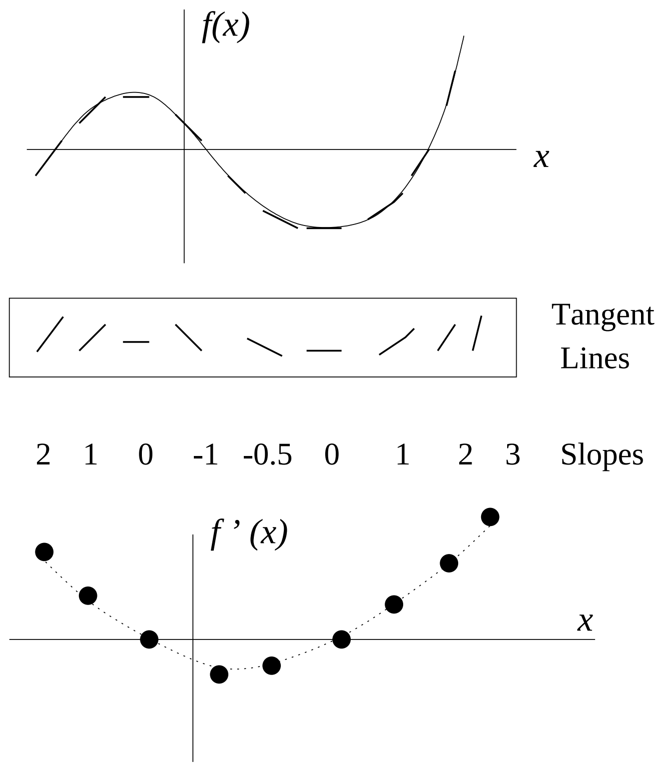



**Figure 3.5.** *Sketching the derivative of a function*

We notice that the slopes start out positive, decrease to zero, become negative, and then increase again through zero back to positive values. (We see precisely two dashes that are horizontal, and so have slope 0.) Next, we plot the numerical values (for slopes) that we have recorded on a new graph. This is the beginning of the graph of the derivative, $f'(x)$. Only a few points have been plotted in our figure of $f'(x)$; we could add other values if we so chose, but the trend, is fairly clear: The derivative function has two **zeros** (places of intersection with the $x$ axis). It dips down below the axis between these places. In Figure 3.6 we show the original function $f(x)$ and its derivative $f'(x)$ now drawn as a continuous curve. We have aligned these graphs so that the slope of $f(x)$ matches the value of $f'(x)$ shown directly below.

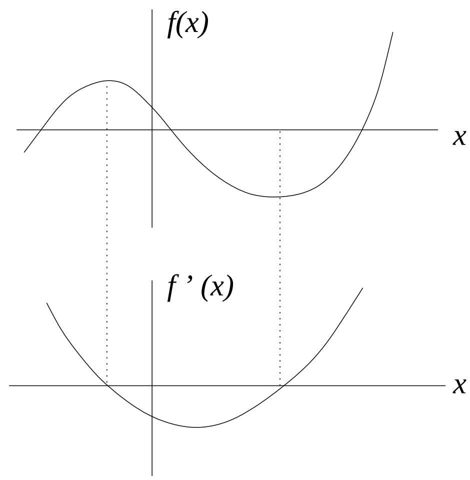



**Figure 3.6.** *A sketch of the function and its derivative*

## 3.4 For further study: tangent line to a circle

**Example 3.5 (A multi-step problem)** Find the area of the triangle AOB in Figure 3.7. The circle shown in the figure has radius 1 and center at the origin. The line AB is tangent to the circle at the point $x = 1/2$. ∎

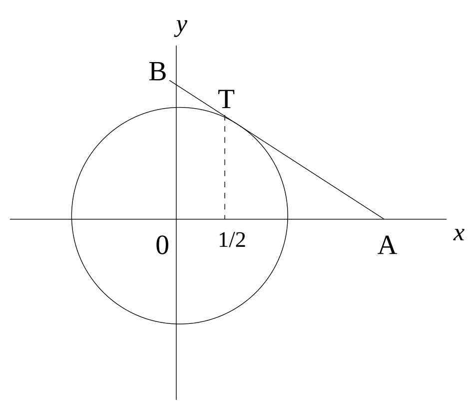



**Figure 3.7.** *Problem: Find the area of triangle AOB, given that the line AB is a tangent to the circle at the point T.*

### 3.4.1 The solution

This problem involves multiple steps, and uses several aspects of the geometry shown in Figure 3.7. Our strategy will be to find the height and the base of the triangle shown, since then we can easily find its area. To do so, we first need to know the slope of the line AB, a line that (we are told) touches the circle at the point T.

#### The equation and the point

The equation of a circle with radius 1 and center at the origin is

$$x^2 + y^2 = 1 \tag{3.1}$$

We are given $x = 1/2$ so we can find the corresponding y value:

$$y^2 = 1 - x^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

Thus

$$y = \frac{\sqrt{3}}{2}$$

so that the point T has coordinates $(x, y) = (1/2, \sqrt{3}/2)$.

#### Property of tangents to circles

Later on in this course we will find clever ways of determining slopes of tangent lines using derivatives. For now, we will use the following property of circles:

*In a circle, the tangent line is always perpendicular to the radius vector*. See Figure 3.8.

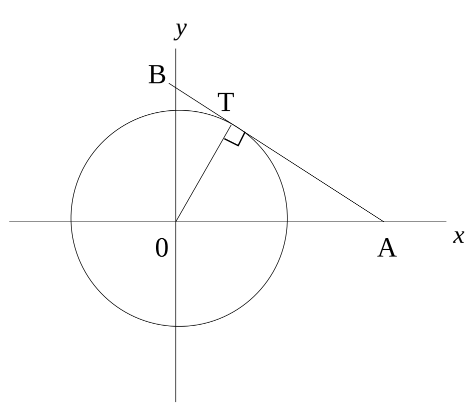



**Figure 3.8.** *In a circle, a tangent line is perpendicular to the radius vector at a given point.*

More specifically, in the picture shown in Figure 3.7, a line from O to T (the radius vector), would be perpendicular to the tangent line AB. This also allows us to determine its slope, using the following geometric fact:

*If $L_1$ is a line having slope $m_1$, and $L_2$ is a line perpendicular to $L_1$, then the slope of line $L_2$ is $m_2 = -1/m_1$.*

### Slopes

The slope $m_{\mathrm{OT}}$ of the line OT can be found simply as follows:

$$m_{\mathrm{OT}} = \frac{\Delta y}{\Delta x} = \frac{\sqrt{3}/2 - 0}{1/2 - 0} = \sqrt{3}.$$

Thus the slope of the tangent line, (using the perpendicular property) is

$$m_{\mathrm{AB}} = -\frac{1}{m_{\mathrm{OT}}} = -\frac{1}{\sqrt{3}}.$$

### Where to go from here?

Now that we have found the slope of the line AB, we are close to our goal. However, we still need to get the actual points of intersection of line AB with the axes so that we can determine the base and height of the triangle AOB. To do this, we will find the equation of the tangent line, using the fact that it has a known slope and goes through a known point $(x, y) = (1/2, \sqrt{3}/2)$

### Equation of the tangent line AB

We use the following facts: The line AB has slope $-1/\sqrt{3}$ and goes through $(1/2, \sqrt{3}/2)$. Thus

$$\frac{y - \sqrt{3}/2}{x - 1/2} = -\frac{1}{\sqrt{3}}. \tag{3.2}$$

We could use this relationship directly to find the desired intercepts: plugging in $y = 0$ and solving for $x$ would lead to the $x$ intercept, and plugging in $x = 0$ would similarly lead to the $y$ intercept. However, for practice, we will first determine the equation of the tangent line in standard form:

Rearranging equation (3.2) leads to

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}\left(x - \frac{1}{2}\right)$$

$$y = -\frac{1}{\sqrt{3}}x + \frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{2}$$

or simply

$$y = -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}. \tag{3.3}$$

### Intercepts

With the equation of the tangent line (3.3) in hand, we can obtain the desired coordinates of the points A and B:

Point B is simply the y-intercept, which, from equation (3.3) is: $2/\sqrt{3}$.

Point A, the x-intercept can be found be setting $y = 0$ and solving for $x$ in equation (3.3):

$$0 = -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}.$$

We find that $x = (2/\sqrt{3})(\sqrt{3}) = 2$.

### The desired area

We now have the height of the triangle, namely $2/\sqrt{3}$ and the base, i.e. $2$ as shown in Figure 3.9.

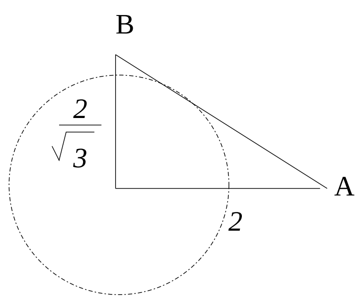



**Figure 3.9.** *We find the area by first determining the equation of the tangent line and, thus, its x and y intercepts. This gives us the height and base of the triangle, from which its area is easily computed. ($A = hb/2$).*

We now obtain

$$\text{Area}_\Delta = \frac{1}{2}\ \text{height} \cdot \text{base} = \frac{1}{2} \cdot 2 \cdot \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

### Comments

This problem illustrates that the strategy for solving a problem of this type is to break it down into a series of steps, each one simple and straightforward, together getting us to the goal. Many problems in science and mathematics involve multiple steps, not just one simple formula or method. We have here used some geometric facts about circles, some knowledge of properties of slopes and straight lines, as well as simple algebraic manipulations to get to our final destination.

## Exercises

3.1. You are given the following information about the signs of the derivative of a function, $f(x)$. Use this information to sketch a (very rough) graph of the function for $-3 < x < 3$.

| $x$ | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | 0 | + | 0 | - | 0 | + | + |

3.2. You are given the following information about the the values of the derivative of a function, $g(x)$. Use this information to sketch (very rough) graph the function for $-3 < x < 3$.

| $x$ | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $g'(x)$ | -1 | 0 | 2 | 1 | 0 | -1 | -2 |

3.3. What is the slope of the tangent line to the function $y = f(x) = 5x + 2$ when $x = 2$? when $x = 4$ ? How would this slope change if a negative value of $x$ was used? Why?

3.4. Find the equation of the tangent line to the function $y = f(x) = |x + 1|$ at:

(a) $x = -1$,

(b) $x = -2$,

(c) $x = 0$.

If there is a problem finding a tangent line at one of these points, indicate what the problem is.

3.5. A function $f(x)$ has as its derivative $f'(x) = 2x^2 - 3x$

(a) In what regions is $f$ increasing or decreasing?

(b) Find any local maxima or minima.

(c) Is there an absolute maximum or minimum value for this function?

3.6. Sketch the graph of a function $f(x)$ whose derivative is shown in Figure 3.10. Is there only one way to draw this sketch? What difference might occur between the sketches drawn by two different students?

3.7. Shown in Figure 3.11 is the graph of some function $f(x)$. Sketch the graph of its derivative, $f'(x)$.

3.8. Shown in Figure 3.12 below are three functions, $f(x)$ (dotted lines). Sketch the derivatives of these functions, $f'(x)$.

3.9. A function $f(x)$ satisfies $f(1) = -1$ and $f'(1) = 2$. What is the equation of the tangent line of $f(x)$ at $x = 1$?

3.10. Sketch the graph of the derivative of the function shown in Figure 3.13.

3.11. For each of the following functions, sketch the graph for $-1 < x < 1$, find $f'(0)$, $f'(1)$, $f'(-1)$ and identify any local minima and maxima.

(a) $y = x^2$,

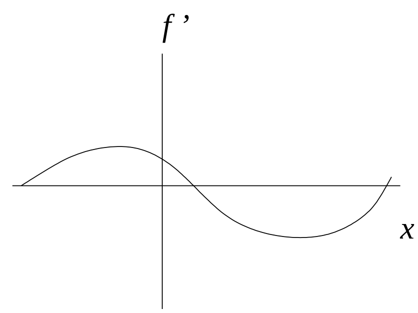



**Figure 3.10.** *Figure for Problem 6*

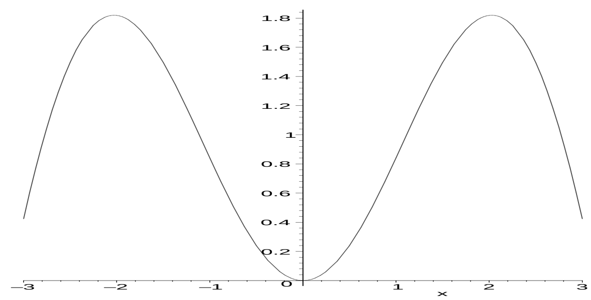


**Figure 3.11.** *Figure for Problem 7*

(b) $y = -x^3$,

(c) $y = -x^4$

(d) Using your observations above, when can you conclude that a function whose derivative is zero at some point has a local maximum at that point?

3.12. (a) Given the function in Figure 3.14(a), graph its derivative.

(b) Given the function in Figure 3.14(b), graph its derivative

(c) Given the derivative $f'(x)$ shown in Figure 3.14(c) graph the function $f(x)$.

(d) Given the derivative $f'(x)$ shown in Figure 3.14(d) graph the function $f(x)$.

3.13. Given the derivative $f'(x)$ shown in Figure 3.14(c), graph the second derivative $f''(x)$.

3.14. Shown in Figure 3.15 is the graph of the velocity of a particle moving in one dimension. Indicate directly on the graph any time(s) at which the particle's acceleration is zero.

3.15. Use the definition of the derivative to compute the slope of the tangent line to the graph of the function $y = 3t^2 - t + 2$ at the point $t = 1$.

3.16. Shown in Figure 3.16 is the function $f(x) = x^3$ with a tangent line at the point $(1, 1)$.

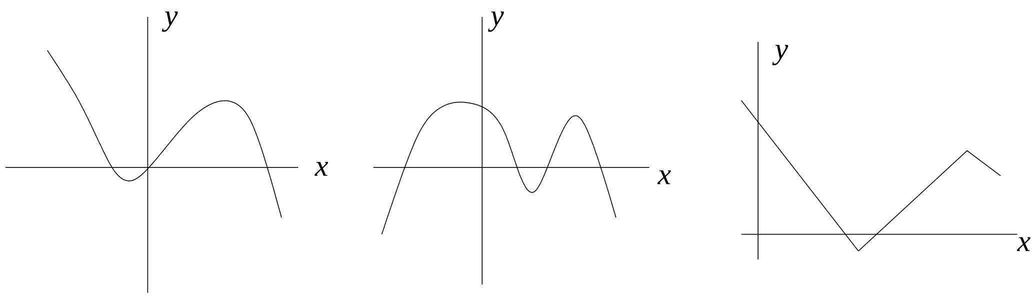



**Figure 3.12.** *Figure for problem 8*

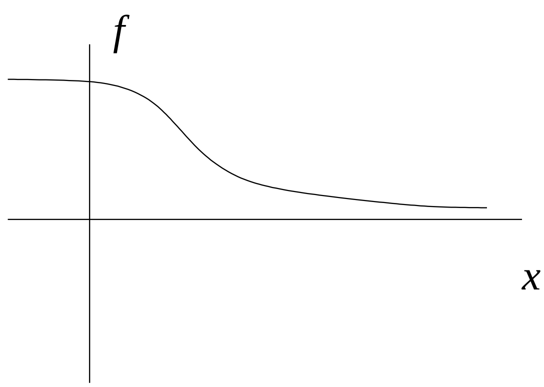



**Figure 3.13.** *Figure for Problem 10*

(a) Find the equation of the tangent line.

(b) Determine the point at which the tangent line intersects the $x$ axis.

(c) Compute the value of the function at $x = 1.1$. Compare this with the value of $y$ on the tangent line at $x = 1.1$. (This latter value is the *linear approximation* of the function at the desired point based on its known value and known derivative at the nearby point $x = 1$.)

3.17. Shown in Figure 3.17 is the function $f(x) = 1/x^4$ together with its tangent line at $x = 1$.

(a) Find the equation of the tangent line.

(b) Determine the points of intersection of the tangent line with the $x$ and the $y$ axes.

(c) Use the tangent line to obtain a linear approximation to the value of $f(1.1)$. Is this approximation larger or smaller than the actual value of the function at $x = 1.1$?

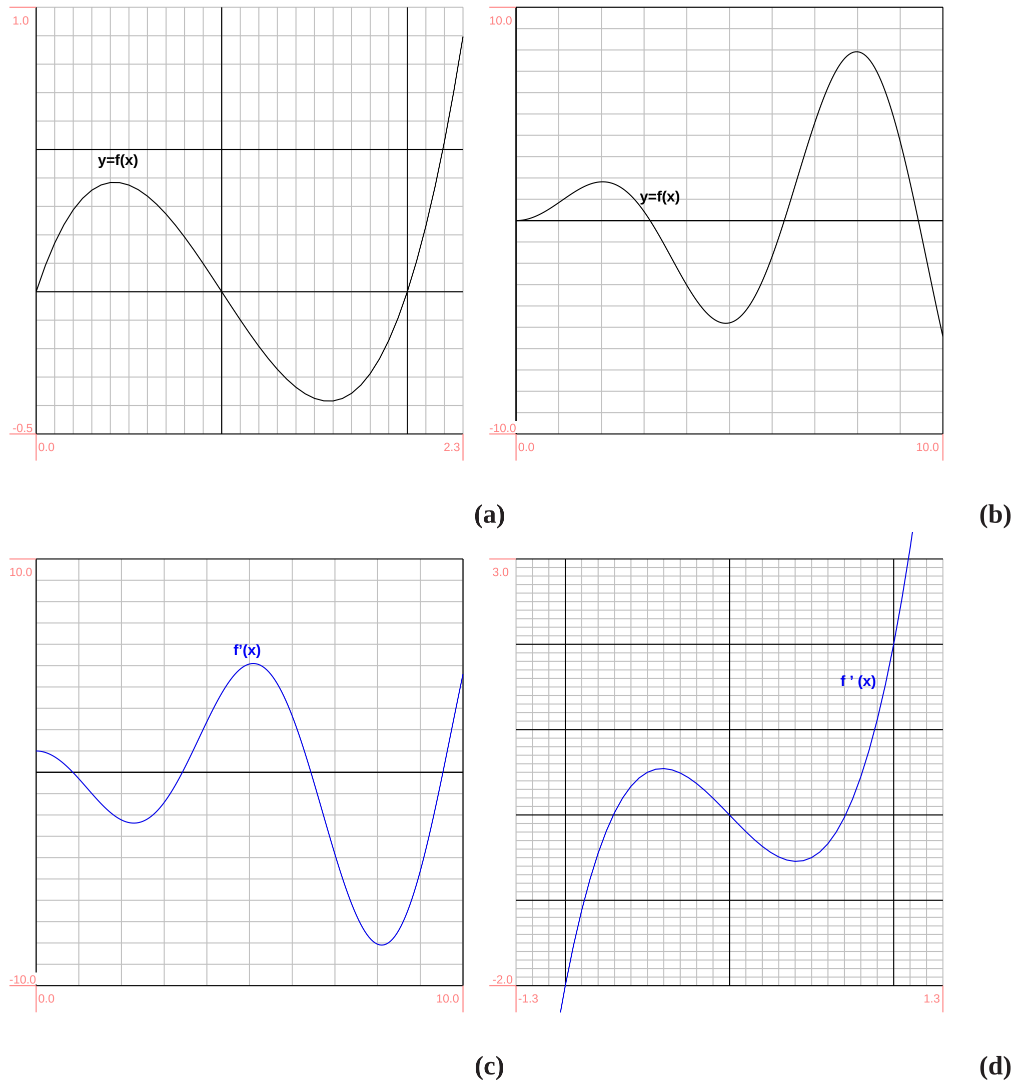



**Figure 3.14.** *Figures for Problem 12.*

3.18. Shown in Figure 3.18 is the graph of a function and its tangent line at the point $x_0$.

(a) Find the equation of the tangent line expressed in terms of $x_0$, $f(x_0)$ and $f'(x_0)$.

(b) Find the coordinate $x_1$ at which the tangent line intersects the $x$ axis.

3.19. Shown in Figure 3.19 is the graph of $f'(x)$, the derivative of some function. Use this to sketch the graphs of the two related functions, $f(x)$ and $f''(x)$

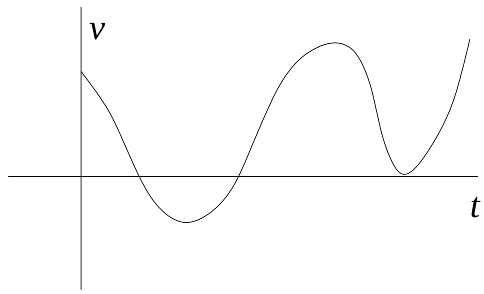



**Figure 3.15.** *Figure for Problem 14*

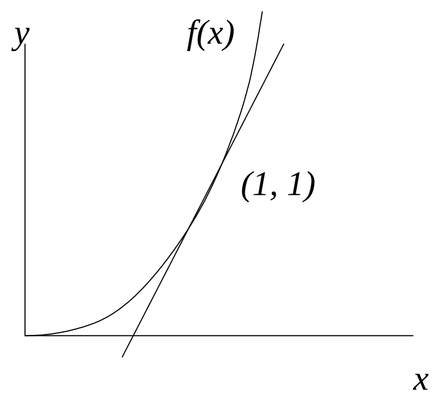



**Figure 3.16.** *Figure for Problem 16*

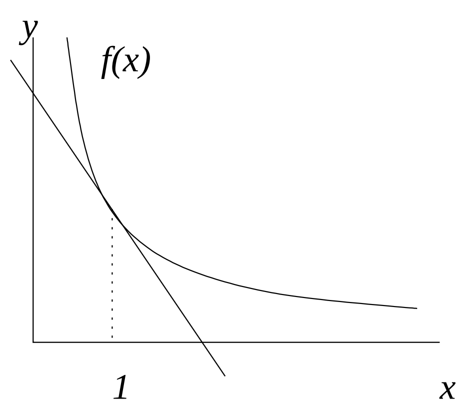



**Figure 3.17.** *Figure for problem 17*

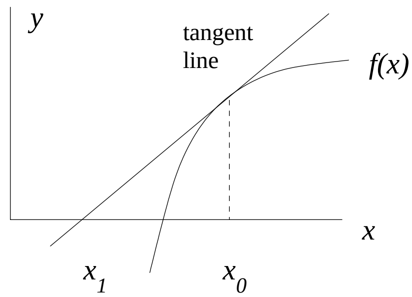



**Figure 3.18.** *Figure for problem 18*

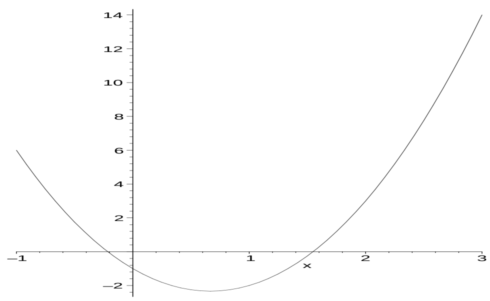



**Figure 3.19.** *Figure for Problem 19*

3.20. **Concentration gradient:** Certain types of tissues, called epithelia are made up of thin sheets of cells. Substances are taken up on one side of the sheet by some active transport mechanism, and then diffuse down a concentration gradient by a mechanism called facilitated diffusion on the opposite side. Shown in Figure 3.20 is the concentration profile $c(x)$ of some substance across the width of the sheet ($x$ represents distance). Sketch the corresponding concentration gradient, i.e. sketch $c'(x)$, the derivative of the concentration with respect to $x$.

3.21. The vertical height of a ball, $d$ (in meters) at time $t$ (seconds) after it was thrown upwards was found to satisfy $d(t) = 14.7t - 4.9t^2$ for the first 3 seconds of its motion.

(a) What is the initial velocity of the ball (i.e. the instantaneous velocity at $t = 0$) ?

(b) What is the instantaneous velocity of the ball at $t = 2$ seconds?

3.22. Shown in Figure 3.21 is the graph of $y = x^2$ with one of its tangent lines.

(a) Show that the slope of the tangent to the curve $y = x^2$ at the point $x = a$ is $2a$.

(b) Suppose that the tangent line intersects the $x$ axis at the point (1,0). Find the coordinate, $a$, of the point of tangency.

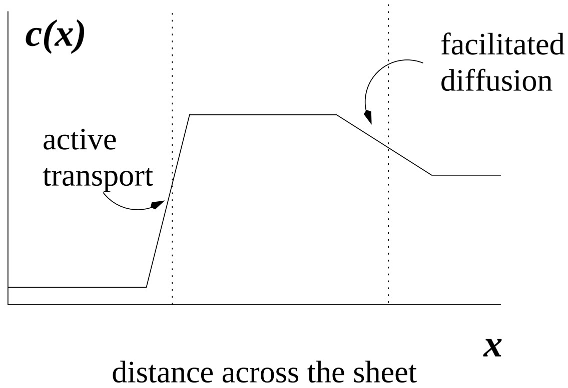



**Figure 3.20.** *Figure for Problem 20*

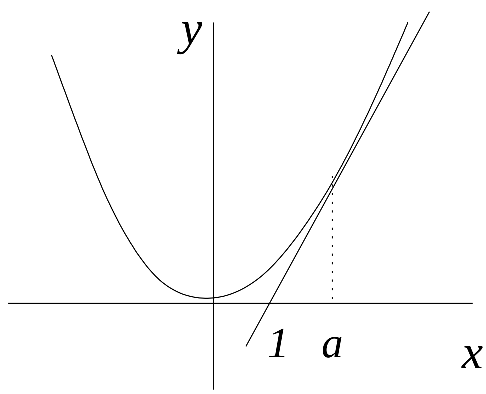



**Figure 3.21.** *Figure for Problem 22*

3.23. The parabola $y = x^2$ has two tangent lines that intersect at the point $(2, 3)$. These are shown as the dark lines in Figure 3.22. [Remark: note that the point $(2, 3)$ is not on the parabola]. Find the coordinates of the two points at which the lines are tangent to the parabola.

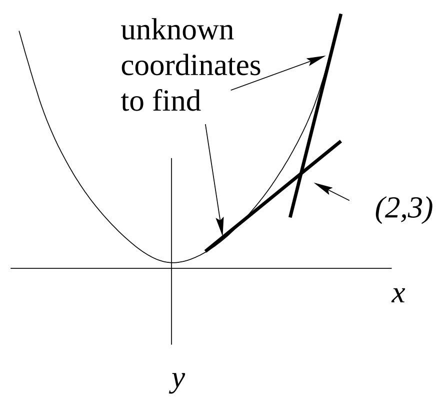



**Figure 3.22.** *Figure for Problem 23*

## Chapter 4

# The Derivative

In our investigation so far, we have defined the notion of an instantaneous rate of change, and called this the derivative. We have also identified this mathematical concept with the slope of a tangent line to the graph of a function. Recall that our definition for the derivative of a function, $y = f(x)$ is

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In this chapter, we will use this definition to establish how to compute the derivatives of power functions. In our previous discussions, we observed that power functions are building blocks of polynomials, a family of well-behaved functions that are exceedingly useful in approximations. Using some further elementary properties of derivatives we will arrive at a simple way of calculating the derivative of any polynomial. This will permit interesting and useful calculations, on a variety of applied problems.

In this and the following sections, we will gain experience with the many-faceted properties of derivatives that use relatively simple differentiation calculations. (Some of the problems we address will be challenging nevertheless, but all of them will be based on polynomial and power function forms.) Using the definition of the derivative, we can compute derivatives of a power function. While we here show a specific example for the cubic, the general idea can be extended to other cases, and leads to a pattern that we will call the **power rule** of differentiation.

## 4.1 The derivative of power functions: the power rule

We have already computed the derivatives of several of the power functions. See Example 3.1 for $y = x^0 = 1$ and Example 3.2 for $y = x^1$. See also Example 2.8 for $y = x^2$. We tabulate these results in Table 4.1. Let us extend our set of results by another calculation of the derivative of a cubic function.

**Example 4.1 (Derivative of the cubic power function)** Compute the derivative of the function $y = f(x) = Kx^3$. ∎

**solution:** For $y = f(x) = Kx^3$ we have

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{K(x+h)^3 - Kx^3}{h}\\ &= \lim_{h\to 0} K\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}\\ &= \lim_{h\to 0} K\frac{(3x^2h+3xh^2+h^3)}{h}\\ &= \lim_{h\to 0} K(3x^2+3xh+h^2)\\ &= K(3x^2) = 3Kx^2\end{aligned}$$

We see from simple experimentation that a derivatives of a power function consists of reducing the power (by 1) and multiplying the result by the original power. See Table 4.1, where we have taken all the coefficients to be 1 for simplicity. We refer to this pattern as the **power rule** of differentiation.

| Function $f(x)$ | Derivative $f'(x)$ |
|---|---|
| 1 | 0 |
| $x$ | 1 |
| $x^2$ | $2x$ |
| $x^3$ | $3x^2$ |
| $\vdots$ | $\vdots$ |
| $x^n$ | $nx^{n-1}$ |

**Table 4.1.** *The* **Power Rule** *of differentiation states that the derivative of the power function* $y = x^n$ *is* $nx^{n-1}$. *For now, we have established this result for integer* $n$. *Later, we will find that this result holds for other values of* $n$.

We can show that this rule applies for any power function of the form $y = f(x) = x^n$ where $n$ is an integer power. The calculation is essentially the same as the examples we have shown, but the step of expanding the binomial $(x+h)^n$ entails lengthier algebra. Such expansion contains terms of the form $x^{n-k}h^k$ multiplied by *binomial coefficients*, and we omit the details here. From now on, we will use this convenient result to simply write down the derivative of a power function, without having to recalculate it from the definition.

**Example 4.2** Find the equation of the tangent line to the graph of the power function $y = f(x) = 4x^5$ at $x = 1$, and determine the $y$ intercept of that tangent line. ■

**Solution:** The derivative of this function is

$$f'(x) = 20x^4.$$

At the point $x = 1$, we have $dy/dx = f'(1) = 20$ and $y = f(1) = 4$. This means that the tangent line goes through the point $(1, 4)$ and has slope $20$. Thus, its equation is

$$\frac{y-4}{x-1} = 20$$

$$y = 4 + 20(x-1) = 20x - 16.$$

(At this point is is a good idea to do a quick check that the point $(1, 4)$ satisfies this equation, and that the slope of the line is 20.) Thus, we find that the $y$ intercept of the tangent line is $y = -16$.

Next, we find that the result for derivatives of power functions can be extended to derivatives of polynomials, using further simple properties of the derivative.

## 4.2 The derivative is a linear operation

The derivative satisfies several convenient properties: The sum of two functions or the constant multiple of a function has a derivative that is related simply to the original function(s). The derivative of a sum is the same as the sum of the derivatives. A constant multiple of a function can be brought outside the differentiation.

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{df}{dx} + \frac{dg}{dx} \tag{4.1}$$

$$\frac{d}{dx}Cf(x) = C\frac{df}{dx} \tag{4.2}$$

We can summarize these observations by saying that the derivative is a **linear operation**. In general, a linear operation $L$ is a rule or process that satisfies two properties: (1) $L[f + g] = L[f] + L[g]$ and $L[cf] = cL[f]$, where $f, g$ are objects (such as functions, vectors, etc) on which $L$ acts, and $c$ is a constant multiple. We will refer to (4.1) and (4.2) as the "linearity" properties of the derivative.

## 4.3 The derivative of a polynomial

Using the properties (4.1) and (4.2), we can extend our differentiation power rule to compute the derivative of any polynomial. Recall that polynomials are sums of power functions multiplied by constants. A polynomial of **degree** $n$ has the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0 \tag{4.3}$$

where the coefficients, $a_i$ are constant and $n$ is an integer. Thus, by the above two properties, the derivative of a polynomial is just the sum of derivatives of power functions (multiplied by constants). Thus the derivative of (4.3) is

$$p'(x) = \frac{dy}{dx} = a_n \cdot nx^{n-1} + a_{n-1} \cdot (n-1)x^{n-2} + \ldots a_1 \tag{4.4}$$

(Observe that each term consists of the coefficient times the derivative of a power functions. The constant term $a_0$ has disappeared since the derivative of any constant is zero.) The derivative, $p'(x)$, is apparently also a function, and a polynomial as well. Its degree is $n-1$, one less than that of $p(x)$. In view of this observation, we could ask what is the derivative of the derivative, which we henceforth call the **second derivative**., written in the notation $p''(x)$ or, equivalently $\frac{d^2y}{dx^2}$. Using the same rules, we can compute this easily, obtaining

$$p''(x) = \frac{d^2y}{dx^2} = a_n \cdot n(n-1)x^{n-2} + a_{n-1} \cdot (n-1) \cdot (n-2)x^{n-3} + \dots a_2 \tag{4.5}$$

We demonstrate the idea with a few examples

**Example 4.3** Find the first and second derivatives of the function (a) $y = f(x) = 2x^5 + 3x^4 + x^3 - 5x^2 + x - 2$ with respect to $x$ and (b) $y = f(t) = At^3 + Bt^2 + Ct + D$ with respect to $t$. ■

**Solution:** We obtain the results (a) $f'(x) = 10x^4 + 12x^3 + 3x^2 - 10x + 1$ and $f''(x) = 40x^3 + 36x^2 + 6x - 10$. (b) $f'(t) = 3At^2 + 2Bt + C$ and $f''(t) = 6At + 2B$. In (b) the independent variable is $t$, but, of course, the rules of differentiation are the same.

**Example 4.4** Find the equation of the tangent line to the graph of $y = f(x) = x^3 - x$ at the point $x = 1.5$. ■

**Solution:** In Example 3.4, we approximated the slope of the tangent line shown in Fig. 3.1, and used that value to solve the problem. We return to this problem using the rules of differentiation to get an exact result. As we have learned, the derivative of the polynomial $f(x) = x^3 - x$ is $f'(x) = 3x^2 - 1$. At $x = 1.5$ we have $f'(1.5) = 3(1.5)^2 - 1 = 5.75$, so the slope of the tangent line is 5.75 (an exact result, better than our eyeball approximation in Example 3.4). The coordinates of the point of interest are $f(1.5) = 1.875$), as before. Thus the equation of the tangent line is

$$\frac{y - 1.875}{x - 1.5} = 5.75 \quad \Rightarrow \quad y = 1.875 + 5.75(x - 1.5) \quad \Rightarrow \quad y = 5.75x - 6.75.$$

## 4.4 Antiderivatives of polynomials

As seen above, when we differentiate a polynomial, we obtain a polynomial of a lower degree, that is, the highest power decreases by 1. We now consider the idea of "antidifferentiation", which reverses the operation of the derivative. Suppose we are given that the second derivative of some function is

$$y''(t) = c_1 t + c_2.$$

(This is a polynomial of degree 1.) Evidently, this function resulted by taking the derivative of $y'(t)$, which had to be a polynomial of degree 2. We can check that

$$y'(t) = \frac{c_1}{2}t^2 + c_2 t$$

could be such a function, but so could

$$y'(t) = \frac{c_1}{2}t^2 + c_2 t + c_3$$

for any constant $c_3$. In turn, the function $y(t)$ had to be a polynomial of degree 3. We can see that one such function is

$$y(t) = \frac{c_1}{6}t^3 + \frac{c_2}{2}t^2 + c_3 t + c_4$$

where $c_4$ is any constant. (This can be checked by differentiating.) The steps we have just illustrated are "antidifferentiation". In short, the relationship is:

$$\text{for differentiation} \quad y(t) \to y'(t) \to y''(t)$$

whereas

$$\text{for antidifferentiation} \quad y''(t) \to y'(t) \to y(t).$$

(Arrows denote what is done to one function to arrive at the next.) We also note an important result that holds for functions other than polynomials:

**Given a function,** $f(x)$ **we can only determine its antiderivative up to some (additive) constant**.

We apply these ideas in the following section.

## 4.5 Position, velocity, and acceleration

As an example of the relation between a function and its first and second derivative, we return to the discussion of displacement, velocity and acceleration of an object falling under the force of gravity. Here we will use the notation $y(t)$ to denote the position of the object at time $t$. From now on, we will refer to the instantaneous velocity of a particle or object at time $t$ simply as *the velocity*, $v(t)$.

**Definition 4.5 (The velocity).** *Given the position of some particle as a function of time,* $y(t)$*, we define the velocity as the rate of change of the position, i.e. the derivative of* $y(t)$*:*

$$v(t) = \frac{dy}{dt} = y'(t)$$

Here we have just used two equivalent notations for the derivative. In general, $v$ may depend on time, a fact we indicated by writing $v(t)$.

**Definition 4.6 (The acceleration).** *We will also define the acceleration as the (instantaneous) rate of change of the velocity, i.e. as the derivative of* $v(t)$.

$$a(t) = \frac{dv}{dt} = v'(t).$$

*(Acceleration could also depend on time, hence $a(t)$.)*

Since the acceleration is the derivative of a derivative of the original function, we also use the notation

$$a(t) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2} = y''(t)$$

Here we have used three equivalent ways of writing a second derivative. (This notation evolved for historical reasons, and is used interchangeably in science.) The acceleration is hence the second derivative of the position.

In view of our discussion of antidifferentiation, given information about the acceleration as a function of $t$, we can obtain the velocity $v(t)$ (up to some constant) by antidifferentiation. Similarly, we can use the velocity $v(t)$ to determine the position $y(t)$ (up to some constant). The constants must be obtained from other information, as examples that follow will illustrate.

**Example 4.7 (Uniformly accelerated motion)** Suppose that the acceleration of an object is constant in time, i.e. $a(t) = g =$ constant. Use antidifferentiation to determine the velocity and the position of the object as functions of time. ■

**Solution:** We ask: what function of time $v(t)$ has the property that

$$a(t) = v'(t) = g = \text{constant?}$$

The function $a(t) = v'(t)$ is a polynomial of degree 0 in the variable $t$. To find the velocity, we apply antidifferentiation to obtain a polynomial of degree 1,

$$v(t) = gt.$$

This is one antiderivative of the acceleration, but in fact, other functions such as

$$v(t) = gt + c, \tag{4.6}$$

would work for any constant $c$. How can we decide which value of the constant $c$ to use? To determine $c$ we need additional information about the velocity, for example at $t = 0$. Suppose we are told that $v(0) = v_0$ is the known value of the **initial velocity**[5]. Then, substituting $t = 0$ into (4.6), we find that $c = v_0$. Thus in general,

$$v(t) = gt + v_0$$

where $v_0$ is the initial velocity of the object.

To now determine the position of the particle as a function of the time $t$, we recall that $v(t) = y'(t)$. Thus, using the result (4.6), we have

$$y'(t) = v(t) = gt + v_0 \tag{4.7}$$

[5]The statement $v(0) = v_0$ will later be called an “initial condition”, since it specifies how fast the particle was moving initially.

Then, by antidifferentiation of (4.7), we obtain a polynomial of degree 2,

$$y(t) = \frac{1}{2}gt^2 + v_0t + k \tag{4.8}$$

where, as before we allow for some additive constant $k$. It is a simple matter to check that the derivative of this function is the given expression for $v(t)$. By reasoning as before, the constant $k$ can be determined from the initial position of the object $y(0) = y_0$. A before, (plugging $t = 0$ into (4.9)) we find that $k = y_0$, so that

$$y(t) = \frac{1}{2}gt^2 + v_0t + y_0. \tag{4.9}$$

Here we use the acceleration due to gravity, $g$, but any other motion with constant acceleration would be treated in the same way.

**Summary, uniformly accelerated motion:** If an object moves with constant acceleration $g$, then given its initial velocity $v_0$ and initial position $y_0$ at time $t = 0$, the position at any later time is described by:

$$y(t) = \frac{1}{2}gt^2 + v_0t + y_0.$$

This powerful and general result is a direct result of the assumption that the acceleration is constant, using the elementary rules of calculus, and the definitions of velocity and acceleration as first and second derivatives of the position. We further illustrate these ideas with examples of motion under the influence of gravity.

**Example 4.8 (The motion of a falling object, revisited)** A falling object experiences uniform acceleration (downwards) with $a(t) = -g =$ constant [6]. Suppose that an object is thrown upwards at initial velocity $v_0$ from a building of height $h_0$.

(a) Find the velocity and the acceleration of the object at any time $t$.

(b) When does the object hit the ground?

(c) Determine when the object reaches its highest point, and what is its velocity at that time.

(d) Find the velocity of the object when it hits the ground.

$\blacksquare$

**Solution:** By previous reasoning, the height of the object at time $t$, denoted $y(t)$ is given by

$$y(t) = -\frac{1}{2}gt^2 + v_0t + h_0.$$

[6] Here we have chosen a coordinate system in which the positive direction is "upwards", and so the acceleration, which is in the opposite direction, is negative. On Earth, $g = 9.8$ m /s$^2$.

(a) The velocity is given by:

$$v(t) = y'(t) = v_0 - 2(\frac{1}{2}gt) = v_0 - gt.$$

We may observe that at $t = 0$, the *initial velocity* is $v(0) = v_0$. If the object was thrown upwards then $v_0 > 0$, i.e., it is initially heading up. Differentiating one more time, we find that the acceleration is:

$$a(t) = v'(t) = -g.$$

We observe that the acceleration is constant. The negative sign means that the object is accelerating downwards, in the direction opposite to the positive direction of the $y$ axis. This makes sense, since the force of gravity acts downwards, causing this acceleration.

(b) We will assume that the object hits the ground at level $y = 0$. Then we must solve for $t$ in the equation:

$$y(t) = h_0 + v_0 t - \frac{1}{2}gt^2 = 0.$$

Here we must observe that the highest power of the independent variable is 2, so that $y$ is a quadratic function of $t$, and solving for $t$ requires us to solve a quadratic equation. This is a quadratic equation, which could be written in the form

$$\frac{1}{2}gt^2 - v_0 t - h_0 = 0, \quad \Rightarrow \quad gt^2 - 2v_0 t - 2h_0 = 0.$$

Using the quadratic formula, we obtain

$$t_{\text{ground}} = \frac{2v_0 \pm \sqrt{4v_0^2 + 8gh_0}}{2g} \quad \Rightarrow \quad t_{\text{ground}} = \frac{v_0}{g} \pm \frac{\sqrt{v_0^2 + 2gh_0}}{g}.$$

We have found two roots. One is positive and the other is negative. Since we are interested in $t > 0$, we will reject the negative root, so

$$t_{\text{ground}} = \frac{v_0}{g} + \frac{\sqrt{v_0^2 + 2gh_0}}{g}.$$

(c) To find when the object reaches its highest point, we note that the object shoots up, but it slows down with time. Eventually, it can no longer continue to go up: this happens precisely when its velocity is zero. From then on it will start to fall to the ground. The top of its trajectory is determined by finding when the velocity of the object is zero. Equating

$$v(t) = v_0 - gt = 0$$

we solve for $t$, to get

$$t_{\text{top}} = \frac{v_0}{g}.$$

(d) To find the velocity of the object when it hits the ground. we need to use the time determined in part (b). Substituting $t_{\text{ground}}$ into the expression for velocity, we obtain:

$$v(t_{\text{ground}}) = v_0 - gt_{\text{ground}} = v_0 - g\left(\frac{v_0}{g} + \frac{\sqrt{v_0^2 + 2gh_0}}{g}\right).$$

After some algebraic simplification, we obtain

$$v(t_{\text{ground}}) = -\sqrt{v_0^2 + 2gh_0}.$$

We observe that this velocity is negative, indicating (as expected) that the object is falling *down*.

Figure 4.1 illustrates the relationship between the three functions.

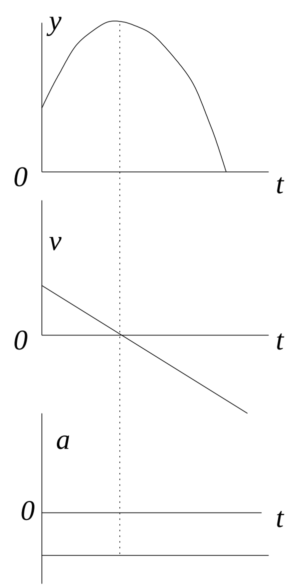



**Figure 4.1.** *The position, velocity, and acceleration of an object that is thrown upwards and falls under the force of gravity.*

## 4.6 Sketching skills

We have already encountered the idea of sketching the derivative of a function, given a sketch of the original function. Here we practice this skill further. In the examples below, we make no attempt to be accurate about heights of peaks and valleys in our sketches (as would be certainly possible using numerical methods like a spreadsheet). Rather, we are aiming for qualitative features, where the most important aspects of the graphs (locations of key points such as peaks and troughs) are indicated.

**Example 4.9 (Sketching the derivative from the original function)** Use the function shown in Figure 4.2 to sketch the first and second derivatives. ■

**Solution:** See the panels of Figure 4.2 for the function $y(t)$, its first derivative, $v(t) = y'(t)$, and its second derivative, $a(t) = y''(t)$. (This was done in two steps: in each case, we determined the slopes of tangent lines as a first step.) An important feature to notice is that wherever a tangent line to a curve is horizontal, e.g. at the "tops of peaks" (local maxima) or "bottoms of valleys"(local minima), the derivative is zero. This is indicated at several places in Figure 4.2.

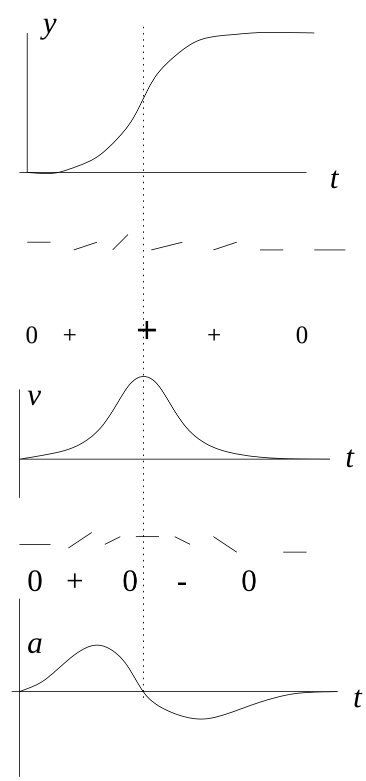



**Figure 4.2.** *Figure for Example 4.9.*

**Example 4.10 (Sketching a function from a sketch of its derivative)** Use the sketch of $f'(x)$ in the top panel of Figure 4.3 to sketch the original function $f(x)$ ■

**Solution:** See the bottom panel of Figure 4.3. An important point is that there are many possible ways to draw $f(x)$ given $f'(x)$, because $f'(x)$ only contains information about *changes* in $f(x)$, not about how high the function is at any point. This means that:

**Given the derivative of a function, $f'(x)$, we can only determine $f(x)$ up to some (additive) constant**.

In Figure 4.3 we show a number of possibilities for $f(x)$. If we were given an *additional* piece of information, for example that $f(0) = 0$, we would be able to select out one specific curve out of this family of solutions.

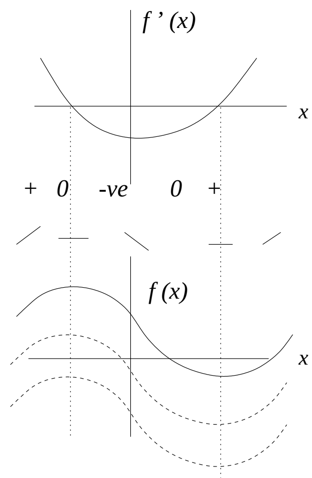



**Figure 4.3.** *Using the sketch of a function $f'(x)$ to sketch the function $f(x)$.*

## 4.7 A biological speed machine

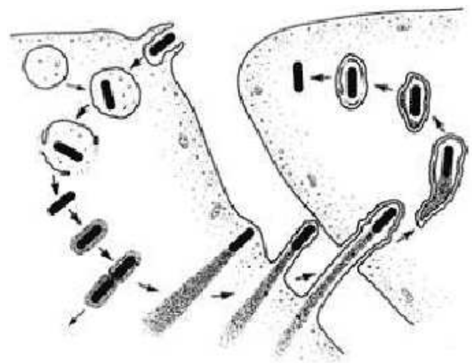


**Figure 4.4.** *The parasite* Lysteria *lives inside a host cell. It assembles a "rocket-like" tail made up of actin, and uses this assembly to move around the cell, and to pass from one host cell to another.*

*Lysteria monocytogenes* is a parasite that lives inside cells of the host, causing a nasty infection. It has been studied by cellular biologists for its amazingly fast propulsion, which uses the host's actin filaments as "rocket fuel". Actin is part of the structural component of all animal cells, and is known to play a major role in cell motility. Lysteria manages to "hijack" this cellular mechanism, assembling it into its own comet tail, which can be used to propel inside the cell and pass from one cell to the next. Figure 4.4 illustrates part of these curious traits.

Researchers in cell biology use Lysteria to find out more about motility at the cellular level. It has been discovered that certain proteins on the external surface of this parasite (ActA) are responsible for the ability of Lysteria to assemble an actin filament tail. Surprisingly, even small plastic beads artificially coated in Lysteria's ActA proteins can perform the same "trick": they assemble an actin tail which pushes the bead like a tiny rocket.

A recent paper in the literature (Bernheim-Groswasser A, Weisner S, Goldsteyn RM, Carlier M-F, Sykes C (2002) The dynamics of actin-based motility depend on surface parameters Nature 417: 308-311.) describes the motion of these beads, shown in Figure 4.5. When the position of the bead is plotted on a graph with time as the horizontal axis, (see Figure 4.6) we find that the trajectory is not a simple one: it appears that the bead slows down periodically, and then accelerates.

With the techniques of this chapter, we can analyze the experimental data shown in Figure 4.6 to determine both the average velocity of the beads, and the instantaneous velocity over the course of the motion.

### Average velocity of the bead

We can get a rough idea of how fast the micro-beads are moving by computing an average velocity over the time interval shown on the graph. We can use two (approximate) data

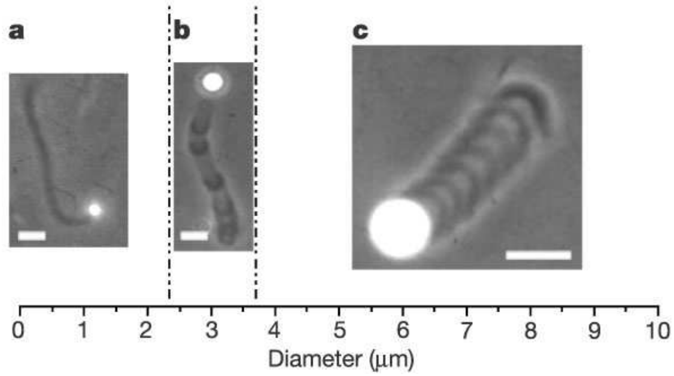



**Figure 4.5.** *Small spherical beads coated with part of Lysteria's special actin-assembly kit also gain the ability to swim around. Based on Bernheim-Groswasser et al, 2002.*

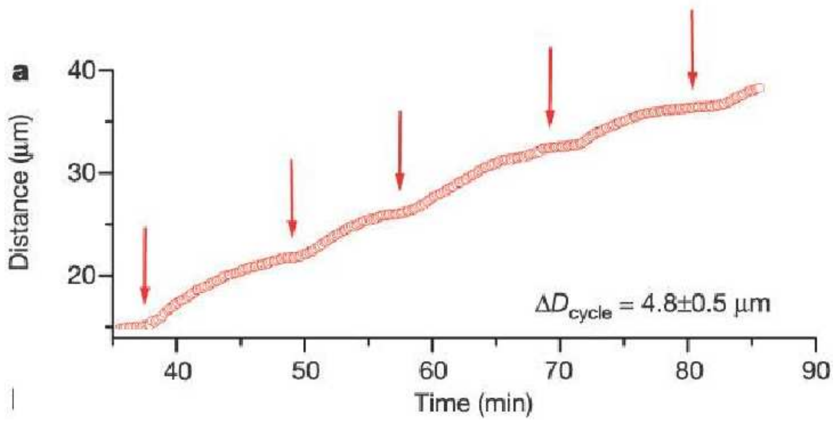



**Figure 4.6.** *The distance traveled by a little bead is shown as a function of time. The arrows point to times when the particle slowed down or stopped. We can use this data to analyze the velocity of the particles. Based on Bernheim-Groswasser et al, 2002.*

points $(t, D(t)$, at the beginning and end of the run, for example (45,20) and (80,35): Then the average velocity is

$$\bar{v} = \frac{\Delta D}{\Delta t}$$

$$\bar{v} = \frac{35 - 20}{80 - 45} \approx 0.43 \mu \,\mathrm{min}^{-1}$$

so the beads move with average velocity 0.43 microns per minute. (One micron is $10^{-6}$ meters.)

### The changing instantaneous velocity:

Because the actual data points are taken at finite time increments, the curve shown in Figure 4.6 is not smooth. We will smoothen it, as shown in Figure 4.7 for a simpler treatment. In Figure 4.8 we sketch this curve together with a collection of lines that represent the slopes of tangents along the curve. A horizontal tangent has slope zero: this means that at all such points (also indicated by the arrows for emphasis), the velocity of the beads is zero. Between these spots, the bead has picked up speed and moved forward until the next time in which it stops.

We show the velocity $v(t)$, which is the derivative of the original function $D(t)$ in Figure 4.9. As shown here, the velocity has periodic increases and decreases.

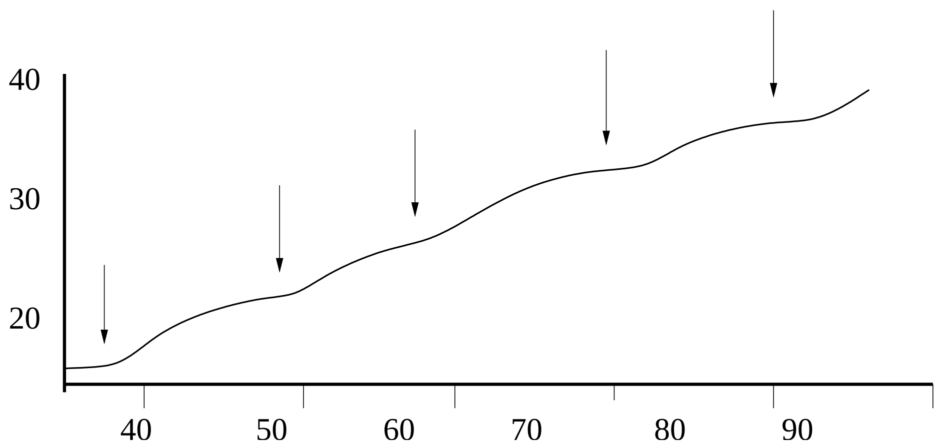



**Figure 4.7.** *The (slightly smoothened) bead trajectory is shown here.*

## 4.8 Additional problems and examples

We now turn to a number of problems based on derivatives, tangent lines, and slopes of polynomials. We use these to build up our problem-solving skills in examples where the calculations are relatively straight-forward. In the two examples below, we use information about a function to identify the slope and/or equation of its tangent line.

**Example 4.11** (a) Find the equation of the tangent line to

$$y = f(x) = x^3 - ax$$

for $a > 0$ a constant, at the point $x = 1$. (b) Find where that tangent line intersects the $x$ axis.

**Solution:** The function given in the example is a simple polynomial, so we easily calculate its derivative. The idea is very similar to that of the previous example, but the constant $a$

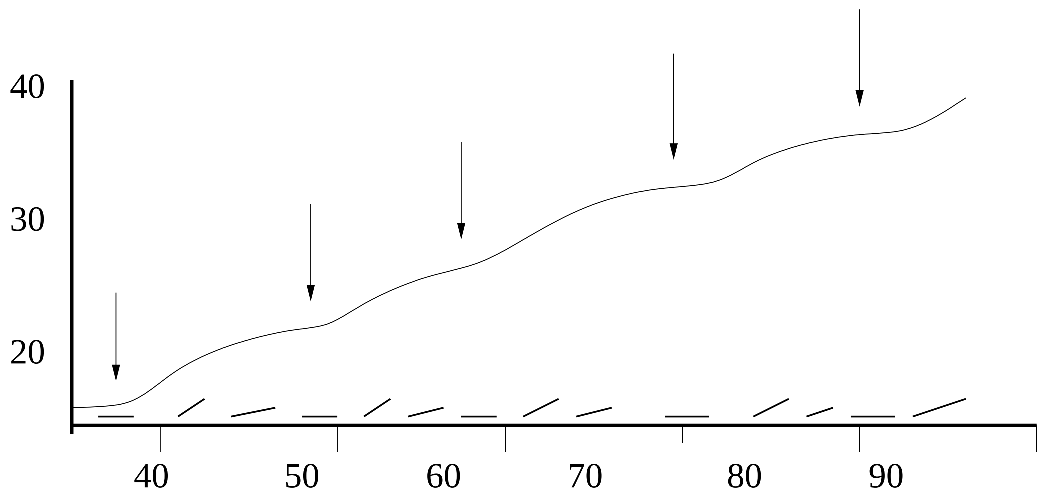



**Figure 4.8.** *We have inserted a sketch of the tangent line configurations along the trajectory from beginning to end. We observe that some of these tangent lines are horizontal, implying a zero derivative, and, thus, a zero instantaneous velocity at that time.*

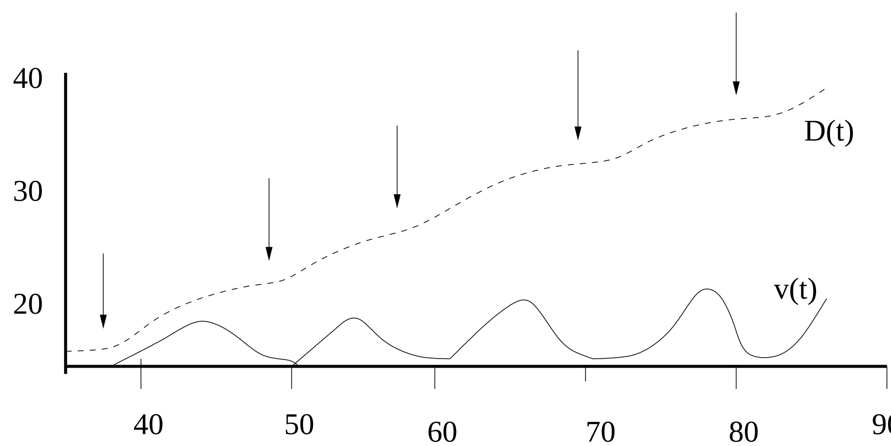



**Figure 4.9.** *Here we have sketched the velocity on the same graph.*

makes this calculation a little less straightforward. (a) $y = f(x) = x^3 - ax$ so the derivative is

$$\frac{dy}{dx} = f'(x) = 3x^2 - a$$

and at $x = 1$ the slope (in terms of the constant $a$) is $f'(1) = 3 - a$. The point of interest on the curve has coordinates $x = 1, y = 1^3 - a \cdot 1 = 1 - a$.

We look for a line through $(1, 1-a)$ with slope $m = 3-a$. That, is,

$$\frac{y-(1-a)}{x-1} = 3-a.$$

Simplifying algebraically leads to

$$y = (3-a)(x-1) + (1-a)$$

or simply

$$y = (3-a)x - 2.$$

[Remark: at this point is is wise to check that the tangent line goes through the desired point and has the slope we found. One way to do this is to pick a simple value for $a$, e.g. $a = 1$ and do a quick check that the answer matches what we have found.]

(b) To find the point of intersection, we set

$$y = (3-a)x - 2 = 0$$

and solve for $x$. We find that

$$x = \frac{2}{3-a}.$$

**Example 4.12** Find any value(s) of the constant $a$ such that the line $y = ax$ is tangent to the curve

$$y = f(x) = -x^2 + 3x - 2.$$

$\blacksquare$

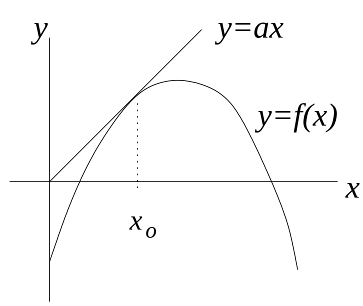



**Figure 4.10.** *Figure for Example 4.12*

**Solution:** This example, too, revolves around the properties of a polynomial, but the problem is somewhat more challenging. We must use some geometric properties of the function and the tentative candidate for a tangent line to determine the value of the unknown constant $a$.

As shown in Figure 4.10, there may be one (or more) points at which tangency occurs. We do not know the coordinate of any such point, but we will label it $x_0$ to denote that it is some definite (as yet to be determined) value. Notation can sometimes be confusing. We must remember that while we can compute the derivative of $f$ at any point, only the specific point at which the tangent touches the curve will have special properties that we will outline below. Finding that point of tangency, $x_0$, will be part of the problem.

What we know is that, at $x_0$,

- The straight line and the graph of the function $f(x)$ go through the same point.
- The straight line $y = ax$ and the tangent line to the graph coincide, i.e. the derivative of $f(x)$ at $x_0$ is the same as the slope of the straight line, which is clearly $a$

Using these two facts, we can write down the following equations:

- Equating slopes:
$$f'(x_0) = -2x_0 + 3 = a$$
- Equating $y$ values on line and graph of $f(x)$:
$$f(x_0) = -x_0^2 + 3x_0 - 2 = ax_0$$

We now have two equations for two unknowns, ($a$ and $x_0$). We can solve this system easily by substituting the value of $a$ from the first equation into the second, getting

$$-x_0^2 + 3x_0 - 2 = (-2x_0 + 3)x_0.$$

Simplifying:

$$-x_0^2 + 3x_0 - 2 = -2x_0^2 + 3x_0$$

so

$$x_0^2 - 2 = 0, \quad x_0 = \pm\sqrt{2}.$$

This shows that there are two points at which the conditions would apply. In Figure 4.11 we show two such points.

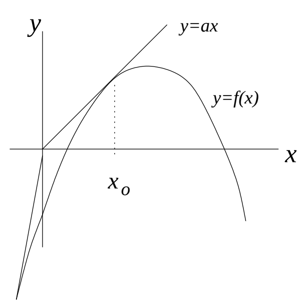



**Figure 4.11.** *Figure for solution to Example 4.12*

We can now find the slope $a$ using $a = -2x_0 + 3$. We get:

$$x_0 = \sqrt{2} \;\; a = -2\sqrt{2} + 3,$$

and

$$x_0 = -\sqrt{2} \;\; a = 2\sqrt{2} + 3.$$

Remark: This problem illustrates the idea that in some cases, we proceed by listing properties that are known to be true, using the information to obtain a set of (algebraic) equations, and then solving those equations. The challenge is to use these sequential steps properly - each step on its own is relatively understandable and clearcut. Most problems encountered in scientific and engineering applications require a whole chain of reasoning, calculation, or logic, so practicing such multi-step problems is an important part of training for science, medicine, engineering, and other fields.

# Chapter 5

# What the Derivative tells us about a function

The derivative of a function contains a lot of important information about the behaviour of a function. In this chapter we will focus on how properties of the first and second derivative can be used to help up refine curve-sketching techniques.

## 5.1 The shape of a function: from $f'(x)$ and $f''(x)$

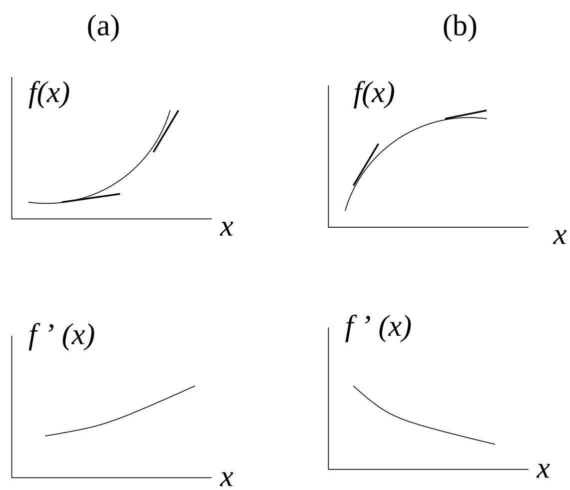



**Figure 5.1.** *In (a) the function is concave up, and its derivative thus increases (in the positive direction). In (b), for a concave down function, we see that the derivative decreases.*

Consider a function given by $y = f(x)$. We first make the following observations:

1. If $f'(x) > 0$ then $f(x)$ is *increasing*.

2. If $f'(x) < 0$ then $f(x)$ is *decreasing*.

Naturally, we read graphs from left to right, i.e. in the direction of the positive $x$ axis, so when we say "increasing" we mean that as we move from left to right, the value of the function gets larger.

We can use the same ideas to relate the second derivative to the first derivative.

1. If $f''(x) > 0$ then $f'(x)$ is **increasing**. This means that the slope of the original function is getting steeper (from left to right). The function curves upwards: we say that it is *concave up*. See Figure 5.1(a).
2. If $f''(x) < 0$ then $f'(x)$ is **decreasing**. This means that the slope of the original function is getting shallower (from left to right). The function curves downwards: we say that it is *concave down*. See Figure 5.1(b).

We see examples of the above two types in Figure 5.1. In Figure 5.1(a), $f(x)$ is **concave up**, and its second derivative (not shown) would be positive. In Figure 5.1(b), $f(x)$ is **concave down**, and second derivative would be negative.

To summarize, the second derivative of a function provides information about the curvature of the graph of the function, also called the concavity of the function.

## 5.2 Points of inflection

**Definition 5.1.** *A* **point of inflection** *of a function $f(x)$ is a point $x$ at which the concavity of the function changes.*

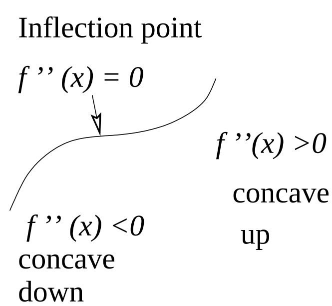



**Figure 5.2.** *An inflection point is a place where the concavity of a function changes.*

We can deduce from the definition and previous remarks that at a point of inflection the *second derivative changes sign*. This is illustrated in Figure 5.2. **Note carefully:** It is not enough to show that $f''(x) = 0$ to conclude that $x$ is an inflection point. We must actually check that $f''(x)$ changes sign at opposite sides of the value $x$, as the following example shows:

**Example 5.2** Show that the function $f(x) = x^4$ does not have a point of inflection at the origin, even though its second derivative is zero at that point. ∎

**Solution:** Consider the function

$$y = f(x) = x^4.$$

The first and second derivatives of $f$ are:

$$\frac{dy}{dx} = f'(x) = 4x^3, \quad \text{and} \quad \frac{d^2y}{dx^2} = f''(x) = 12x^2.$$

Then $f''(x) = 0$ when $x = 0$. However, $x = 0$ is **NOT** an inflection point. In fact, it is a local minimum, as is evident from Figure 5.3.

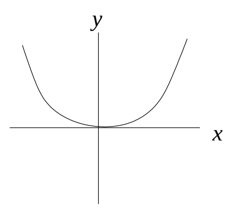



**Figure 5.3.** *The function $y = f(x) = x^4$ has a minimum at $x = 0$. The fact that the second derivative is zero at the origin, $f''(0) = 0$, is clearly NOT associated with an inflection point. This results from the fact that $f''$ does not change sign as we cross $x = 0$.*

## 5.3 Critical points

**Definition 5.3.** *A* **critical point** *of the function $f(x)$ is any point $x$ at which the first derivative is zero, i.e. $f'(x) = 0$.*

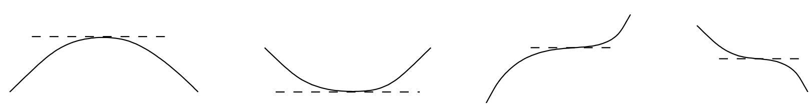


**Figure 5.4.** *A critical point (place where $f'(x) = 0$) can be a local maximum, local minimum, or neither.*

Clearly, this will occur whenever the slope of the tangent line to the graph of the function is zero, i.e. the tangent line is horizontal. Figure 5.4 shows several possible shapes of the graph of function close to a critical point.

We will call the first of these (on the left) a **local maximum**, the second a **local minimum**, and the last two cases (which are bends in the curve) inflection points.

In many scientific applications, critical points play a very important role. (We will see examples of this sort shortly.) We would like criteria for determining whether a critical point is a local maximum, minimum, or neither. We will develop such diagnoses in the next section.

## 5.4 What happens close to a critical point

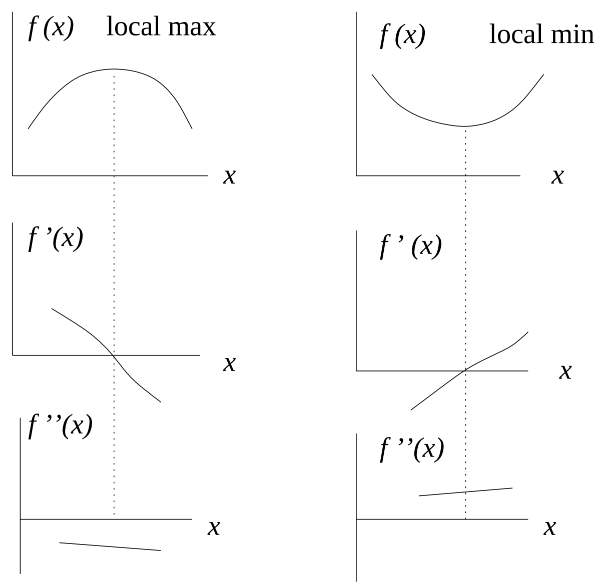



**Figure 5.5.** *Close to a local maximum,* $f(x)$ *is concave down,* $f'(x)$ *is decreasing, so that* $f''(x)$ *is negative. Close to a local minimum,* $f(x)$ *is concave up,* $f'(x)$ *is increasing, so that* $f''(x)$ *is positive.*

From Figure 5.5 we see the behaviour of the first and second derivatives of a function close to critical points. We already know that at the point in question, $f'(x) = 0$, so clearly the graph of $f'(x)$ crosses the $x$ axis at each critical point. However, note that next to a local maximum, (and reading from left to right, as is the convention in any graph) the slope of $f(x)$ is first positive (to the left), then becomes zero (at the critical point) and then becomes negative (to the right of the point). This means that the derivative is *decreasing* from left to right, as indicated in Figure 5.5.

Since the changes in the first derivative are measured by *its* derivative, i.e. by $f''(x)$, we can say, equivalently that the second derivative is negative at a local maximum.

The converse is true near any local minimum. This is shown on the right column of Figure 5.5. We conclude from this discussion that the following diagnosis would distinguish a local maximum from a local minimum:

### Test for maxima and minima

- **First derivative test**: Near a local maximum, the first derivative has a transition from positive to zero to negative values reading across the graph from left to right. Near a local minimum, the first derivative goes from negative to zero to positive values.

- **Second derivative test**: Near a local maximum, the second derivative is negative.

Near a local minimum, the second derivative is positive.

### Summary: first derivative

| $f'(x) < 0$ | $f'(x_0) = 0$ | $f'(x) > 0$ |
|---|---|---|
| decreasing function | critical point at $x_0$ | increasing function |

### Summary: second derivative

| $f''(x) < 0$ | $f''(x_0) = 0$ | $f''(x) > 0$ |
|---|---|---|
| curve concave down | check for inflection point at $x_0$ if $f''$ changes sign | curve concave up |

### Summary: type of critical point

Here we assume that $x_0$ is a critical point, i.e. a point at which $f'(x_0) = 0$. Then the following table summarizes what happens at that point

| $f''(x_0) < 0$ | $f''(x_0) = 0$ | $f''(x_0) > 0$ |
|---|---|---|
| local maximum | inconclusive | local minimum |

### Important note:

The fact that $f''(x_0) = 0$ may alert us to look for an inflection point. However, only if $f''$ changes sign from left to right of $x_0$ can we conclude that $x_0$ is an inflection point.

We will apply some of these ideas to a number of examples and applications.

## 5.5 Sketching the graph of a function

**Example 5.4** Sketch the graph of the function defined by $B(x) = C(x^2 - x^3)$. ■

**Solution:** To prepare the way, we compute the derivatives:

$$B'(x) = C(2x - 3x^2), \quad B''(x) = C(2 - 6x).$$

The following set of steps will be a useful way to proceed:

1. We can easily find the *zeros* of the function by setting $B(x) = 0$. We find that

$$C(x^2 - x^3) = 0, \quad \Rightarrow \quad x^2 = x^3$$

so $x = 0$ or $x = 1$ are the solutions.

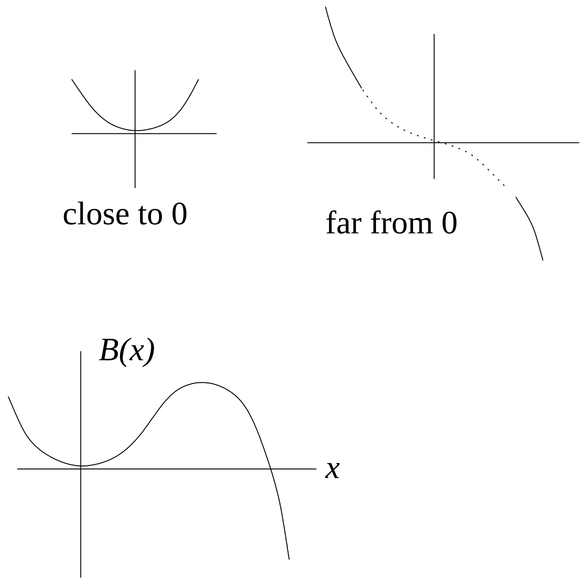



**Figure 5.6.** *Figure for Example 5.4 showing which power dominates.*

2. By considering powers, we note that close to the origin, the power $x^2$ would dominate (so we expect to see something resembling a parabola opening upwards close to the origin), whereas, far away, where the term $-x^3$ dominates, we expect an (upside down) cubic curve, as shown in a preliminary sketch in Figure 5.6.

3. To find the critical points, we set $B'(x) = 0$, obtaining

$$B'(x) = C(2x - 3x^2) = 0, \quad \Rightarrow \quad 2x - 3x^2 = 0, \quad \Rightarrow \quad 2x = 3x^2$$

   so either $x = 0$ or $x = 2/3$. From the sketch in Figure 5.6 it is clear that the first is a local minimum, and the second a local maximum. (But we will also get a confirmation of this fact from the second derivative.)

4. From the second derivative we find that $B''(0) = 2 > 0$ so that $x = 0$ is indeed a local minimum. Further, $B''(2/3) = 2 - 6 \cdot (2/3) = -2 < 0$ so that $x = 2/3$ is a local maximum. This is the confirmation that our sketch makes sense.

5. Now identifying where $B''(x) = 0$, we find that

$$B''(x) = C(2 - 6x) = 0, \quad \text{when} \quad 2 - 6x = 0 \quad \Rightarrow \quad x = \frac{2}{6} = \frac{1}{3}$$

   we also note that the second derivative changes sign here: i.e. for $x < 1/3$, $B''(x) > 0$ and for $x > 1/3$, $B''(x) < 0$. Thus there is an inflection point at $x = 1/3$. The final sketch would be as given in Figure 5.7.

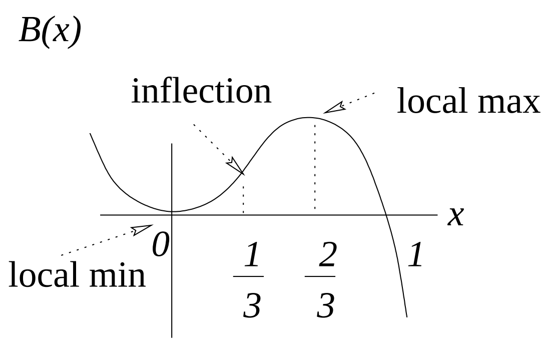



**Figure 5.7.** *Figure for Example 5.4.*

**Example 5.5** Sketch the graph of the function $y = f(x) = 8\,x^5 + 5\,x^4 - 20\,x^3$ ∎

**Solution:**

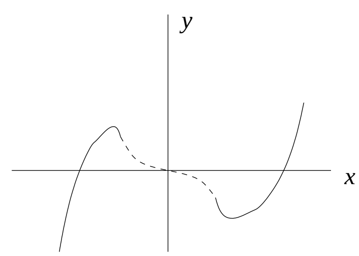



**Figure 5.8.** *The function* $y = f(x) = 8\,x^5 + 5\,x^4 - 20\,x^3$ *of Example 5.5 behaves roughly like the negative cubic near the origin, and like* $8\,x^5$ *for large* $x$.

1. **Consider the powers:**

   The highest power is $8\,x^5$ so that far from the origin we expect a typical positive odd function behavior.

   The lowest power is $-20\,x^3$, which means that close to zero, we would expect to see a negative cubic. This already indicates to us that the function "turns around", and so, must have some local maxima and minima. We draw a rough sketch in Figure 5.8.

2. **Zeros:** Factoring the expression for $y$ leads to

$$y = x^3(8x^2 + 5x - 20).$$

   Using the quadratic formula, we can find the places where $y = 0$, i.e. the *zeros* of the function. They are

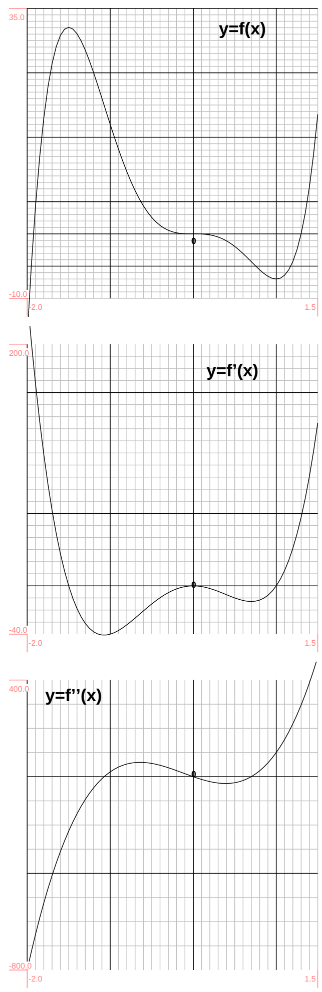



**Figure 5.9.** *The function* $y = f(x) = 8\,x^5 + 5\,x^4 - 20\,x^3$*, and its first and second derivatives,* $f'(x)$ *and* $f''(x)$

$$x = 0,\ 0,\ 0,\ -\frac{5}{16} + \frac{1}{16}\sqrt{665},\ -\frac{5}{16} - \frac{1}{16}\sqrt{665}$$

In decimal form, these are approximately $x = 0, 0, 0, 1.3, -1.92$

3. **First derivative:** Calculating the derivative of $f(x)$ and then factoring leads to

$$\frac{dy}{dx} = f'(x) = 40\,x^4 + 20\,x^3 - 60\,x^2 = 20x^2(2x+3)(x-1)$$

so that the places where this derivative is zero are: $x = 0, 0, 1, -3/2$. We expect critical points at these places.

4. **Second derivative:** We calculate the second derivative and factor to obtain

$$\frac{d^2y}{dx^2} = f''(x) = 160\,x^3 + 60\,x^2 - 120\,x = 20x(8x^2 + 3x - 6)$$

Thus, we can find places where the second derivative is zero. This occurs at

$$x = 0,\ -\frac{3}{16} + \frac{1}{16}\sqrt{201},\ -\frac{3}{16} - \frac{1}{16}\sqrt{201}$$

The values of these roots can be approximated by: $x = 0, 0.69, -1.07$

5. **Classifying the critical points:** To identify the types of critical points, we can use the second derivative test, i.e. determine the sign of the second derivative at each of the critical points.

At $x = 0$ we see that $f''(0) = 0$ so the test is inconclusive. At $x = 1$, we have $f''(1) = 20(8 + 3 - 6) > 0$ implying that this is a local minimum. At $x = -3/2$ we have $f''(-1.5) = -225 < 0$ so this is a local maximum. In fact we find that the value of the function at $x = -1.5$ is $y = f(-1.5) = 32.0625$, whereas at $x = 1$ $f(1) = -7$.

The table below summarizes what we have found, and what we concluded. Each of the values of $x$ across its top row has some significance in terms of the behaviour of the function.

| $x =$ | $-1.92$ | $-1.5$ | $-1.07$ | $0$ | $0.69$ | $1$ | $1.3$ |
|---|---|---|---|---|---|---|---|
| $f(x) =$ | $0$ | $32.0$ | | $0$ | | $-7$ | $0$ |
| $f'(x) =$ | | $0$ | | $0$ | | $0$ | |
| $f''(x) =$ | | $< 0$ | $0$ | $0$ | $0$ | $> 0$ | |
| | zero | max | inflection | | inflection | min | zero |

We can now sketch the shape of the function, and its first and second derivatives in Figure 5.9.

## 5.6 Product and Quotient rules for derivatives

So far, using a single “rule” for differentiation, the power rule, together with properties of the derivative such as additivity and constant multiplication (described in Section 4.2), we were able to calculate derivatives of polynomials. here we state without proof, two other rules of differentiation that will prove to be useful in due time.

**The product rule:** If $f(x)$ and $g(x)$ are two functions, each differentiable in the domain of interest, then

$$\frac{d[f(x)g(x)]}{dx} = \frac{df(x)}{dx}g(x) + \frac{dg(x)}{dx}f(x).$$

Another notation for this rule is

$$[f(x)g(x)]' = f'(x)g(x) + g'(x)f(x).$$

**Example 5.6** Find the derivative of the product of the two functions $f(x) = x$ and $g(x) = 1 + x$. ∎

**Solution:** Using the product rule leads to

$$\frac{d[f(x)g(x)]}{dx} = \frac{d[x(1+x)]}{dx} = \frac{d[x]}{dx} \cdot (1+x) + \frac{d[(1+x)]}{dx} \cdot x = 1 \cdot (1+x) + 1 \cdot x = 2x+1.$$

(This can be easily checked by noting that $f(x)g(x) = x(1+x) = x+x^2$, whose derivative agrees with the above.)

**The quotient rule:** If $f(x)$ and $g(x)$ are two functions, each differentiable in the domain of interest, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{df(x)}{dx}g(x) - \frac{dg(x)}{dx}f(x)}{[g(x)]^2}.$$

We can also write this in the form

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}.$$

**Example 5.7** Find the derivative of the function $y = ax^{-n} = a/x^n$ where $a$ is a constant and $n$ is a positive integer. ∎

**Solution:** We can rewrite this as the quotient of the two functions $f(x) = a$ and $g(x) = x^n$. Then $y = f(x)/g(x)$ so, using the quotient rule leads to the derivative

$$\frac{dy}{dx} = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2} = \frac{0 \cdot x^n - (nx^{n-1}) \cdot a}{(x^n)^2} = \frac{-anx^{n-1}}{x^{2n}}$$

After algebraic simplification, we obtain $dy/dx = a(-n)x^{n-1-2n} = a(-n)x^{-n-1}$. This is an interesting result: **The power rule of differentiation holds for negative integer powers.**

## 5.7 Global maxima and minima: behaviour at the endpoints of an interval

**Global (absolute) maxima and minima:**

A global (or absolute) maximum of a function $y = f(x)$ over some interval is the largest value that the function attains on that interval. Similarly a global (or absolute) minimum is the smallest value.

Comment: If the function is defined on a closed interval, we must check both the local maxima and minima as well as the endpoints of the interval to determine where the global maxima and minima occur.

**Example 5.8** Consider the function $y = f(x) = \frac{2}{x} + x^2 \quad 0.1 < x < 4$. Find the largest and smallest values that this function takes over the given interval. ■

**Solution:** We first compute the derivatives:

$$f'(x) = -2\frac{1}{x^2} + 2x,$$

$$f''(x) = 4\frac{1}{x^3} + 2.$$

We now determine where critical points $f'(x) = 0$ occur:

$$-2\frac{1}{x^2} + 2x = 0.$$

Simplifying, we find $-2\frac{1}{x^2} = 2x$, so $x^3 = 1$ and the critical point is at $x = 1$. Observe that the second derivative at this point is

$$f''(1) = 4\frac{1}{1^3} + 2 = 6 > 0,$$

so that $x = 1$ is a local minimum.

We now calculate the value of the function at the endpoints $x = 0.1$ and $x = 4$ as well as at the critical point $x = 1$ to determine where global and local minima and/or maxima occur:

| $f(0.1) = 20.01$ | $f(1) = 3$ | $f(4) = 16.5$ |
|---|---|---|
| global maximum | global minimum | |

We see that the global minimum occurs at $x = 1$. There are no local maxima. The global maximum occurs at the left endpoint.

## 5.8 For further study: Automatic landing system

We now discuss a practical example that uses many of the ideas so far. Our goal is to design an automatic landing system for an airplane.

We would like to find a polynomial of smallest degree that would describe the trajectory of an airplane as it makes its final approach to a landing site. We will assume that the landing trajectory starts when the plane is at a distance $x_0$ from the site where it touches down (at $x = 0$) and that it has been flying at a level (horizontal) direction up to that point. (See Figure 5.10.) We also want to make sure that the landing is as smooth as possible!

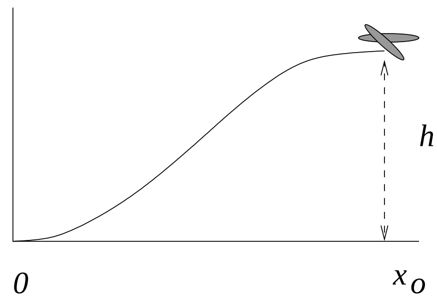



**Figure 5.10.** *The trajectory of a plane landing*

### 5.8.1 Solution:

Let $x$ stand for the distance of the plane from its touch down site at any point in its landing maneuver. A polynomial is a function of the form

$$y = p(x) = Ax^n + Bx^{n-1} + \dots.$$

"Finding a polynomial" is the same as determining the coefficients of the various powers. To do so, we will make following observations:

1. The plane it at height $h$ when it is at position $x_0$. Thus
$$p(x_0) = h.$$

2. The plane is moving horizontally before it starts to descend. This means that, precisely at the point $x_0$, at which the descent starts, the tangent line to the curve is horizontal. The derivative of $p(x)$ is zero at $x_0$:
$$p'(x_0) = 0.$$

3. The plane is supposed to be on the ground by the time it gets to $x = 0$. Thus
$$p(0) = 0.$$

4. If the plane lands at an angle to the ground, we'd have a bumpy and dangerous landing. The nose of the plane might get damaged too! To prevent this, we want the plane to be moving horizontally on the final part of its approach. Thus at $x = 0$ we want
$$p'(0) = 0.$$

The above observations have resulted in four conditions that the polynomial is to satisfy. The polynomial of smallest degree in which we can select four coefficients is:

$$p(x) = ax^3 + bx^2 + cx + d.$$

Below, we show how to solve for each of these coefficients in terms of the height $h$ and the location $x_0$ at which descent starts. (Note: if we use a polynomial of higher degree, our four conditions would not suffice to completely determine all the coefficients. Since we are asked to find the polynomial of least degree, we shall settle on this cubic polynomial.)

We will need to use the fact that the derivative of this polynomial is

$$p'(x) = 3ax^2 + 2bx + c.$$

From the above conditions we get:

1. $p(x_0) = h$. Thus
$$p(x_0) = ax_0^3 + bx_0^2 + cx_0 + d = h.$$

2. $p'(x_0) = 0$. Thus
$$p'(x_0) = 3ax_0^2 + 2bx_0 + c = 0.$$

3. $p(0) = 0$. Thus
$$p(0) = a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d = 0.$$
All the terms disappear but one, from which we conclude that $d = 0$.

4. $p'(0) = 0$. Thus
$$p'(0) = 3a \cdot 0^2 + 2b \cdot 0 + c = 0.$$
Again, all terms but one disappear, leaving $c = 0$.

Using the fact that $c, d = 0$, and returning to the first two equations, we are left with two equations and two unknowns:

$$ax_0^3 + bx_0^2 = h,$$

$$3ax_0^2 + 2bx_0 = 0$$

(Remember that here we are trying to find the values of the unknown coefficients $a, b$, and the quantities $x_0$ and $h$ are known fixed constants that represent the starting point of the plane as it initiates landing procedures.)

Solving for $a$ and $b$ we get, from the second equation:

$$3ax_0^2 = -2bx_0, \quad a = -\frac{2b}{3x_0}.$$

Plugging into the first we arrive at:

$$-\frac{2b}{3x_0}x_0^3 + bx_0^2 = h$$

$$-\frac{2b}{3}x_0^2 + bx_0^2 = h$$

$$bx_0^2\left(-\frac{2}{3} + 1\right) = h$$

so, after simplification, we get

$$b = \frac{3h}{x_0^2}, \quad a = -\frac{2b}{3x_0} = -\frac{2h}{x_0^3}.$$

Thus, we have arrived at the desired result:

$$p(x) = \left(-\frac{2h}{x_0^3}\right)x^3 + \left(\frac{3h}{x_0^2}\right)x^2.$$

(The expression looks cumbersome, but we remember that the terms in brackets are actually constants.)

To illustrate how this would work in a specific example, suppose that the plane starts its descent 20 km away from the airport at a height of 1 km. Then $x_0 = 20, h = 1$, so

$$b = \frac{3}{20^2} = 0.0075, \quad a = -\frac{2}{20^3} = -0.00025.$$

In this case, the polynomial that describes the landing trajectory would be:

$$p(x) = -0.00025x^3 + 0.0075x^2$$

## Exercises

5.1. A zero of a function is a place where $f(x) = 0$.

(a) Find the zeros, local maxima, and minima of the polynomial $y = f(x) = x^3 - 3x$

(b) Find the local minima and maxima of the polynomial $y = f(x) = (2/3)x^3 - 3x^2 + 4x$.

(c) A point of inflection is a point at which the second derivative changes sign. Determine whether each of the polynomials given in parts (a) and (b) have an inflection point.

5.2. Find the absolute maximum and minimum values on the given interval:

(a) $y = 2x^2$ on $-3 \le x \le 3$

(b) $y = (x-5)^2$ on $0 \le x \le 6$

(c) $y = x^2 - x - 6$ on $1 \le x \le 3$

(d) $y = \dfrac{1}{x} + x$ on $-4 \le x \le -\dfrac{1}{2}$.

5.3. Sketch the graph of $x^4 - x^2 + 1$ in the range $-3$ to $3$. Find its minimum value.

5.4. Identify all the critical points of the following function.
$y = x^3 - 27$

5.5. Consider the function $g(x) = x^4 - 2x^3 + x^2$. Determine locations of critical points and inflection points.

5.6. Consider the polynomial $y = x^3 + 3x^2 + ax + 1$. Show that when $a > 3$ this polynomial has no critical points.

5.7. Find the values of $a$, $b$, and $c$ if the parabola $y = ax^2 + bx + c$ is tangent to the line $y = -2x + 3$ at $(2, -1)$ and has a critical point when $x = 3$.

5.8. The position of a particle is given by the function $y = f(t) = t^3 + 3t^2$.

(a) Find the velocity and the acceleration of the particle.

(b) A second particle has position given by the function $y = g(t) = at^4 + t^3$ where $a$ is some constant and $a > 0$. At what time(s) are the particles in the same position?

(c) At what times do the particles have the same velocity?

(d) When do the particles have the same acceleration?

5.9. **Double Wells and Physics:** In physics, a function such as

$$f(x) = x^4 - 2x^2$$

is often called a *double well potential*. Physicists like to think of this as a "landscape" with hills and valleys. They imagine a ball rolling along such a landscape: with friction, the ball eventually comes to rest at the bottom of one of the valleys in this potential. Sketch a picture of this landscape and use information about the derivative of this function to predict where the ball might be found, i.e. where the valley bottoms are located.

5.10. A ball is thrown from a tower of height $h_0$. The height of the ball at time $t$ is

$$h(t) = h_0 + v_0 t - (1/2)gt^2$$

where $h_0, v_0, g$ are positive constants.

(a) When does the ball reach its highest point?

(b) How high is it at that point?

(c) What is the instantaneous velocity of the ball at its highest point ?

5.11. (From Final Exam, Math 100 Dec 1996) Find the first and second derivatives of the function

$$y = f(x) = \frac{x^3}{1-x^2}.$$

Use information about the derivatives to determine any local maxima and minima, regions where the curve is concave up or down, and any inflection points.

5.12. Find all the critical points of the function

$$y = f(x) = 2x^3 + 3ax^2 - 12a^2x + 1$$

and determine what kind of critical point each one is. Your answer should be given in terms of the constant $a$, and you may assume that $a > 0$.

5.13. (From Final Exam Dec 1995) The function $f(x)$ is given by

$$y = f(x) = x^5 - 10kx^4 + 25k^2x^3$$

where $k$ is a positive constant.

(a) Find all the intervals on which $f$ is either increasing or decreasing. Determine all local maxima and minima.

(b) Determine intervals on which the graph is either concave up or concave down. What are the inflection points of $f(x)$ ?

5.14. **Muscle shortening:** In 1938 Av Hill proposed a mathematical model for the rate of shortening of a muscle, $v$, (in cm/sec) when it is working against a load $p$ (in gms). His so called force-velocity curve is given by the relationship

$$(p + a)v = b(p_0 - p)$$

where $a$, $b$, $p_0$ are positive constants.

(a) Sketch the shortening velocity versus the load, i.e., $v$ as a function of $p$. (Note: the best way to do this is to find the intercepts of the two axes, i.e. find the value of $v$ corresponding to $p = 0$ and vice versa.)

(b) Find the rate of change of the shortening velocity with respect to the load, i.e. calculate $dv/dp$.

(c) What is the largest load for which the muscle will contract? (Hint: A contracting muscle has a positive shortening velocity, whereas a muscle with a very heavy load will stretch, rather than contract, i.e. will have a negative value of $v$.)

5.15. **Reaction kinetics:** Chemists often describe the rate of a saturating chemical reaction by using simplified expressions. Two examples of such expressions are:

Michaelis-Menten kinetics: $R_m(c) = \dfrac{Kc}{k_n + c}$, Sigmoidal kinetics: $R_s(c) = \dfrac{Kc^2}{k_n^2 + c^2}$

where $c$ is the concentration of the reactant, $K > 0$, $k_n > 0$ are constants. $R(c)$ is the speed of the reaction (Observe that the speed of the reaction depends on the concentration of the reactant).

(a) Sketch the two curves. To do this, you should analyze the behaviour for $c = 0$, for small $c$, and for very large $c$. You will find a horizontal asymptote in both cases. We refer to that asymptote as the "maximal reaction speed". What is the "maximal reaction speed" for each of the functions $R_m,\ R_s$ ? (Note: express your answer in terms of the constants $K$, $k_n$.)

(b) Show that the value $c = k_n$ leads to a half-maximal reaction speed.

For the questions below, you may assume that $K = 1$ and $k_n = 1$.

(c) Sketch the curves $R_m(c),\ R_s(c)$.

(d) Show that sigmoidal kinetics, but not Michaelis Menten kinetics has an inflection point.

(e) Explain how these curves would change if $K$ is increased; if $k_n$ is increased.

5.16. **Checking the endpoints !:** Find the absolute maximum and minimum values of the function

$$f(x) = x^2 + \frac{1}{x^2}$$

on the interval $[\frac{1}{2}, 2]$. Be sure to verify if any critical points are maxima or minima and to check the endpoints of the interval.

5.17. Find the first derivative for each of the following functions.

(a) $f(x) = (2x^2 - 3x)(6x + 5)$

(b) $f(x) = (x^3 + 1)(1 - 3x)$

(c) $g(x) = (x - 8)(x^2 + 1)(x + 2)$

(d) $f(x) = (x - 1)(x^2 + x + 1)$

(e) $f(x) = \dfrac{x^2 - 9}{x^2 + 9}$

(f) $f(x) = \dfrac{2 - x^3}{1 - 3x}$

(g) $f(b) = \dfrac{b^3}{2 - b^{\frac{2}{3}}}$

(h) $f(m) = \dfrac{m^2}{3m - 1} - (m - 2)(2m - 1)$

(i) $f(x) = \dfrac{(x^2 + 1)(x^2 - 2)}{3x + 2}$

# Chapter 6

# Optimization

In this chapter, we collect a variety of problems in which the ideas developed in earlier material are put to use. In particular, we will use calculus to find local (and global) maxima, and minima so as to get the best (optimal) values of some desirable quantity. Setting up these problems, from first verbal description, to clear cut mathematical formulation is the main challenge we will face. Often, we will use geometric ideas to express relationships between variables leading to our solution.

## 6.1 Density dependent (logistic) growth in a population

Biologists often notice that the growth rate of a population depends not only on the size of the population, but also on how crowded it is. When individuals have to compete for resources, nesting sites, mates, or food, they cannot reproduce as quickly, leading to a decline in the rate of growth of the population.

The rule that governs this growth, called **logistic growth** assumes that the growth rate $G$ depends on the density of the population $N$ as follows:

$$G(N) = rN\left(\frac{K-N}{K}\right)$$

where $r > 0$ is a constant, called the **intrinsic growth rate** and $K > 0$ is a constant called the **carrying capacity** of the environment for the population. We will soon see that the largest population that would grow at all is $N = K$. We show a sketch of this function in Figure 6.1. But how would we arrive at such a sketch?

**Example 6.1** Answer the following questions:

- Find the population density that leads to the maximal growth rate.
- What is the maximal growth rate?
- For what population size is the growth rate zero?

$\blacksquare$

**Solution:**

$$G(N) = rN\left(\frac{K-N}{K}\right) = rN - \frac{r}{K}N^2$$

To find the maximal growth rate we differentiate $G$ with respect to the variable $N$, remembering that $K, r$ are here treated as constants. We get

$$G'(N) = r - 2\frac{r}{K}N.$$

Setting $G'(N) = 0$ and solving for $N$ leads to

$$r = 2\frac{r}{K}N$$

so

$$N = \frac{K}{2}.$$

By taking a second derivative we find that

$$G''(N) = -2\frac{r}{K}$$

which is negative for all population sizes. This tells us that the function $G(N)$ is concave down, and that $N = K/2$ is a local maximum. Thus the density leading to largest growth rate is one half of the carrying capacity.

The growth rate at this density is

$$G(\frac{K}{2}) = r\left(\frac{K}{2}\right)\left(\frac{K-\frac{K}{2}}{K}\right) = r\frac{K}{2}\frac{1}{2} = \frac{rK}{4}.$$

To find the population size at which the growth rate is zero, we set $G = 0$ and solve for $N$:

$$G(N) = rN\left(\frac{K-N}{K}\right) = 0.$$

The two solutions are $N = 0$ (which is not very interesting, since when there is no population there is no growth) and $N = K$.

We will have many more things to say about this type of density dependent growth a little later on in this course.

## 6.2 Cell size and shape

Consider a spherical cell that is absorbing nutrients at a rate proportional to its surface area and consuming them at a rate proportional to its volume. Determine the size of the cell for which the net rate of increase of nutrients is largest.

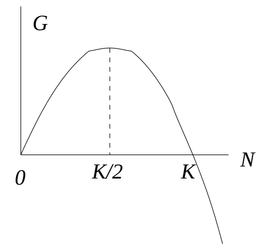



**Figure 6.1.** *The growth rate $G$ depends on population size $N$ as shown here for logistic growth.*

**Solution:**

We have seen in a previous chapter that the absorption ($A$ and consumption ($C$) rates for this simple spherical cell are:

$$A = k_1 S = 4k_1 \pi r^2,$$

$$C = k_2 V = \frac{4}{3}\pi k_2 r^3,$$

where $k_1, k_2 > 0$ are constants and $r$ is the radius of the cell.
The net rate of increase of nutrients is just the rate of absorption minus the rate of consumption, and it follows that it depends on the size of the cell:

$$N = A - C = 4k_1 \pi r^2 - \frac{4}{3}\pi k_2 r^3.$$

To find the size for greatest net nutrient increase rate, we find critical points of this function:

$$N'(r) = 8k_1 \pi r - 4k_2 \pi r^2.$$

Critical points occur when $N'(r) = 0$, i.e.

$$N'(r) = 8k_1 \pi r - 4k_2 \pi r^2 = 0.$$

Simplifying leads to

$$4\pi r(2k_1 - k_2 r) = 0.$$

This is satisfied (trivially) when $r = 0$, and also when

$$r = 2\frac{k_1}{k_2}.$$

We need to check that this is a local maximum. We obtain the second derivative

$$N''(r) = 8k_1 \pi - 8k_2 \pi r = 8\pi(k_1 - k_2 r).$$

plugging in $r = 2k_1/k_2$ we get

$$N'' = 8\pi(k_1 - k_2 \frac{2k_1}{k_2}) = -8\pi k_1 < 0.$$

This verifies that we have a local maximum.

## 6.3 A cylindrical cell

Not all cells are spherical. Some are skinny cylindrical filaments, or sausage shapes. Some even grow as helical tubes, but we shall leave such complicated examples aside here. We will explore how minimization of surface area would determine the overall shape of a cylindrical cell.

Consider a cell shaped like a cylinder with a circular cross-section. The volume of the cell will be assumed to be fixed, because the cytoplasm in its interior cannot be "compressed". However, suppose that the cell has a "rubbery" membrane that tends to take on the smallest surface area possible. (In physical language, the elastic energy stored in the membrane tends to a minimum.) We want to find the proportions of the cylinder (e.g. the ratio of length to radius) so that the cell has minimal surface area.

Recall the following properties for a cylinder:

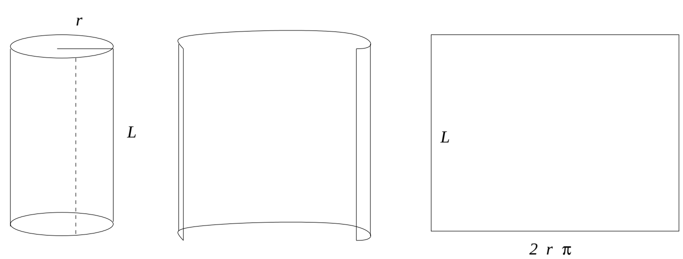



**Figure 6.2.** *Properties of a cylinder*

- the volume of a cylinder is the product of its base area $A$ and its height, $h$. That is, $V = Ah$. For a cylinder with circular cross-section: $V = \pi r^2 L$.
- A cylinder can be "cut and unrolled" into a rectangle. One side of the rectangle has length $L$ and the other has length that made the perimeter of the circle, $2\pi r$. The surface area of the unrolled rectangle is then $S_{\text{side}} = 2\pi r L$. See Figure 6.2
- If the "ends" of the cylinder are two flat circular caps, then the sum of the areas of these two ends is $S_{\text{ends}} = 2\pi r^2$.
- The total surface area of the cylinder with flat ends is then

$$S = 2\pi r L + 2\pi r^2.$$

We would expect that in a cell surrounded by a rubbery membrane, the end caps would not really be flat. However for simplicity, we will here neglect this issue and assume that the ends are flat and circular. Then, mathematically, our problem can be restated as follows

**Example 6.2** Minimize the surface area $S = 2\pi r L + 2\pi r^2$ given that the volume $V = \pi r^2 L = K$ is constant. ∎

**Solution:** The shape of the cell depends on both the length $L$, and the radius $r$ of the cylinder. However, these are not independent . They are related to one another because the volume of the cell has to be constant. This is an example of an optimization problem with *a constraint*, i.e. a condition that has to be satisfied. The constraint will allow us to eliminate one of the variables, as we show below.

The constraint is “the volume is fixed”, i.e.,

$$V = \pi r^2 L = K$$

where $K > 0$ is a constant that represents the volume of the given cell. We can use this to express one variable in terms of the other. For example, we can solve for $L$.

$$L = \frac{K}{\pi r^2}. \tag{6.1}$$

The function to minimize is

$$S = 2\pi r L + 2\pi r^2.$$

We eliminate $L$ by using the previous relationship, (6.1) to obtain $S$ as a function of $r$ alone:

$$S(r) = 2\pi r \frac{K}{\pi r^2} + 2\pi r^2$$

Simplification leads to

$$S(r) = 2\frac{K}{r} + 2\pi r^2.$$

observe that $S$ is now clearly a function of the single variable, $r$, ($K$ and $\pi$ are constants).

In order to find local minima, we will look for critical points of the function $S(r)$. We compute the relevant derivatives:

$$S'(r) = -2\frac{K}{r^2} + 4\pi r,$$

The second derivative will also be useful.

$$S''(r) = 4\frac{K}{r^3} + 4\pi.$$

From the last calculation, we observe that the second derivative is always positive since $K, r > 0$, so the function $S(r)$ is concave up. Any critical point we find thus will be a minimum automatically.

To find a critical point, set $S'(r) = 0$:

$$S'(r) = -2\frac{K}{r^2} + 4\pi r = 0.$$

Solving for $r$:

$$2\frac{K}{r^2} = 4\pi r, \quad r^3 = \frac{K}{2\pi}$$

so

$$r = \left(\frac{K}{2\pi}\right)^{1/3}.$$

We also find the length of this cell using Eqn. 6.1.

$$L = \left(\frac{4K}{\pi}\right)^{1/3}.$$

This comes about from a manipulation of powers:

$$L = \frac{K}{\pi r^2} = K\pi^{-1}\left(\frac{K}{2\pi}\right)^{-2/3} = K^{1-2/3}\pi^{-1+2/3}2^{2/3} = K^{1/3}\,\pi^{-1/3}\,4^{1/3}.$$

We can finally characterize the shape of the cell. One way to do this is to specify the ratio of its radius to its length. Based on our previous results, we can compute that ratio as follows:

$$\frac{L}{r} = \frac{(4K/\pi)^{1/3}}{(K/2\pi)^{1/3}} = 8^{1/3} = 2$$

Thus, the length of this cylinder is the same as its diameter (which is twice the radius). This means that in a cylindrical cell with a rubbery membrane, we find a short and fat shape. In order for the cell to grow as a long skinny cylinder, it has to have some structural support that prevents the surface area from contracting to the smallest possible area. An example of this type occurs in fungal cells. These grow as long branched filaments. The outer cell wall contains structural components that prevent the cell surface from contracting elastically.

*I would like to thank Prof Nima Geffen (Tel Aviv University) with providing the inspiration for this example.*

## 6.4 Geometric optimization

We consider several other examples of optimization where volumes, lengths, and/or surface areas are considered.

**Example 6.3 (Wrapping a rectangular box:)** A box with square base and arbitrary height has string tied around each of its perimeter. The total length of string so used is 10 inches. Find the dimensions of the box with largest surface area. (That is, figure out what is the largest amount of wrapping paper needed to wrap this box.) ■

**Solution:** The total length of string shown in Figure 6.3, consisting of three perimeters of the box is as follows:

$$L = 2(x + x) + 2(x + y) + 2(x + y) = 8x + 4y = 10$$

This total length is to be kept constant, so the above equation is the constraint in this problem. This means that $x$ and $y$ are related to one another. We will use this fact to eliminate one of them from the formula for surface area.

The surface area of the box is

$$S = 4(xy) + 2x^2$$

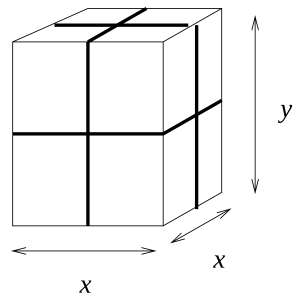



**Figure 6.3.** *A rectangular box is to be wrapped with paper*

since there are two faces (top and bottom) which are squares (area $x^2$) and four rectangular faces with area $xy$. At the moment, the total area is expressed in terms of both variables.

Suppose se eliminate $y$ by rewriting the constraint in the form:

$$y = \frac{5}{2} - 2x.$$

Then

$$S(x) = 4x\left(\frac{5}{2} - 2x\right) + 2x^2 = 10x - 8x^2 + 2x^2 = 10x - 6x^2.$$

We show the shape of this function in Figure 6.4. Note that $S(x) = 0$ at $x = 0$ and at $10 - 6x = 0$ which occurs at $x = 5/3$. Now that $S$ is expressed as a function of one

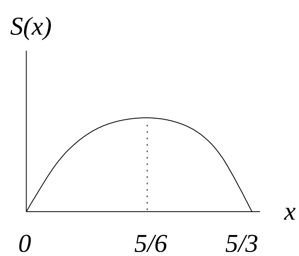



**Figure 6.4.** *Figure for Example 6.3.*

variable, we can find its critical points by setting $S'(x) = 0$, i.e., solving

$$S'(x) = 10 - 12x = 0$$

for $x$: We get $x = 10/12 = 5/6$. To find the corresponding value of $y$ we can substitute our result back into the constraint. We get

$$y = \frac{5}{2} - 2\left(\frac{5}{6}\right) = \frac{15 - 10}{6} = \frac{5}{6}.$$

Thus the dimensions of the box of interest are all the same, i.e. it is a cube with side length 5/6.

We can verify that

$$S''(x) = -12 < 0,$$

(indeed this holds for all $x$), which means that $x = 5/6$ is a local maximum.

Further, we can find that

$$S = 4\left(\frac{5}{6}\right)\left(\frac{5}{6}\right) + 2\left(\frac{5}{6}\right)^2 = \frac{25}{6}$$

square inches. Figure 6.4 shows how the surface area varies as the dimension $x$ of the box is varied.

## 6.5 Checking endpoints

In some cases, the optimal value of a function will not occur at any of its local maxima, but rather at one of the endpoints of an interval.

The following example illustrates this point:

**Example 6.4 (maximal perimeter)** The area of a rectangle having sides of length $x$ and $y$ is $A = xy$. Suppose that that the variable $x$ is only allowed to take values in the range $0.5 \leq x \leq 4$ Find the dimensions of the rectangle having largest perimeter whose area is fixed. (The perimeter of a rectangle is the total length of its outer edge.)

**Solution:** The perimeter of a rectangle whose sides are length $x, y$ is

$$P = x + y + x + y = 2x + 2y.$$

We are asked to maximize this quantity, where $xy = 1$ is our constraint.

Using the constraint, we can solve for $y$ and eliminate it:

$$y = \frac{1}{x}.$$

Then

$$P(x) = 2x + \frac{2}{x}.$$

To find critical points, we set

$$P'(x) = 2\left(1 - \frac{1}{x^2}\right) = 0.$$

Thus, $x^2 = 1$ or $x = \pm 1$. We reject the negative root as it is irrelevant for the (positive) side length of the rectangle. Checking if this is a maximum we find that

$$P''(x) = \frac{4}{x^3} > 0$$

so we have found a local *minimum*! This is clearly not the maximum we were looking for.

We must thus check the endpoints of the interval for the maximal value of the function. We find that $P(4) = 8.5$ and $P(0.5) = 5$. The largest perimeter for the rectangle will thus occur when $x = 4$, indeed at the endpoint of the domain, as shown in Figure 6.5.

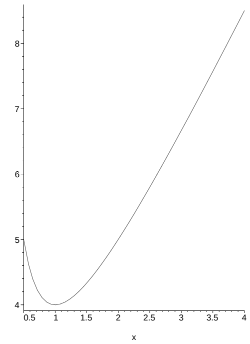



**Figure 6.5.** *In Example 6.4, the critical point we found is a local minimum. To maximize the perimeter of the rectangle, we must consider the end points of the interval* $0.5 \le x \le 4$.

## 6.6 Kepler's wedding

In 1613, Kepler set out to purchase a few barrels of wine for his wedding party. The merchant selling the wine had an interesting way of computing the cost of the wine: He would plunge a measuring rod through a hole in the barrel, as shown in Figure 6.6. The price was proportional to the length of the "wet" part of rod. We will refer to that length as $L$ in what follows.

Kepler noticed that barrels come in different shapes. Some are tall and skinny, while others are squat and fat. He conjectured that some shapes would contain larger volumes for a given length of the measuring rod, i.e. would contain more wine for the same price. Knowing mathematics, he set out to determine which barrel shape would be the best bargain for his wedding.

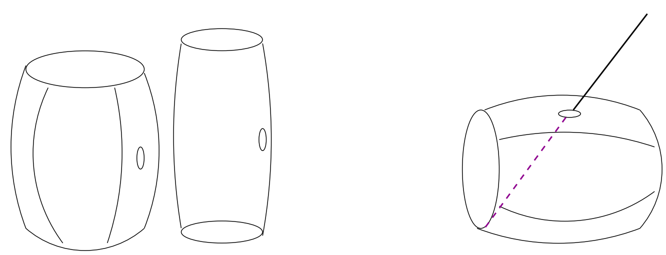


**Figure 6.6.** *Barrels come in various shapes. But the cost of a barrel of wine was determined by the length of the wet portion of the rod inserted into the barrel diagonally. Some barrels contain larger volume, but have identical cost.*

Suppose we ask what shape of barrel will contain the most wine for a given cost. This is equivalent to asking *which cylinder has the largest volume for a fixed (constant) length* $L$. Below, we show how this optimization problem can be solved.

### Solution

To simplify the problem, we will assume that the barrel is a simple cylinder, as shown in Figure 6.7. We also assume that the tap-hole (normally covered to avoid leaks) is half-way up the height of the barrel. We will define $r$ as the radius and $h$ as the height of the barrel. These two variables uniquely determine the shape as well as the volume of the barrel. We'll also assume that the barrel is full up to the top with delicious wine, so that the volume of the cylinder is the same as the volume of wine.

The volume of a cylinder is

$$V = \text{base area} \times \text{height}.$$

The base is a circle of area $A = \pi r^2$, so that the volume of the barrel is:

$$V = \pi r^2 h.$$

The rod used to "measure" the amount of wine (and hence determine the cost of the barrel) is shown as the diagonal of length $L$ in Figure 6.7. Because the cylinder walls are perpendicular to its base, the length $L$ is the hypotenuse of a right-angle triangle whose other sides have lengths $2r$ and $h/2$. (This follows from the assumption that the tap hole is half-way up the side.) Thus, by the Pythagorean theorem,

$$L^2 = (2r)^2 + \left(\frac{h}{2}\right)^2.$$

The problem can be restated: maximize $V$ subject to a fixed value of $L$. The fact that $L$ is fixed means that we have a constraint. That constraint will be used to reduce the number of variables in the problem.

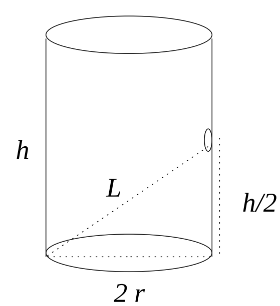



**Figure 6.7.** *Here we simplify and idealize the problem to consider a cylindrical barrel with diameter $2r$ and height $h$. We assumed that the tap-hole is at height $h/2$. The length $L$ denotes the "wet" portion of the merchant's rod, used to determine the cost of this barrel of wine. We observe that the dotted lines form a Pythagorian triangle.*

The function to be maximized is:

$$V = \pi r^2 h.$$

After expanding the squares, the constraint is:

$$L^2 = 4r^2 + \frac{h^2}{4}.$$

We can use the constraint to eliminate one variable; in this case the simplest way is to replace $r^2$ using:

$$r^2 = \frac{1}{4}\left(L^2 - \frac{h^2}{4}\right).$$

Then

$$V = \pi r^2 h = \frac{\pi}{4}\left(L^2 - \frac{h^2}{4}\right)h = \frac{\pi}{4}\left(L^2 h - \frac{1}{4}h^3\right).$$

We now have a function of one variable, namely

$$V(h) = \frac{\pi}{4}\left(L^2 h - \frac{1}{4}h^3\right).$$

For this function, the variable $h$ could sensibly take on any value in the range $0 \leq h \leq 2L$. Outside this range, the volume is negative, and at the two endpoints the volume is zero. Thus, we anticipate that somewhere inside this range of values we should find the desired optimum.

To find any critical points of the function $V(h)$, we calculate the derivative $V'(h)$ and set it to zero:

$$V'(h) = \frac{\pi}{4}\left(L^2 - \frac{3}{4}h^2\right) = 0$$

This implies that $L^2 - \frac{3}{4}h^2 = 0$, i.e.

$$3h^2 = 4L^2,$$

$$h^2 = 4\frac{L^2}{3},$$

$$h = 2\frac{L}{\sqrt{3}}.$$

Now we must check whether this solution is a local *maximum* (or a minimum).

The second derivative is:

$$V''(h) = \frac{\pi}{4}\left(0 - 2 \cdot \frac{3}{4}h\right) = -\frac{3}{8}\pi h < 0.$$

From this we see that $V''(h) < 0$ for any positive value of $h$. The the function $V(h)$ is concave down when $h > 0$. This verifies that the solution above is a local maximum. According to the discussion of the relevant range of values of $h$, this local maximum is also the optimal solution we need. i.e. there are no larger values at endpoints of the interval $0 \leq h \leq 2L$.

To finish the problem, we can find the radius of the barrel having this height by plugging this result for $h$ into the constraint equation, i.e. using

$$r^2 = \frac{1}{4}\left(L^2 - \frac{h^2}{4}\right) = \frac{1}{4}\left(L^2 - \frac{L^2}{3}\right) = \frac{1}{4}\left(\frac{2}{3}L^2\right).$$

After simplifying and rewriting, we get

$$r = \frac{1}{\sqrt{3}\sqrt{2}}L.$$

The shape of the wine barrel with largest volume for the given price can now be specified. One way to do this is to specify the ratio of height to radius. (Tall skinny barrels have a high ratio $h/r$ and squat fat ones have a low ratio.) By the above reasoning, the ratio of $h/r$ for the optimal barrel is

$$\frac{h}{r} = \frac{2\frac{L}{\sqrt{3}}}{\frac{1}{\sqrt{3}\sqrt{2}}L} = 2\sqrt{2}$$

The height of the barrel should be $2\sqrt{2} \approx 3$ times the radius in these most economical wine barrels.

## 6.7 Additional examples: A cylinder in a sphere

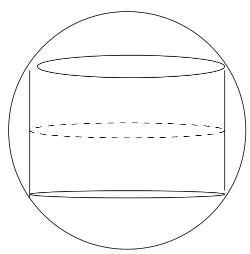


**Figure 6.8.** *The largest cylinder that fits inside a sphere of radius* $R$

**Example 6.5 (Fitting a cylinder inside a sphere)** Find the cylinder of maximal volume that would fit inside a sphere of radius $R$. See Figure 6.8. ■

**Solution:**

We label Figure 6.9 and define the following:

$$h = \text{height of cylinder,}$$
$$r = \text{radius of cylinder,}$$
$$R = \text{radius of sphere.}$$

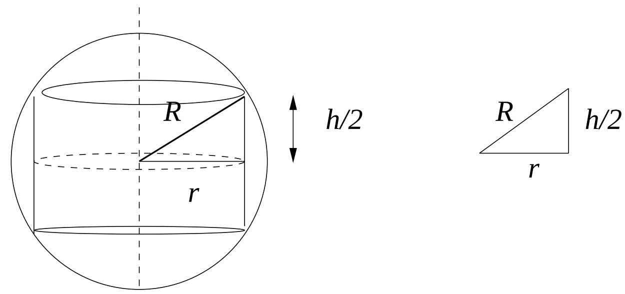



**Figure 6.9.** *Definition of variables and geometry to consider*

Then $R$ is assumed a given fixed positive constant, and $r$ and $h$ are dimensions of the cylinder to be determined.

From Figure 6.9 we see that the cylinder will fit if the top and bottom rims touch the circle. When this occurs, the dark line in Figure 6.9 will be a radius of the sphere, and so would have length $R$.

The connection between the variables (which will be our constraint) is given from Pythagoras' theorem by:

$$R^2 = r^2 + \left(\frac{h}{2}\right)^2.$$

We would like to maximize the volume of the cylinder,

$$V = \pi r^2 h$$

subject to the above constraint.

Eliminating $r^2$ leads to

$$V(h) = \pi(R^2 - \frac{h^2}{4})h.$$

We see that the problem is very similar to our previous discussion. The reader can show by working out the steps that

$$V'(h) = 0$$

occurs at the critical point

$$h = \frac{2}{\sqrt{3}}R$$

and that this is a local maximum.

## 6.8 For further study: Optimal foraging

### Biological background

Animals need to spend a considerable part of their time searching for food. There is a limited time available for this activity, since when the sun goes down, risk of becoming food (to a predator) increases, and chances of finding more food items decreases. There are also limited resources, so those who are most successful at finding and utilizing these over the available time will likely survive, produce offspring, and have an adaptive advantage. It is argued by biologists that evolution tends to optimize animal behaviour by selecting in favour of those that are faster, more efficient, stronger, or more fit. In this section we investigate how foraging behaviour is optimized.

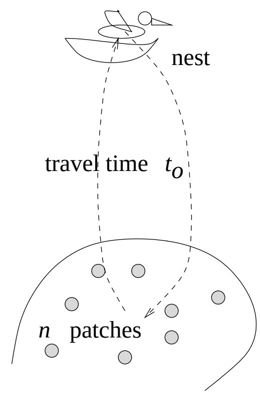



**Figure 6.10.** *A bird travels daily to forage in food patches. We want to determine how long it should stay in the patch to optimize its efficiency.*

We will assume that animals try to maximize the efficiency of collecting food. According to Charnov (1976), the efficiency of foraging is defined by the following ratio:

$$R(t) = \frac{\text{Total energy gained}}{\text{total time spent}}$$

i.e., $R$, is *e*nergy gain per unit time. This quantity will depend on the amount of time $t$ that is spent foraging during a day. The question we ask is whether there is an optimal foraging time (i.e. a value of the time, $t$), that maximizes $R(t)$. As we show below, whether or not an optimum exists depends greatly on how hard it is to extract food from a food patch. When an optimal foraging time exists, we will see that it also depends on how much time is wasted in transit to such foraging sites.

### Notation for our model

The following notation will be useful in discussing this problem:

- $t_0$ = travel time between nest and food patches. (This is considered as time that is unavoidably wasted.)
- $n$ = number of patches encountered on average per day,
- $t$ = residence time in patch (i.e. how long to spend foraging in one patch), also called foraging time,

- $f(t)$ = energy gained by foraging in a patch for time $t$,
- $R(t)$ = efficiency of foraging, i.e. total energy gained per unit time over the day.

### Energy gain in food patches

In some patches, it is easy to quickly load up on resources: this would be true if it is easy to find the nectar (or hunt the prey) or spot the berries. In other places, it may take some effort to locate the food items or process them so they can be eaten. This is reflected by a gain function $f(t)$, that may have one of several shapes. Some examples are shown in Figure 6.11.

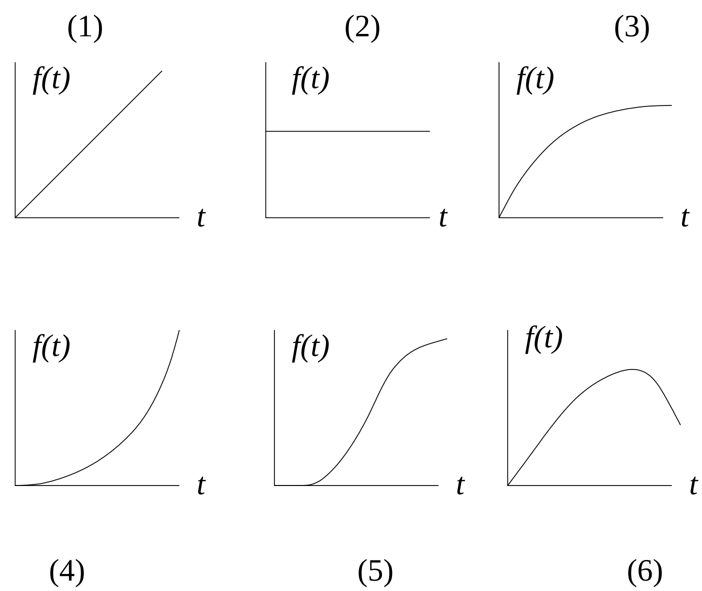



**Figure 6.11.** *Examples of various total energy gain $f(t)$ for a given foraging time $t$. The shapes of these functions determine how hard or easy it is to extract food from a food patch. See text for details about what these functions imply about the given food patch.*

In the examples shown in Figure 6.11 we see the following assortment of cases:

1. The energy gain is linearly proportional to time spent in the patch. In this case it appears that the patch has so much food in it that it is never depleted. It would make sense to stay in such a patch as long as possible, we might suspect.
2. Here the energy gain is independent of time spent. The animal gets the full quantity as soon as it gets to the patch. (This is not very realistic from a biological perspective.)
3. In this case, the food is gradually depleted in a given patch, (the total gain levels off to some constant level as $t$ increases). There is diminishing return for staying longer.

Here, we may expect to have some choice to make as to when to leave and look for food elsewhere.

4. In this example, the rewards for staying longer actually multiply: the net energy gain has an increasing slope (or, otherwise stated, $f''(t) > 0$). We will see that in this case, there is no optimal residence time: some other strategy, such as staying in just one patch would be optimal.

5. It takes some time to begin to gain energy but later on the gain increases rapidly. Eventually, the patch is depleted.

6. Here we have the case where staying too long in a patch is actually disadvantageous in that it leads to a net loss of energy. This might happen if the animal spends more energy looking for food that is already depleted. Here it is clear that leaving the patch early enough is the best strategy.

### The optimal residence time

We now turn to the task of finding the optimal **residence time**, i.e. time to spend in the patch. We will make a simplifying assumption that all the patches are identical, making it equally easy to utilize each one. Now suppose on average, the time spent in a patch is $t$. Then, the total energy gained during the day, after visiting $n$ patches is $nf(t)$. It takes a time $t_0$ to get from the nest to the food, and a time $t$ in each of $n$ patches to feed, so that the total time spent is $t_0 + nt$. Thus

$$R(t) = \frac{nf(t)}{t_0 + nt}.$$

We wish to maximize this function with respect to the residence time, i.e. find the time $t$ such that $R(t)$ is as large as possible.

Differentiating, we find the first derivative,

$$R'(t) = \frac{nf'(t)(t_0 + nt) - n^2 f(t)}{(t_0 + nt)^2} = \frac{G(t)}{H(t)}$$

(For our own convenience, we have defined two functions that represent the numerator and the denominator of $R'(t)$.)

$$G(t) = nf'(t)(t_0 + nt) - n^2 f(t),$$

$$H(t) = (t_0 + nt)^2.$$

We will find a later calculation easier with this notation.

To maximize $R(t)$ we set

$$R'(t) = 0$$

which can occur only when the numerator of the above equation is zero, i.e.

$$G(t) = 0.$$

This means that

$$nf'(t)(t_0 + nt) - n^2 f(t) = 0$$

so that, after simplifying algebraically,

$$f'(t) = \frac{nf(t)}{t_0 + nt},$$

$$f'(t) = \frac{f(t)}{(t_0/n) + t}. \tag{6.2}$$

### A geometric argument

In practice, we would need to specify a function for $f(t)$ in order to solve for the optimal time $t$. However, we can also solve this problem using a geometric argument. The last equation equates two quantities that can be interpreted as slopes. On the right is the slope of a tangent line, On the left is the slope (rise over run) of some right triangle whose height is $f(t)$ and whose base length is $(t_0/n) + t$. In Figure 6.12, we show each slope on its own: In the right panel, $f'(t)$ is the slope of the tangent line to the graph of $f(t)$. In the central panel, we have constructed some triangle with the property that its hypotenuse has slope $f(t)/[(t_0/n) + t]$. On the left panel we have superimposed both, selecting a value of $t$ for which the slope of the triangle is the same as the slope of the tangent line. Notice that in order to fit the triangle on the same diagram, we had to place its tip at the point $-(t_0/n)$ along the horizontal axis. When these slopes coincide, it means that we have satisfied equation (6.2), and we have found the desired time $t$ for optimal foraging.

We can use this observation in general to come up with the following steps to solve an optimal foraging problem:

1. A biologist conducts some field experiments to determine the mean number of patches that the organism can visit daily, $n$, the mean travel time from food to nest, $t_0$, and the shape of the energy gain function $f(t)$. (This may require capturing the animal and examining the contents of its stomach… an unappetizing thought; we will leave this to task to our brave biological colleagues.)

2. We draw a sketch of $f(t)$ as shown in rightmost panel of Figure 6.12 and extend the $t$ axis in the negative direction. At the point $-t_0/n$ we draw a line that just touches the curve $f(t)$ at some point (i.e. a tangent line). The slope of this line is $f'(t)$ for some value of $t$.

3. The value of $t$ at the point of tangency is the optimal time to spend in the patch!

The diagram drawn in our geometric solution (right panel in Figure 6.12 is often called a "rooted tangent").

We have shown that the point labeled $t$ indeed satisfies the condition that we derived above for $R'(t) = 0$, and hence is a critical point.

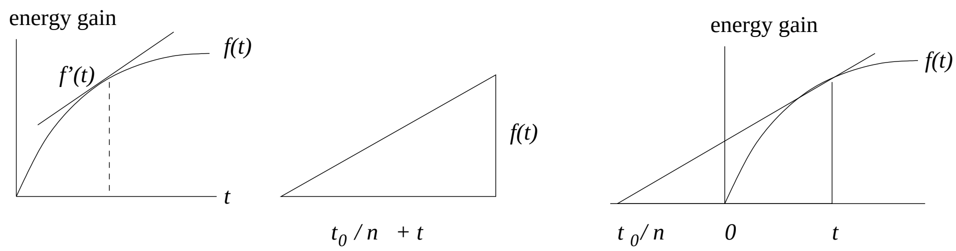



**Figure 6.12.** *The solution to the optimal foraging problem can be expressed geometrically in the form shown in this figure. The tangent line at the (optimal) time $t$ should have the same slope as the hypotenuse of the right triangle shown above. The diagram on the far right is sometimes termed the "rooted tangent" diagram.*

### Checking the type of critical point

We still need to show that this solution leads to a maximum efficiency, (rather than, say a minimum or some other critical point). We will do this by examining $R''(t)$.

Recall that

$$R'(t) = \frac{G(t)}{H(t)}$$

in terms of the notation used above. Then

$$R''(t) = \frac{G'(t)H(t) - G(t)H'(t)}{H^2(t)}.$$

But, according to our remark above, at the patch time of interest (the candidate for optimal time)

$$G(t) = 0$$

so that

$$R''(t) = \frac{G'(t)H(t)}{H^2(t)} = \frac{G'(t)}{H(t)}.$$

Now we substitute the derivative of $G'(t), H(t)$ into this ratio:

$$G(t) = nf'(t)(t_0 + nt) - n^2 f(t)$$

i.e.

$$G'(t) = nf``(t)(t_0 + nt) + n^2 f'(t) - n^2 f'(t) = nf''(t)(t_0 + nt)$$

We find that

$$R``(t) = \frac{nf''(t)(t_0 + nt)}{(t_0 + nt)^2} = \frac{nf''(t)}{(t_0 + nt)}.$$

The denominator of this expression is always positive, so that the sign of $R''(t)$ will be the same as the sign of $f''(t)$. But in order to have a maximum efficiency at some residence time, we need $R''(t) < 0$. This tells us that the gain function has to have the property that $f''(t) < 0$, i.e. has to be concave down at the optimal residence time.

Going back to some of the shapes of the function $f(t)$ that we discussed in our examples, we see that only some of these will lead to an optimal solution. In cases (1), (2), (4) the function $f(t)$ has *no* points of downwards concavity on its graph. This means that in such cases there will be no local maximum. The optimal efficiency would then be attained by spending as much time as possible in just one patch, or as little time as possible in any patch, i.e. it would be attained at the endpoints.

### 6.8.1 References:

1. Stephens DW, Krebs J R (1986) Foraging Theory, Princeton University Press, Princeton, NJ.

2. Charnov EL (1976) Optimal Foraging: the marginal value theorem; Theor. Pop. Biol. 9 : 129-136.

## Exercises

6.1. The sum of two positive number is $20$. Find the numbers

(a) if their product is a maximum.

(b) if the sum of their squares is a minimum.

(c) if the product of the square of one and the cube of the other is a maximum.

6.2. A tram ride at Disney World departs from its starting place at $t = 0$ and travels to the end of its route and back. Its distance from the terminal at time $t$ can be approximately described by the expression

$$S(t) = 4t^3(10 - t)$$

(where $t$ is in minutes, $0 < t < 10$, and $S$ is distance in meters.)

(a) Find the velocity as a function of time.

(b) When is the tram moving at the fastest rate?

(c) At what time does it get to the furthest point away from its starting position?

(d) Sketch the acceleration, the velocity, and the position of the tram on the same set of axes.

6.3. At $9A.M.$, car $B$ is $25$ km west of another car $A$. Car $A$ then travels to the south at $30$ km/h and car $B$ travels east at $40$ km/h. When will they be the closest to each other and what is this distance?

6.4. A cannonball is shot vertically upwards from the ground with initial velocity $v_0 = 15\text{m/sec}$. It is determined that the height of the ball, $y$ (in meters), as a function of the time, $t$ (in sec) is given by

$$y = v_0 t - 4.9t^2$$

Determine the following:

(a) The time at which the cannonball reaches its highest point,

(b) The velocity and acceleration of the cannonball at $t = 0.5$ s, and $t = 1.5$ s.

(c) The time at which the cannonball hits the ground.

6.5. (From Final Exam, Math 100, Dec 1997) A closed 3-dimensional box is to be constructed in such a way that its volume is 4500 cm$^3$. It is also specified that the length of the base is 3 times the width of the base. Find the dimensions of the box which satisfy these conditions and have the minimum possible surface area. Justify your answer.

6.6. A box with a square base is to be made so that its diagonal has length $1$. See Figure 6.13.

(a) What height $y$ would make the volume maximal?

(b) What is the maximal volume?
[Hint: A box having side lengths $\ell$, $w$, $h$ has diagonal length $D$ where $D^2 = \ell^2 + w^2 + h^2$ and volume $V = \ell wh$.]

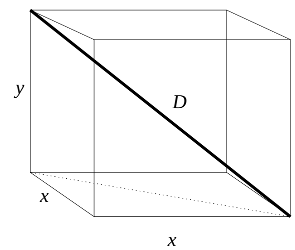



**Figure 6.13.** *Figure for Problem 6*

6.7. Find the minimum distance from a point on the positive $x$-axis $(a, 0)$ to the parabola $y^2 = 8x$.

6.8. **The largest garden:** You are building a fence to completely enclose part of your backyard for a vegetable garden. You have already purchased material for a fence of length 100 ft. What is the largest rectangular area that this fence can enclose?

6.9. **"Good Fences make Good Neighbors":** A fence of length 100 ft is to be used to enclose two gardens. One garden is to have a circular shape, and the other to be square. Find out how the fence should be cut so that the sum of the areas inside both gardens is as large as possible.

6.10. A rectangular piece of cardboard with dimension $12$ cm by $24$ cm is to be made into an open box (i.e., no lid) by cutting out squares from the corners and then turning up the sides. Find the size of the squares that should be cut out if the volume of the box is to be a maximum.

6.11. Find the shortest path that would take a milk-maid from her house at $(10, 10)$ to fetch water at the river located along the $x$ axis and then to the thirsty cow at $(3, 5)$.

6.12. **Water and ice:** Why does ice float on water? Because the density of ice is lower! In fact, water is the only common liquid whose maximal density occurs above its freezing temperature. (This phenomenon favors the survival of aquatic life by preventing ice from forming at the bottoms of lakes.) According to the *Handbook of Chemistry and Physics*, a mass of water that occupies one liter at $0°$C occupies a volume (in liters) of

$$V = -aT^3 + bT^2 - cT + 1$$

at $T°$C where $0 \le T \le 30$ and where the coefficients are

$$a = 6.79 \times 10^{-8}, \;\; b = 8.51 \times 10^{-6}, \;\; c = 6.42 \times 10^{-5}.$$

Find the temperature between $0°$C and $30°$C at which the density of water is the greatest. (Hint: maximizing the density is equivalent to minimizing the volume. Why is this?)

6.13. **Drug doses and sensitivity:** The *Reaction* $R(x)$ of a patient to a drug dose of size $x$ depends on the type of drug. For a certain drug, it was determined that a good description of the relationship is:

$$R(x) = Ax^2(B - x)$$

where $A$ and $B$ are positive constants. The *Sensitivity* of the patient's body to the drug is defined to be $R'(x)$.

(a) For what value of $x$ is the reaction a maximum, and what is that maximum reaction value?

(b) For what value of $x$ is the sensitivity a maximum? What is the maximum sensitivity?

6.14. **Thermoregulation in a swarm of bees::** In the winter, honeybees sometimes escape the hive and form a tight swarm in a tree, where, by shivering, they can produce heat and keep the swarm temperature elevated. Heat energy is lost through the surface of the swarm at a rate proportional to the surface area ($k_1 S$ where $k_1 > 0$ is a constant). Heat energy is produced inside the swarm at a rate proportional to the mass of the swarm (which you may take to be a constant times the volume). We will assume that the heat production is $k_2 V$ where $k_2 > 0$ is constant. Swarms that are not large enough may lose more heat than they can produce, and then they will die. The heat depletion rate is the loss rate minus the production rate. Assume that the swarm is spherical. Find the size of the swarm for which the rate of depletion of heat energy is greatest.

6.15. A right circular cone is circumscribed about a sphere of radius $5$. Find the dimension of this cone if its volume is to be a minimum. (Remark: this is a rather challenging geometric problem.)

6.16. **Optimal Reproductive Strategy:** Animals that can produce many healthy babies that survive to the next generation are at an evolutionary advantage over other, competing, species. However, too many young produce a heavy burden on the parents (who must feed and care for them). If this causes the parents to die, the advantage is lost. Also, competition of the young with one another for food and parental attention jeopardizes the survival of these babies. Suppose that the evolutionary **Advantage** $A$ to the parents of having litter size $x$ is

$$A(x) = ax - bx^2.$$

Suppose that the **Cost** $C$ to the parents of having litter size $x$ is

$$C(x) = mx + e.$$

The **Net Reproductive Gain** $G$ is defined as

$$G = A - C.$$

(a) Explain the expressions for $A, C$ and $G$.

(b) At what litter size is the advantage, $A$, greatest?

(c) At what litter size is there least cost to the parents?

(d) At what litter size is the Net Reproductive Gain greatest?.

6.17. **Behavioural Ecology::** Social animals that live in groups can spend less time scanning for predators than solitary individuals. However, they do waste time fighting

with the other group members over the available food. There is some group size at which the net benefit is greatest because the animals spend least time on these unproductive activities, and thus can spend time on feeding, mating, etc.
Assume that for a group of size $x$, the fraction of time spent scanning for predators is

$$S(x) = A\frac{1}{(x+1)}$$

and the fraction of time spent fighting with other animals over food is

$$F(x) = B(x+1)^2$$

where $A, B$ are constants. Find the size of the group for which the time wasted on scanning and fighting is smallest.

6.18. **Logistic growth:** Consider a fish population whose density (individuals per unit area) is $N$. The rate of growth $R$ of this population is found to satisfy

$$R(N) = rN(1 - N/K)$$

where $r$ and $K$ are positive constants. This type of growth rate is called *logistic* (or density dependent) growth.

(a) Sketch $R$ and a function of $N$.

(b) For what density of fish is the growth rate maximal?

6.19. **Logistic growth with harvesting:** Consider a fish population of density $N$ growing logistically, i.e. with rate of growth $R(N) = rN(1 - N/K)$ where $r$ and $K$ are positive constants. The rate of harvesting (i.e. removal) of the population is

$$h(N) = qEN$$

where $E$, the effort of the fishermen, and $q$, the catchability of this type of fish, are positive constants.
At what density of fish does the growth rate exactly balance the harvesting rate ? (This density is called the maximal sustainable yield: MSY.)

6.20. **Conservation of a harvested population:** Conservationists insist that the density of fish should never be allowed to go below a level at which growth rate of the fish exactly balances with the harvesting rate. (At this level, the harvesting is at its maximal sustainable yield. If more fish are taken, the population will keep dropping and the fish will eventually go extinct.) What level of fishing effort should be used to lead to the greatest harvest at this maximal sustainable yield? [Remark: you should first do the previous problem.]

# Chapter 7

# The Chain Rule, Related Rates, and Implicit Differentiation

## 7.1 Function composition

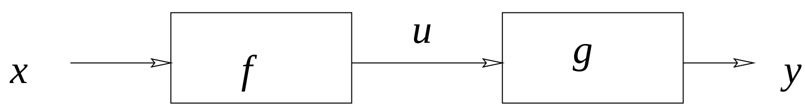



**Figure 7.1.** *Function composition*

Shown in the diagram above is an example of function composition: An independent variable, $x$, is used to evaluate a function, and the result, $u = f(x)$ then acts as an input to a second function, $g$. The result, $y = g(u) = g(f(x))$ can be related to the original variable, and we are interested in understanding how changes in that original variable affect the final outcome: That is, we want to know how $y$ changes when we change $x$. The chain rule will apply to this situation.

## 7.2 The chain rule

The **chain rule** of differentiation helps to calculate the result of this chain of effects. Basically, this rule states that the change in $y$ with respect to $x$ is a product of two rates of change: (1) the rate of change of $y$ with respect to its immediate input $u$, and (2) the rate of change of $u$ with respect to its input, $x$.

If $y = g(u)$ and $u = f(x)$ are both differentiable functions (meaning that their derivatives exist everywhere), and we consider the composite function $y = g(f(x))$ then the chain rule says that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

It is common to use the notation $d/dx$ as shown here when stating the chain rule, simply because this notation helps to remember the rule. Although the derivative is not merely a quotient, we can recall that it is arrived at from a quotient through a process of shrinking an interval. If we write

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}$$

then it is apparent that the "cancellation" of terms $\Delta u$ in numerator and denominator lead to the correct fraction on the left. The proof of the chain rule (optional) uses this essential idea, but care is taken to ensure that the quantity $\Delta u$ is nonzero, to avoid the embarrassment of dealing with the nonsensical ratio $0/0$.

The most important aspect of the chain rule to students of this course is an understanding of why it is needed, and how to use it in practical examples. The following intuitive examples may help to motivate why the chain rule is based on a product of two rates of change. Later in this chapter, we discuss examples of applications of this rule.

**Example 7.1 (Pollution level in a lake)** A species of fish is sensitive to pollutants in its lake. As humans settle and populate the area adjoining the lake, one may see a decline in the population of these fish due to increased levels of pollution. Quantify the rate at which the pollution level changes with time based on the pollution produced per human and the rate of increase of the human population. ∎

**Solution:** The rate of decline of the fish would depend on the rate of change in the human population around the lake, and the rate of change in the pollution created by each person. If either of these factors increases, one would expect an increase in the effect on the fish population and their possible extinction. The chain rule says that the net effect is a product of the two interdependent rates. To be more specific, we could think of time $t$ in years, $x = f(t)$ as the number of people living at the lake in year $t$, and $p = g(x)$ as the pollution created by $x$ people. Then the rate of change of the pollution $p$ over the years will be a product in the rate of change of pollution per human, and the rate of increase of humans over time:

$$\frac{dp}{dt} = \frac{dp}{dx}\frac{dx}{dt}$$

**Example 7.2 (Population of carnivores, prey, and vegetation)** The population of large carnivores, $C$, on the African Savannah depends on the population of gazelles that are prey, $P$. The population of these gazelles, in turn, depends on the abundance of vegetation $V$, and this depends on the amount of rain in a given year, $r$. Quantify the rate of change of the carnivore population with respect to the rainfall. ∎

**Solution:** We can express these dependencies through functions; for instance, we could write $V = g(r)$, $P = f(V)$ and $C = h(P)$, where we understand that $g, f, h$ are some functions (resulting from measurement or data collection on the savanna).

As one specific example, shown in Figure 7.2, consider the case that

$$C = h(P) = P^2, \quad P = f(V) = 2V, \quad V = g(r) = r^{1/2}.$$

If there is a drought, and the rainfall changes, then there will be a change in the vegetation. This will result in a change in the gazelle population, which will eventually

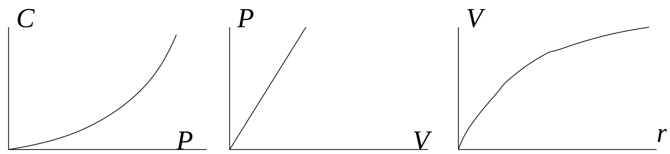



**Figure 7.2.** *An example in which the population of carnivores, $C = h(P) = P^2$ depends on prey $P$, while the prey depend on vegetation $P = f(V) = 2V$, and the vegetation depends on rainfall $V = g(r) = r^{1/2}$.*

affect the population of carnivores on the savanna. We would like to compute the rate of change in the carnivores population with respect to the rainfall, $dC/dr$.

According to the chain rule,

$$\frac{dC}{dr} = \frac{dC}{dP}\frac{dP}{dV}\frac{dV}{dr}.$$

The derivatives we need are

$$\frac{dV}{dr} = \frac{1}{2}r^{-1/2}, \quad \frac{dP}{dV} = 2, \quad \frac{dC}{dP} = 2P.$$

so that

$$\frac{dC}{dr} = \frac{dC}{dP}\frac{dP}{dV}\frac{dV}{dr} = \frac{1}{2}r^{-1/2}(2)(2P) = \frac{2P}{r^{1/2}}.$$

We can simplify this result by using the fact that $V = r^{1/2}$ and $P = 2V$. Plugging these in, we obtain

$$\frac{dC}{dr} = \frac{2P}{V} = \frac{2(2V)}{V} = 4.$$

This example is simple enough that we can also express the number of carnivores explicitly in terms of rainfall, by using the fact that $C = h(P) = h(f(V)) = h(f(g(r)))$. We can eliminate all the intermediate variables and express $P$ in terms of $r$ directly:

$$C = P^2 = (2V)^2 = 4V^2 = 4(r^{1/2})^2 = 4r.$$

(This may be much more cumbersome in more complicated examples.) We can compute the desired derivative in the simple old way, i.e.

$$\frac{dC}{dr} = 4.$$

We can see that our two answers agree.

**Example 7.3 (Budget for coffee)** The budget spent on coffee depends on the number of cups consumed per day and on the price per cup. The total budget might change if the price goes up or if the consumption goes up (e.g. during late nights preparing for midterm exams). Quantify the rate at which your budget for coffee would change if both consumption and price change. ■

**Solution:** The total rate of change of the coffee budget is a product of the change in the price and the change in the consumption. (In this example, we might think of time $t$ in days as the independent variable, $x = f(t)$ as the number of cups of coffee consumed on day $t$, and $y = g(x)$ as the price for $x$ cups of coffee.)

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

## 7.3 Applications of the chain rule to "related rates"

| | |
|---|---|
| Volume of sphere | $V = \frac{4}{3}\pi r^3$ |
| Surface area of sphere | $S = 4\pi r^2$ |
| Area of circle | $A = \pi r^2$ |
| Perimeter of circle | $P = 2\pi r$ |
| Volume of cylinder | $V = \pi r^2 h$ |
| Volume of cone | $V = \frac{1}{3}\pi r^2 h$ |
| Area of rectangle | $A = xy$ |
| Perimeter of rectangle | $P = 2x + 2y$ |
| Volume of box | $V = xyz$ |
| Sides of Pythagorean triangle | $c^2 = a^2 + b^2$ |

**Table 7.1.** *Common relationships on which problems about related rates are often based.*

In most of the applications given below, we are interested in processes that take place over time. We ask how the relationships between certain geometric (or physical) variables affects that rates at which they change over time. Many of these examples are given as word problems, and we are called on to assemble the required geometric or other relationships in solving the problem.

A few relationships that we will find useful are concentrated in Table 7.1 shown below

**Example 7.4 (Tumor growth:)** A tumor grows so that its radius expands at a constant rate, $k$. Determine the rate of growth of the volume of the tumor when the radius is one centimeter. Assume that the shape of the tumor is well approximated by a sphere. ∎

**Solution:** The volume of a sphere of radius $r$ is $V = (4/3)\pi r^3$. Here both $r$ and $V$ are changing with time so that

$$V(t) = \frac{4}{3}\pi[r(t)]^3.$$

Thus

$$\frac{d}{dt}V(t) = \frac{4}{3}\pi\frac{d}{dt}[r(t)]^3.$$

$$\frac{dV}{dt} = \frac{4}{3}\pi 3[r(t)]^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}.$$

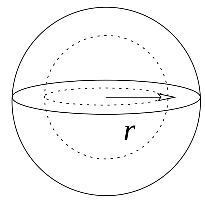



**Figure 7.3.** *Growth of a spherical tumor*

But we are told that the radius expands at a constant rate, $k$, so that

$$\frac{dr}{dt} = k.$$

Hence

$$\frac{dV}{dt} = 4\pi r^2 k.$$

We see that the rate of growth of the volume actually goes as the square of the radius. (Indeed a more astute observation is that the volume grows at a rate proportional to the surface area, since the quantity $4\pi r^2$ is precisely the surface area of the sphere.

In particular, for $r = 1$ cm we have

$$\frac{dV}{dt} = 4\pi k.$$

**Example 7.5 (A spider's thread:)** A spider moves horizontally across the ground at a constant rate, $k$, pulling a thin silk thread with it. One end of the thread is tethered to a vertical wall at height $h$ above ground and does not move. The other end moves with the spider. Determine the rate of elongation of the thread. ■

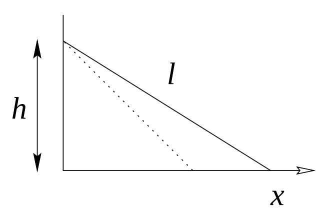



**Figure 7.4.** *The length of a spider's thread*

**Solution:** We use the Pythagorean Theorem to relate the height of the tether point $h$, the position of the spider $x$, and the length of the thread $\ell$:

$$\ell^2 = h^2 + x^2.$$

We note that $h$ is constant, and that $x, \ell$ are changing so that

$$[\ell(t)]^2 = h^2 + [x(t)]^2.$$

Differentiating with respect to $t$ leads to

$$\frac{d}{dt}\left([\ell(t)]^2\right) = \frac{d}{dt}\left(h^2 + [x(t)]^2\right)$$

$$2\ell\frac{d\ell}{dt} = 0 + 2x\frac{dx}{dt}.$$

Thus

$$\frac{d\ell}{dt} = \frac{2x}{2\ell}\frac{dx}{dt}.$$

Simplifying and using the fact that

$$\frac{dx}{dt} = k$$

leads to

$$\frac{d\ell}{dt} = \frac{x}{\ell}k = k\frac{x}{\sqrt{h^2+x^2}}.$$

**Example 7.6 (A conical cup:)** Water is leaking out of a conical cup of height $H$ and radius $R$. Find the rate of change of the height of water in the cup at the instant that the cup is full, if the volume is decreasing at a constant rate, $k$. ■

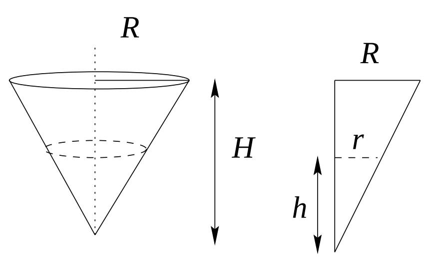



**Figure 7.5.** *The geometry of a conical cup*

**Solution:** Let us define $h$ and $r$ as the height and radius of water inside the cone. Then we know that the volume of this (conically shaped) water in the cone is

$$V = \frac{1}{3}\pi r^2 h,$$

or, in terms of functions of time,

$$V(t) = \frac{1}{3}\pi[r(t)]^2 h(t).$$

We are told that

$$\frac{dV}{dt} = -k,$$

where the negative sign indicates that volume is decreasing.

By similar triangles, we note that

$$\frac{r}{h} = \frac{R}{H}$$

so that we can substitute

$$r = \frac{R}{H}h$$

and get the volume in terms of the height alone:

$$V(t) = \frac{1}{3}\pi \left[\frac{R}{H}\right]^2 [h(t)]^3.$$

We can now use the chain rule to conclude that

$$\frac{dV}{dt} = \frac{1}{3}\pi \left[\frac{R}{H}\right]^2 3[h(t)]^2 \frac{dh}{dt}$$

Now using the fact that volume decreases at a constant rate, we get

$$-k = \pi \left[\frac{R}{H}\right]^2 [h(t)]^2 \frac{dh}{dt}$$

or

$$\frac{dh}{dt} = \frac{-kH^2}{\pi R^2 h^2}.$$

The rate computed above holds at any time as the water leaks out of the container. At the instant that the cup is full, we have $h(t) = H$ and $r(t) = R$, and then

$$\frac{dh}{dt} = \frac{-kH^2}{\pi R^2 H^2} = \frac{-k}{\pi R^2}.$$

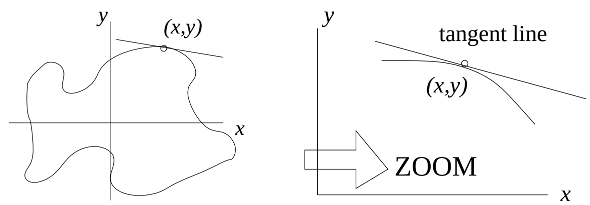
For example, for a cone of height $H = 4$ and radius $R = 3$,

$$\frac{dh}{dt} = \frac{-k}{9\pi}.$$

It is important to remember to plug in the information about the specific instant at the very end of the calculation, after the derivatives are computed.

## 7.4 Implicit differentiation

Often we would like to find the slope of a tangent line to a curve whose equation is not easily expressed in a form where $y$ is a function of $x$. In such cases, **implicit differentiation** is a useful tool to use.




**(a)** **(b)**

**Figure 7.6.** *The curve in (a) cannot be described by a single function (since there are values of $x$ that have more than one corresponding values of $y$). Hence it can only be described implicitly. However, if we zoom in to a point in (b), we can define the derivative as the slope of the tangent line to the curve at the point of interest.*

Shown in Figure 7.6 is a curve in the $xy$ plane. By inspection, we see that it is unlikely that the relationship of the variables can be expressed by a single formula in which $y$ is written explicitly as a function of $x$, such as $y = f(x)$. We have trouble doing so because this curve evidently is not a function: it does not satisfy the vertical line property. Nevertheless, we can reasonably ask what the slope of a tangent to the curve would be at some point along this curve, such as the one shown in the zoom. The slope will still be in the form $\Delta y/\Delta x$, and the slope of the tangent line will be $dy/dx$. We now show how to compute this slope in several examples where is is inconvenient, or impossible to isolate $y$ as a function of $x$.

**Example 7.7 (Tangent to a circle:)** In the first example, we find the slope of the tangent line to a circle. This example can be done in a number of different ways, but here we focus on the method of implicit differentiation.

(a) Find the slope of the tangent line to the point $x = 1/2$ in the first quadrant on a circle of radius 1 and center at the origin.

(b) Find the second derivative $d^2y/dx^2$ at the above point.

∎

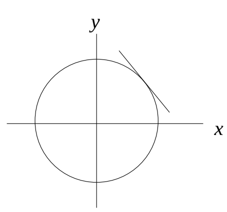



**Figure 7.7.** *Tangent line to a circle by implicit differentiation*

**Solution:**

(a) The equation of a circle with radius 1 and center at the origin is

$$x^2 + y^2 = 1.$$

When $x = 1/2$ we have $y = \pm\sqrt{1-(1/2)^2} = \pm\sqrt{1-(1/4)} = \pm\sqrt{3}/2$. However only one of these two values is in the first quadrant, i.e. $y = +\sqrt{3}/2$, so we are concerned with the behaviour close to this point.

In the original equation of the circle, we see that the two variables are linked in a symmetric relationship: although we could solve for $y$, we would not be able to express the relationship as a single function. Indeed, the top of the circle can be expressed as

$$y = f_1(x) = \sqrt{1-x^2}$$

and the bottom as

$$y = f_2(x) = -\sqrt{1-x^2}.$$

However, this makes the work of differentiation more complicated than it needs be.

Here is how we can handle the issue conveniently: We will think of $x$ as the independent variable and $y$ as the dependent variable. That is, we will think of the behaviour close to the point of interest as a small portion of the upper part of the circle, in which $y$ varies locally as $x$ varies. Then the equation of the circle would look like this:

$$x^2 + [y(x)]^2 = 1.$$

Now differentiate each side of the above with respect to $x$:

$$\frac{d}{dx}\left(x^2 + [y(x)]^2\right) = \frac{d1}{dx} = 0. \quad \Rightarrow \quad \left(\frac{dx^2}{dx} + \frac{d}{dx}[y(x)]^2\right) = 0.$$

We now apply the chain rule to the second term, and obtain

$$2x + 2[y(x)]\frac{dy}{dx} = 0$$

Thus

$$2y\frac{dy}{dx} = -2x \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

Here the slot of the tangent line to the circle is expressed as a ratio of the coordinates of the point of the circle. We could, in this case, simplify to

$$y'(x) = \frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}.$$

(This will not always be possible. In many cases we will not have an easy way to express $y$ as a function of $x$ in the final equation).

The point of interest is $x = 1/2$ (and here $y = \sqrt{3}/2$). Thus

$$y' = \frac{dy}{dx} = -\frac{1/2}{\sqrt{3}/2} = \frac{-1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}.$$

(b) The second derivative can be computed by differentiating

$$y' = \frac{dy}{dx} = -\frac{x}{y}$$

We use the quotient rule:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{x}{y}\right)$$

$$\frac{d^2y}{dx^2} = -\frac{1y - xy'}{y^2} = -\frac{y - x\frac{-x}{y}}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{1}{y^3}.$$

Substituting $y = \sqrt{3}/2$ from part (a) yields

$$\frac{d^2y}{dx^2} = -\frac{1}{(\sqrt{3}/2)^3} = -\frac{8}{3(3/2)}\ .$$

We have used the equation of the circle, and our previous result for the first derivative in simplifying the above. We can see from this last expression that the second derivative is negative for $y > 0$, i.e. for the top semi-circle, indicating that this part of the curve is concave down (as expected). Similarly, for $y < 0$, the second derivative is positive, and this agrees with the concave up property of that portion of the circle.

As in the case of simple functions, the second derivative can thus help identify concavity of curves.

## 7.5 The power rule for fractional powers

Implicit differentiation can help in determining the derivatives of a number of new functions. In this case, we use what we know about the integer powers to determine the derivative for a fractional power such as $1/2$. A similar idea will recur several times later on in this course, when we encounter a new type of function and its inverse function.

**Example 7.8 (Derivative of $\sqrt{x}$:)** Consider the function

$$y = \sqrt{x}$$

Use implicit differentiation to compute the derivative of this function. ∎

**Solution:** We can re-express this function in the form

$$y = x^{1/2}.$$

In this example, we will show that the power rule applies in the same way to fractional powers: That is, we show that

$$y'(x) = \frac{1}{2}x^{-1/2}$$

We rewrite the function $y = \sqrt{x}$ in the form

$$y^2 = x$$

but we will continue to think of $y$ as the dependent variable, i.e. when we differentiate, we will remember that

$$[y(x)]^2 = x$$

Taking derivatives of both sides leads to

$$\frac{d}{dx}\left([y(x)]^2\right) = \frac{d}{dx}(x)$$

$$2[y(x)]\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

We now use the original relationship to eliminate $y$, i.e. we substitute $y = \sqrt{x}$. We find that

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}.$$

This verifies the power law for the above example.

A similar procedure can be applied to a power function with fractional power. When we apply similar steps, we find that

**Derivative of fractional-power function:** The derivative of

$$y = f(x) = x^{m/n}$$

is

$$\frac{dy}{dx} = \frac{m}{n}x^{\left(\frac{m}{n}-1\right)}.$$

This is left as an exercise for the reader.

**Example 7.9** Compute the derivative of the function

$$y = f(x) = \sqrt{x^2 + a^2}, \quad \text{where } a \text{ is some positive real number}$$

$\blacksquare$

**Solution:** This function can be considered as the composition of $g(u) = \sqrt{u}$ and $u(x) = x^2 + a^2$, That is, we can write $f(x) = g(h(x))$ We rewrite $g$ in the form of a power function and then use the chain rule to compute the derivative. We obtain

$$\frac{dy}{dx} = \frac{1}{2} \cdot (x^2 + d^2)^{-1/2} \cdot 2x = \frac{x}{(x^2 + d^2)^{1/2}} = \frac{x}{\sqrt{x^2 + d^2}}$$

**Example 7.10** Compute the derivative of the function

$$y = f(x) = \frac{x}{\sqrt{x^2 + d^2}}, \quad \text{where } d \text{ is some positive real number}$$

■

**Solution:** We use both the quotient rule and the chain rule for this calculation.

$$\frac{dy}{dx} = \frac{[x]' \cdot \sqrt{x^2 + d^2} - [\sqrt{x^2 + d^2}]' \cdot x}{(\sqrt{x^2 + d^2})^2}$$

Here the $'$ denotes differentiation. Then

$$\frac{dy}{dx} = \frac{1 \cdot \sqrt{x^2 + d^2} - [\frac{1}{2} \cdot 2x \cdot (x^2 + d^2)^{-1/2}] \cdot x}{(x^2 + d^2)}$$

We simplify algebraically by multiplying top and bottom by $(x^2 + d^2)^{1/2}$ and canceling factors of 2 to obtain

$$\frac{dy}{dx} = \frac{x^2 + d^2 - x^2}{(x^2 + d^2)^{1/2}(x^2 + d^2)} = \frac{d^2}{(x^2 + d^2)^{3/2}}$$

**Example 7.11 (The astroid:)** The curve

$$x^{2/3} + y^{2/3} = 2^{2/3}$$

has the shape of an **astroid**. It describes the shape generated by a ball of radius $\frac{1}{2}$ rolling inside a ball of radius 2. Find the slope of the tangent line to a point on the astroid. ■

**Solution:** We use implicit differentiation as follows:

$$\frac{d}{dx}\left(x^{2/3} + y^{2/3}\right) = \frac{d}{dx} 2^{2/3}$$

$$\frac{2}{3} x^{-1/3} + \frac{d}{dy}(y^{2/3}) \frac{dy}{dx} = 0$$

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \frac{dy}{dx} = 0$$

$$x^{-1/3} + y^{-1/3} \frac{dy}{dx} = 0$$

We can rearrange into the form:

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}.$$

We can see from this form that the derivative fails to exist at both $x = 0$ (where $x^{-1/3}$ would be undefined) and at $y = 0$ (where $y^{-1/3}$ would be undefined. This stems from the sharp points that the curve has at these places.

In the next example we put the second derivative to work in an implicit differentiation problem. The goal is as follows:

**Example 7.12 (Horizontal tangent and concavity on a rotated ellipse:)** Find the highest point on the (rotated) ellipse

$$x^2 + 3y^2 - xy = 1$$

$\blacksquare$

**Solution:** The highest point on the ellipse will have a horizontal tangent line, so we should look for the point on this curve at which $dy/dx = 0$. We proceed as follows:

1. **Finding the slope of the tangent line:** By implicit differentiation,

$$\frac{d}{dx}[x^2 + 3y^2 - xy] = \frac{d}{dx}1$$

$$\frac{d(x^2)}{dx} + \frac{d(3y^2)}{dx} - \frac{d(xy)}{dx} = 0.$$

We must use the product rule to compute the derivative of the last term on the LHS:

$$2x + 6y\frac{dy}{dx} - x\frac{dy}{dx} - \frac{dx}{dx}y = 0$$

$$2x + 6y\frac{dy}{dx} - x\frac{dy}{dx} - 1y = 0$$

Grouping terms, we have

$$(6y - x)\frac{dy}{dx} + (2x - y) = 0$$

Thus

$$\frac{dy}{dx} = \frac{(y - 2x)}{(6y - x)}.$$

We can also use the notation

$$y'(x) = \frac{(y - 2x)}{(6y - x)}$$

to denote the derivative. Setting $dy/dx = 0$, we obtain $y - 2x = 0$ so that $y = 2x$ at the point of interest. However, we still need to find the coordinates of the point satisfying this condition.

2. **Determining the coordinates of the point we want:** To do so, we look for a point that satisfies the equation of the curve as well as the condition $y = 2x$. Plugging into the original equation of the ellipse, we get:

$$x^2 + 3y^2 - xy = 1$$

$$x^2 + 3(2x)^2 - x(2x) = 1.$$

After simplifying, this equation becomes $11x^2 = 1$, leading to the two possibilities

$$x = \pm\frac{1}{\sqrt{11}}, \quad y = \pm\frac{2}{\sqrt{11}}.$$

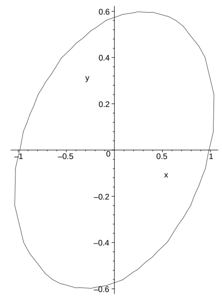



**Figure 7.8.** *A rotated ellipse*

We need to figure out which one of these two points is the top. (Evidently, the other point would also have a horizontal tangent, but would be at the "bottom" of the ellipse.)

3. **Finding which point is the one at the top:** The top point on the ellipse will be located at a portion of the curve that is concave down. We can determine the concavity close to the point of interest by using the second derivative, which we will compute (from the first derivative) using the quotient rule:

$$y''(x) = \frac{d^2y}{dx^2} = \frac{[y-2x]'(6y-x) - [6y-x]'(y-2x)}{(6y-x)^2}$$
$$y''(x) = \frac{[y'-2](6y-x) - [6y'-1](y-2x)}{(6y-x)^2}.$$

In the above, we have used the "prime" notation (') to denote a derivative.

4. **Plugging in information about the point:** Now that we have set down the form of this derivative, we make some important observations about the specific point of interest: (Note that this is done as a final step, only after all derivatives have been calculated!)

- We are only concerned with the sign of this derivative. The denominator is always positive (since it is squared) and so will not affect the sign. (It is possible to work with the sign of the numerator alone, though, in the interest of providing detailed steps, we go through the entire calculation below.)
- At the point of interest (top of ellipse) $y' = 0$, simplifying some of the terms above.
- At the point in question, $y = 2x$ so the term $(y - 2x) = 0$.

We can thus simplify the above to obtain

$$y''(x) = \frac{[-2](6y-x) - [-x](0)}{(6y-x)^2} = \frac{[-2](6y-x)}{(6y-x)^2} = \frac{-2}{(6y-x)}.$$

Using again the fact that $y = 2x$, we get the final form

$$y''(x) = \frac{-2}{(6(2x) - x)} = \frac{-2}{11x}.$$

We see directly from this result that the second derivative is negative (implying concave down curve) whenever $x$ is positive. This tells us that at the point with positive $x$ value, $x = 1/\sqrt{11}$, we are at the top of the ellipse. A graph of this curve is shown in Figure 7.8.

## 7.6 Food choice and attention

The example described in this section is taken from actual biological research. It has several noteworthy features: First, we do put the chain rule to use in the problem. Second, we encounter a surprise in some of our elementary calculations. Third, we find that not every problem has an elegant or analytically simple solution. Finally, we see that some very general observations can provide insight that we do not get as easily from the specific cases. The problem is taken from the study of animal behaviour.

### Paying attention

Behavioural ecologist Reuven Dukas (McMaster U) studies the choices that animals make when deciding which food to look for. His work has resulted in both theoretical and experimental conclusions about choices and strategies that animals follow. The example described below is based on his work with blue jays described in several publications. (See references.)

Many types of food are **cryptic**, i.e. hidden in the environment, and require time and attention to find. Some types of food are more easy to detect than other types, and some foods provide more nourishment than other types. Clearly, the animal that succeeds in gaining the greatest nourishment during a typical day will have a greater chance of surviving and out-competing others. Thus, it makes sense that animals should chose to divide their time and attention between food types in such a way as to maximize the total gain over the given time period available for foraging.

### Setting up a model

Suppose that there are two types of food available in the environment. We will define a variable that represents the attention that an animal can devote to finding a given food type.

- Let $x$ = attention devoted to finding food of some type. Assume that $0 < x < 1$, with $x = 0$ representing no attention at all to that type of food and $x = 1$ full attention devoted to finding that item.
- Let $P(x)$ denote the probability of finding the food given that attention $x$ is devoted to the task. Then $0 < P < 1$, as is commonly assumed for a probability. $P = 0$ means that the food is never found, and $P = 1$ means that the food is always found.
- Consider foods that have the property $P(0) = 0, P(1) = 1$. This means that if no attention is payed ($x = 0$) then there is no probability of finding the food ($P = 0$),

whereas if full attention is given to the task $x = 1$ then there is always success ($P = 1$).

- Suppose that there is more than one food type in the animal's environment. Then we will assume that the attentions paid to finding these foods, $x$ and $y$ sum up to 1: i.e. since attention is limited, $x + y = 1$, or, simply, $y = 1 - x$.

In figure 7.9, we show typical examples of the success versus attention curves for four different types of food labeled 1 through 4. On the horizontal axis, we show the attention $0 < x < 1$, and on the vertical axis, we show the probability of success at finding food, $0 < P < 1$. We observe that all the curves share in common the features we have described: Full success for full attention, and no success for no attention.

However the four curves shown here differ in their values at intermediate levels of attention.

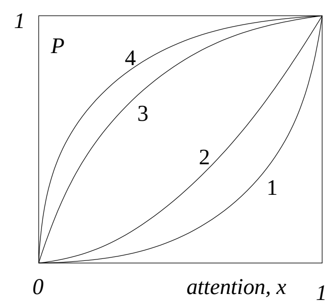



**Figure 7.9.** *The probability, $P$, of finding a food depends on the level of attention $x$ devoted to finding that food. We show possible curves for four types of foods, some easier to find than others.*

### Questions:

1. What is the difference between foods of type 1 and 4?
2. Which food is easier to find, type 3 or type 4?
3. What role is played by the concavity of the curve?

You will have observed that some curves, notably those concave down, such as curves 3 and 4 rise rapidly, indicating that the probability of finding food increases a lot just by increasing the attention by a little: These represent foods that are relatively easy to find. In other cases, where the function is concave up, (curves 1 and 2), we must devote much more attention to the task before we get an appreciable increase in the probability of success: these represent foods that are harder to find, or more cryptic. We now explore what happens when the attention is subdivided between several food types.

Suppose that two foods available in the environment can contribute relative levels of nutrition 1 and N per unit. We wish to determine for what subdivision of the attention, would the total nutritional value gained be as large as possible.

Suppose that $P_1(x)$ and $P_2(y)$ are probabilities of finding food of type 1 or 2 given that we spend attention $x$ or $y$ in looking for that type.

Let $x$ = the attention devoted to finding food of type 1. Then attention $y = 1 - x$ can be devoted to finding food of type 2.

Suppose that the relative nutritional values of the foods are 1 and $N$.

Then the total value gained by splitting up the attention between the two foods is:

$$V(x) = P_1(x) + N\ P_2(1-x).$$

**Example 7.13 ($P_1$ and $P_2$ as power function with integer powers:)** Consider the case that the probability of finding the food types is given by the simple power functions,

$$P_1(x) = x^2, \quad P_2(y) = y^3$$

Find the optimal food value $V(x)$ that can be attained. ■

**Solution:** We note that these functions satisfy $P(0) = 0, P(1) = 1$, in accordance with the sketches shown in Figure 7.9. Further, suppose that both foods are equally nutritious. Then $N = 1$, and the total value is

$$V(x) = P_1(x) + N\ P_2(1-x) = x^2 + (1-x)^3.$$

We look for a maximum value of $V$: Setting $V'(x) = 0$ we get (using the Chain Rule:)

$$V'(x) = 2x + 3(1-x)^2(-1) = 0.$$

We observe that a negative factor $(-1)$ comes from applying the chain rule to the factor $(1-x)^3$.

The above equation can be expanded into a simple quadratic equation:

$$-3x^2 + 8x - 3 = 0$$

whose solutions are

$$x = \frac{4 \pm \sqrt{7}}{3} \approx 0.4514, 2.21.$$

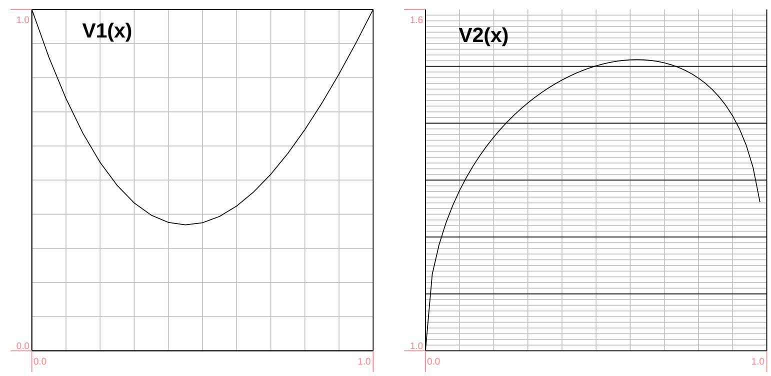
Since the attention must take on a value in $0 < x < 1$, we must reject the second of the two solutions. It would appear that the animal may benefit most by spending a fraction 0.4514 of its attention on food type 1 and the rest on type 2.

However, to confirm our speculation, we must check whether the critical point is a maximum. To do so, consider the second derivative,

$$V''(x) = \frac{d}{dx}\left(2x - 3(1-x)^2\right) = 2 - 3(2)(1-x)(-1) = 2 + 6(1-x).$$

(The factor $(-1)$ that appears in the computation is due to the Chain Rule applied to $(1-x)$ as before.)

Observing the result, and recalling that $x < 1$, we note that the second derivative is *positive* for all values of $x$! This is unfortunate, as it signifies a *local minimum*! The animal gains least by splitting up its attention between the foods in this case. Indeed, from Figure 7.10, we see that the most gain occurs at either $x = 0$ (only food of type 2 sought) or $x = 1$ (only food of type 1 sought). Again we observe the importance of checking for the type of critical point before drawing hasty conclusions.




**(a)** **(b)**

**Figure 7.10.** *(a) Figure for Example 7.13 and (b) for Example 7.14*

**Example 7.14 (Fractional-power functions for $P_1, P_2$:)** As a second example, consider the case that the probability of finding the food types is given by the concave down power functions,

$$P_1(x) = x^{1/2}, \quad P_2(y) = y^{1/3}$$

and both foods are equally nutritious ($N = 1$). Find the optimal food value $V(x)$ ■

**Solution:** These functions also satisfy $P(0) = 0, P(1) = 1$, in accordance with the sketches shown in Figure 7.9. Then

$$V(x) = P_1(x) + P_2(1-x) = \sqrt{x} + (1-x)^{(1/3)}$$

$$V'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3\,(1.0-x)^{(2/3)}}$$

$$V''(x) = -\frac{1}{4\,x^{(3/2)}} - \frac{2}{9\,(1.0-x)^{(5/3)}}$$

Calculations to actually determine the critical point are rather ugly, and best handled numerically. We state without details the fact that a critical point occurs at $x = 0.61977$ (and $y = 1-x = 0.38022$.) within the interval of interest. A plotting program is used to display the Value obtained by splitting up the attention in this way in Figure **??**. It is clear from this figure that a maximum occurs in the middle of the interval, i.e for attention split between finding both foods. We further see from $V''(x)$ that the second derivative is negative for all values of $x$ in the interval, indicating that we have obtained a local maximum, as expected.

### Epilogue

While the conclusions drawn above were disappointing in one specific case, it is not always true that concentrating all one's attention on one type is optimal. We can examine the problem in more generality to find when the opposite conclusion might be satisfied. In the general case, the value gained is

$$V(x) = P_1(x) + N\ P_2(1-x).$$

A critical point occurs when

$$V'(x) = \frac{d}{dx}[P_1(x) + N\ P_2(1-x)] = P_1'(x) + NP_2'(1-x)(-1) = 0$$

(By now you realize where the extra term $(-1)$ comes from - yes, from the Chain Rule!) Suppose we have found a value of $x$ in $0 < x < 1$ at which this is satisfied. We then examine the second derivative:

$$V''(x) = \frac{d}{dx}[V'(x)] = \frac{d}{dx}[P_1'(x) - NP_2'(1-x)] = P_1``(x) - NP_2''(1-x)(-1) = P_1``(x) + NP_2''(1-x).$$

The concavity of the function $V$ is thus related to the concavity of the two functions $P_1(x)$ and $P_2(1-x)$. If these are concave down (e.g. as in food types 3 or 4 in Figure 7.9), then $V''(x) < 0$ and a local maximum will occur at any critical point found by our differentiation.

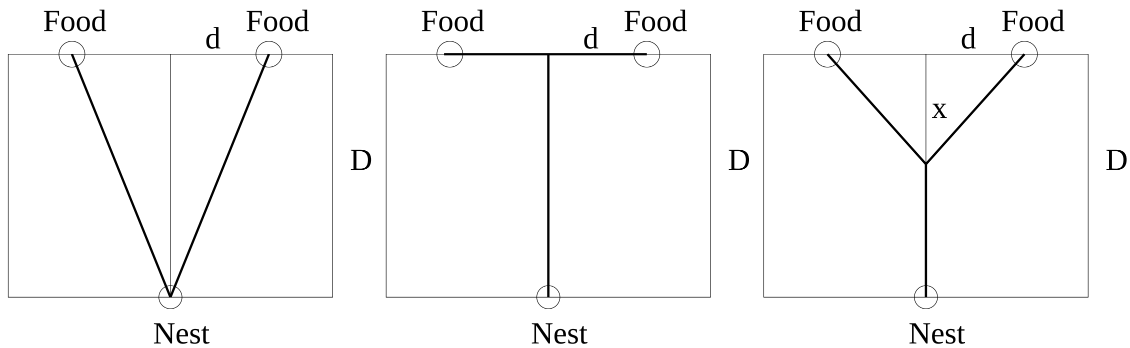
Another way of stating this observation is: if both food types are relatively easy to find, one can gain most benefit by splitting up the attention between the two. Otherwise, if both are hard to find, then it is best to look for only one at a time.

## 7.7 Shortest path from food to nest

Ants are good mathematicians! They are able to find the shortest route that connects their nest to a food source, to be as efficient as possible in bringing the food back home.

But how do they do it? It transpires that each ant secretes a chemical **pheromone** that other ants like to follow. This marks up the trail that they use, and recruits nestmates to food sources. The **pheromone** (chemical message for marking a route) evaporates after a while, so that, for a given number of foraging ants, a longer trail will have a less concentrated chemical marking than a shorter trail. This means that whenever a shorter route is found, the ants will favour it. After some time, this leads to selection of the shortest possible trail.

Shown in the figure below is a common laboratory test scenario, where ants at a nest are offered two equivalent food sources to utilize. We will use the chain rule and other results of this chapter to determine the shortest path that will emerge after the ants explore for some time.




**(a)** **(b)** **(c)**

**Figure 7.11.** *Three ways to connect the ants' nest to two food sources, showing (a) a V-shaped, (b) T-shaped, and (c) Y-shaped paths.*

**Example 7.15 (Minimizing the total path length for the ants)** Use the diagram to determine the length of the shortest path that connects the nest to both food sources. Assume that $d << D$.

**Solution:** We consider two possibilities before doing any calculus. The first is that the shortest path has the shape of the letter **T** whereas the second is that it has the shape of a letter **V**. Then for a T-shaped path, the total length is $D + 2d$ whereas for a V-shaped path it is $2\sqrt{D^2 + d^2}$. Now we consider a third possibility, namely that the path has the shape of the letter **Y**. This means that the ants start to walk straight ahead and then veer off to the food after a while.

It turns out to simplify our calculations if we label the distance from the nest to the Y-junction as $D - x$. Then $x$ is the remaining distance shown in the diagram. The length of the Y-shaped path is then given by

$$L = L(x) = (D - x) + 2\sqrt{d^2 + x^2} \tag{7.1}$$

Now we observe that when $x = 0$, then $L_T = D + 2d$, which corresponds exactly to the T-shaped path, whereas when $x = D$ then $L_V = 2\sqrt{d^2 + D^2}$ which is the length of the V-shaped path. Thus in this problem, we have $0 < x < D$ as the appropriate domain, and we have determined the values of $L$ at the two domain endpoints.

To find the minimal path length, we look for critical points of the function $L(x)$. Differentiating, we obtain (using results of Example 7.9)

$$L'(x) = \frac{dL}{dx} = -1 + 2\frac{x}{\sqrt{x^2 + d^2}}$$

Critical points occur at $L'(x) = 0$, which corresponds to

$$-1 + 2\frac{x}{\sqrt{x^2 + d^2}} = 0$$

We simplify this algebraically to obtain

$$\sqrt{x^2+d^2} = 2x \quad \Rightarrow \quad x^2+d^2 = 4x^2 \quad \Rightarrow \quad 3x^2 = d^2 \quad \Rightarrow \quad x = \frac{d}{\sqrt{3}}.$$

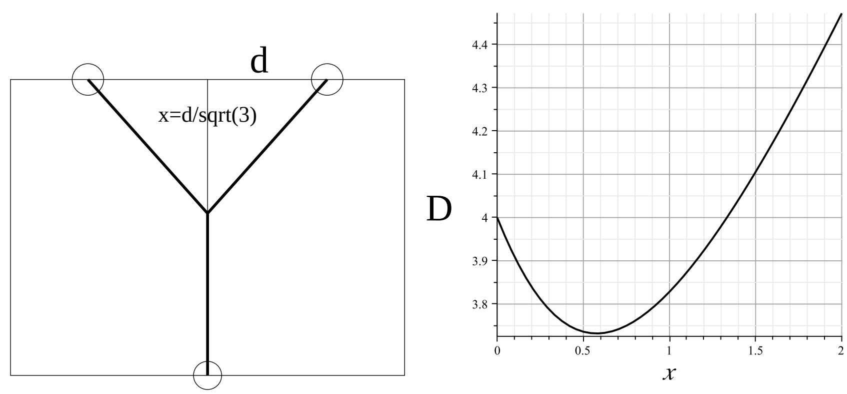
To determine the kind of critical point, we find the second derivative (See Example 7.10). Then

$$L''(x) = \frac{d^2L}{dx^2} = 2\frac{d^2}{(x^2+d^2)^{-3/2}} > 0$$

Thus the second derivative is positive and the critical point is a local maximum.

To determine the actual length of the path, we substitute $x = d/\sqrt{3}$ into the function $L(x)$ and obtain (after simplification, see Exercise 7.28)

$$L = L(x) = D + \sqrt{3}d$$




**(a)** **(b)**

**Figure 7.12.** *(a) In the configuration for the shortest path we found that $x = d/\sqrt{3}$. (b) The total length of the path $L(x)$ as a function of $x$ for $D = 2, d = 1$. The minimal path occurs when $x = 1/\sqrt{3} \approx 0.577$. The length of the shortest path is then $L = D + \sqrt{3}d = 2 + \sqrt{3} \approx 3.73$.*

## 7.8 Optional: Proof of the chain rule

First note that if a function is differentiable, then it is also continuous. This means that when $x$ changes a very little, $u$ can change only by a little. (There are no abrupt jumps). Then $\Delta x \to 0$ means that $\Delta u \to 0$.

Now consider the definition of the derivative $dy/du$:

$$\frac{dy}{du} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}$$

This means that for any (finite) $\Delta u$,

$$\frac{\Delta y}{\Delta u} = \frac{dy}{du} + \epsilon$$

where $\epsilon \to 0$ as $\Delta u \to 0$. Then

$$\Delta y = \frac{dy}{du}\Delta u + \epsilon \Delta u$$

Now divide both sides by some (nonzero) $\Delta x$: Then

$$\frac{\Delta y}{\Delta x} = \frac{dy}{du}\frac{\Delta u}{\Delta x} + \epsilon \frac{\Delta u}{\Delta x}$$

Taking $\Delta x \to 0$ we get $\Delta u \to 0$, (by continuity) and hence also $\epsilon \to 0$ so that as desired,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$


- Dukas R, Kamil A C(2001) Limited attention: The constraint underlying search image. Behavioral Ecology. 12(2): 192-199.
- Dukas R; Kamil A C (2000) The cost of limited attention in blue jays. Behavioral Ecology 11(5): 502-506.
- Dukas R, Ellner S, (1993) Information processing and prey detection. Ecology. 74(5): 1337-1346.

## Exercises

7.1. For each of the following, find the derivative of $y$ with respect to $x$.

(a) $y^6 + 3y - 2x - 7x^3 = 0$

(b) $e^y + 2xy = \sqrt{3}$

(c) $y = x^{\cos x}$

7.2. Consider the growth of a cell, assumed spherical in shape. Suppose that the radius of the cell increases at a constant rate per unit time. (Call the constant $k$, and assume that $k > 0$.)

(a) At what rate would the volume, $V$, increase ?

(b) At what rate would the surface area, $S$, increase ?

(c) At what rate would the ratio of surface area to volume $S/V$ change? Would this ratio increase or decrease as the cell grows? [Remark: note that the answers you give will be expressed in terms of the radius of the cell.]

7.3. **Growth of a circular fungal colony:** A fungal colony grows on a flat surface starting with a single spore. The shape of the colony edge is circular (with the initial site of the spore at the center of the circle.) Suppose the radius of the colony increases at a constant rate per unit time. (Call this constant $C$.)

(a) At what rate does the area covered by the colony change ?

(b) The biomass of the colony is proportional to the area it occupies (factor of proportionality $\alpha$). At what rate does the biomass increase?

7.4. **Limb development:** During early development, the limb of a fetus increases in size, but has a constant proportion. Suppose that the limb is roughly a circular cylinder with radius $r$ and length $l$ in proportion

$$l/r = C$$

where $C$ is a positive constant. It is noted that during the initial phase of growth, the radius increases at an approximately constant rate, i.e. that

$$dr/dt = a.$$

At what rate does the mass of the limb change during this time? [Note: assume that the density of the limb is 1 gm/cm$^3$ and recall that the volume of a cylinder is

$$V = Al$$

where $A$ is the base area (in this case of a circle) and $l$ is length.]

7.5. A rectangular trough is 2 meter long, $0.5$ meter across the top and 1 meter deep. At what rate must water be poured into the trough such that the depth of the water is increasing at $1\ \mathrm{m/min}$ when the depth of the water is $0.7\ \mathrm{m}$?

7.6. Gas is being pumped into a spherical balloon at the rate of $3\ cm^3/s$.

(a) How fast is the radius increasing when the radius is $15\ cm$?

(b) Without using the result from (a), find the rate at which the surface area of the balloon is increasing when the radius is $15\ cm$.

7.7. A point moves along the parabola $y = \frac{1}{4}x^2$ in such a way that at $x = 2$ the $x$-coordinate is increasing at the rate of $5$ cm/s. Find the rate of change of $y$ at this instant.

7.8. **Boyle's Law:** In chemistry, Boyle's law describes the behaviour of an ideal gas: This law relates the volume occupied by the gas to the temperature and the pressure as follows:

$$PV = nRT$$

where $n, R$ are positive constants.

(a) Suppose that the pressure is kept fixed, by allowing the gas to expand as the temperature is increased. Relate the rate of change of volume to the rate of change of temperature.

(b) Suppose that the temperature is held fixed and the pressure is decreased gradually. Relate the rate of change of the volume to the rate of change of pressure.

7.9. **Spread of a population:** In 1905 a Bohemian farmer accidentally allowed several muskrats to escape an enclosure. Their population grew and spread, occupying increasingly larger areas throughout Europe. In a classical paper in ecology, it was shown by the scientist Skellam (1951) that the square root of the occupied area increased at a constant rate, $k$. Determine the rate of change of the distance (from the site of release) that the muskrats had spread. For simplicity, you may assume that the expanding area of occupation is circular.

7.10. A spherical piece of ice melts so that its surface area decreases at a rate of 1 cm$^2$/min. Find the rate that the diameter decreases when the diameter is 5 cm.

7.11. **A Convex lens:** A particular convex lens has a focal length of $f = 10$ cm. The distance $p$ between an object and the lens, the distance $q$ between its image and the lens and the focal length $f$ are related by the equation:

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}.$$

If an object is 30 cm away from the lens and moving away at 4 cm/sec, how fast is its image moving and in which direction?

7.12. **A conical cup:** Water is leaking out of a small hole at the tip of a conical paper cup at the rate of 1 cm$^3$/min. The cup has height 8 cm and radius 6 cm, and is initially full up to the top. Find the rate of change of the height of water in the cup when the cup just begins to leak. [Remark: the volume of a cone is $V = (\pi/3)r^2h$.]

7.13. **Conical tank:** Water is leaking out of the bottom of an inverted conical tank at the rate of $\frac{1}{10}\ m^3/min$, and at the same time is being pumped in the top at a constant rate of $k\ m^3/min$. The tank has height $6\ m$ and the radius at the top is $2\ m$. Determine the constant $k$ if the water level is rising at the rate of $\frac{1}{5}\ m/min$ when the height of the water is $2\ m$. Recall that the volume of a cone of radius $r$ and height $h$ is

$$V = \frac{1}{3}\pi r^2 h.$$

7.14. **The gravel pile:** Gravel is being dumped from a conveyor belt at the rate of $30\ ft^3/min$ in such a way that the gravel forms a conical pile whose base diameter and height are always equal. How fast is the height of the pile increasing when the height is $10\ ft$? (Hint: the volume of a cone of radius $r$ and height $h$ is $V = \frac{1}{3}\pi r^2 h$.)

7.15. **The sand pile:** Sand is piled onto a conical pile at the rate of $10 m^3/min$. The sand keeps spilling to the base of the cone so that the shape always has the same proportions: that is, the height of the cone is equal to the radius of the base. Find the rate at which the height of the sandpile increases when the height is 5 m. Note: The volume of a cone with height $h$ and radius $r$ is

$$V = \frac{\pi}{3} r^2 h.$$

7.16. Water is flowing into a conical reservoir at a rate of $4\ \mathrm{m}^3/\mathrm{min}$. The reservoir is 3 m in radius and 12 m deep.

(a) How fast is the radius of the water surface increasing when the depth of the water is $8$ m?

(b) In (a), how fast is the surface rising?

7.17. A ladder $10$ meters long leans against a vertical wall. The foot of the ladder starts to slide away from the wall at a rate of 3 m/s.

(a) Find the rate at which the top of the ladder is moving downward when its foot is $8$ meters away from the wall.

(b) In (a), find the rate of change of the slope of the ladder.

7.18. **Sliding ladder:** A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at the rate of 0.5 meter/min how fast is the top of the ladder sliding down the wall when the base of the ladder is 1 m away from the wall ?

7.19. Ecologists are often interested in the relationship between the area of a region ($A$) and the number of different species $S$ that can inhabit that region. Hopkins (1955) suggested a relationship of the form

$$S = a \ln(1 + bA)$$

where $a$ and $b$ are positive constants. Find the rate of change of the number of species with respect to the area. Does this function have a maximum?

7.20. **The burning candle:** A candle is placed a distance $l_1$ from a thin block of wood of height $H$. The block is a distance $l_2$ from a wall as shown in Figure 7.13. The candle burns down so that the height of the flame, $h_1$ decreases at the rate of 3 cm/hr. Find the rate at which the length of the shadow $y$ cast by the block on the wall increases. (Note: your answer will be in terms of the constants $l_1$ and $l_2$. Remark: This is a challenging problem.)

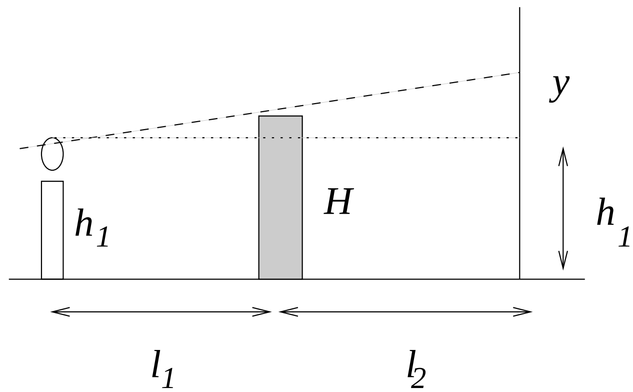



**Figure 7.13.** *Figure for Problem 20*

7.21. Use implicit differentiation to show that the derivative of the function

$$y = x^{1/3}$$

is

$$y' = (1/3)x^{-2/3}.$$

First write the relationship in the form $y^3 = x$, and then find $dy/dx$.

7.22. **Generalizing the Power Law:**

(a) Use implicit differentiation to calculate the derivative of the function

$$y = f(x) = x^{n/m}$$

where $m$ and $n$ are integers. (Hint: rewrite the equation in the form $y^m = x^n$ first.)

(b) Use your result to derive the formulas for the derivatives of the functions $y = \sqrt{x}$ and $y = x^{-1/3}$.

7.23. The equation of a circle with radius $r$ and center at the origin is

$$x^2 + y^2 = r^2$$

(a) Use implicit differentiation to find the slope of a tangent line to the circle at some point $(x, y)$.

(b) Use this result to find the equations of the tangent lines of the circle at the points whose $x$ coordinate is $x = r/\sqrt{3}$.

(c) Use the same result to show that the tangent line at any point on the circle is perpendicular to the radial line drawn from that point to the center of the circle

**Note:** Two lines are perpendicular if their slopes are negative reciprocals.

7.24. The equation of a circle with radius $5$ and center at $(1,1)$ is

$$(x-1)^2+(y-1)^2=25$$

(a) Find the slope of the tangent line to this curve at the point $(4,5)$.

(b) Find the equation of the tangent line.

7.25. **Tangent to a hyperbola:** The curve

$$x^2-y^2=1$$

is a hyperbola. Use implicit differentiation to show that for large $x$ and $y$ values, the slope $dy/dx$ of the curve is approximately $1$.

7.26. **An ellipse:** Use implicit differentiation to find the points on the ellipse

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

at which the slope is -1/2.

7.27. **Motion of a cell:** In the study of cell motility, biologists often investigate a type of cell called a keratocyte, an epidermal cell that is found in the scales of fish. This flat, elliptical cell crawls on a flat surface, and is known to be important in healing wounds. The 2D outline of the cell can be approximated by the ellipse

$$x^2/100+y^2/25=1$$

where $x$ and $y$ are distances in $\mu$ (Note: 1 micron is $10^{-6}$ meters). When the motion of the cell is filmed, it is seen that points on the "leading edge" (top arc of the ellipse) move in a direction perpendicular to the edge. Determine the direction of motion of the point $(x_p, y_p)$ on the leading edge.

7.28. **Shortest path from nest to food sources:**

(a) Use the first derivative test to verify that the value $x=\frac{d}{\sqrt{3}}$ is a local minimum of the function $L(x)$ given by Eqn (7.1)

(b) Show that the shortest path is $L=D+\sqrt{3}d$.

(c) In Section 7.7 we assumed that $d<<D$, so that the food sources were close together relative to the distance from the nest. Now suppose that $D=d/2$. How would this change the solution to the problem?

7.29. **Geometry of the shortest ants' path:** Use the results of Section 7.7 to show that in the shortest path, the angles between the branches of the Y-shaped path are all 120°, You may find it helpful to recall that $\sin(30)=1/2, \sin(60)=\sqrt{3}/2$.

7.30. **The Folium of Descartes:** A famous curve (see Figure 7.14) that was studied historically by many mathematicians (including Descartes) is

$$x^3+y^3=3axy$$

You may assume that $a$ is a positive constant.

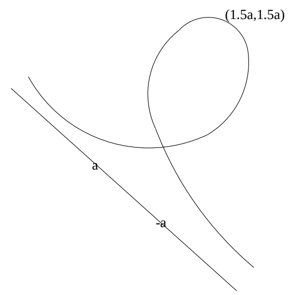



**Figure 7.14.** *The Folium of Descartes in Problem 30*

(a) Explain why this curve cannot be described by a function such as $y = f(x)$ over the domain $-\infty < x < \infty$.

(b) Use implicit differentiation to find the slope of this curve at a point $(x, y)$.

(c) Determine whether the curve has a horizontal tangent line anywhere, and if so, find the $x$ coordinate of the points at which this occurs.

(d) Does implicit differentiation allow you to find the slope of this curve at the point (0,0) ?

7.31. **Isotherms in the Van-der Waal's equation:** In thermodynamics, the Van der Waal's equation relates the mean pressure, $p$ of a substance to its molar volume $v$ at some temperature $T$ as follows:

$$(p + \frac{a}{v^2})(v - b) = RT$$

where $a, b, R$ are constants. Chemists are interested in the curves described by this equation when the temperature is held fixed. (These curves are called isotherms).

(a) Find the slope, $dp/dv$, of the isotherms at a given point $(v, p)$.

(b) Determine where points occur on the isotherms at which the slope is horizontal.

7.32. **The circle and parabola:** A circle of radius 1 is made to fit inside the parabola $y = x^2$ as shown in figure 7.15. Find the coordinates of the center of this circle, i.e. find the value of the unknown constant $c$. [Hint: Set up conditions on the points of intersection of the circle and the parabola which are labeled $(a, b)$ in the figure. What must be true about the tangent lines at these points?]

7.33. Consider the curve whose equation is

$$x^3 + y^3 + 2xy = 4, \ y = 1 \text{ when } x = 1.$$

(a) Find the equation of the tangent line to the curve when $x = 1$.

(b) Find $y''$ at $x = 1$.

(c) Is the graph of $y = f(x)$ concave up or concave down near $x = 1$?
Hint: Differentiate the equation $x^3 + y^3 + 2xy = 4$ twice with respect to $x$.

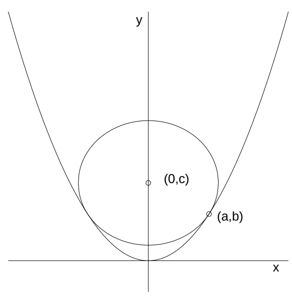



**Figure 7.15.** *Figure for Problem 32*

## Chapter 8

# Exponential functions

## 8.1 The Andromeda Strain

*"The mathematics of uncontrolled growth are frightening. A single cell of the bacterium E. coli would, under ideal circumstances, divide every twenty minutes. That is not particularly disturbing until you think about it, but the fact is that bacteria multiply geometrically: one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth."*

Michael Crichton (1969) The Andromeda Strain, Dell, N.Y. p247

## 8.2 Powers of 2

Note that $2^{10} \approx 1000 = 10^3$. This is a useful approximation in converting binary numbers (powers of 2) to decimal numbers (powers of 10).

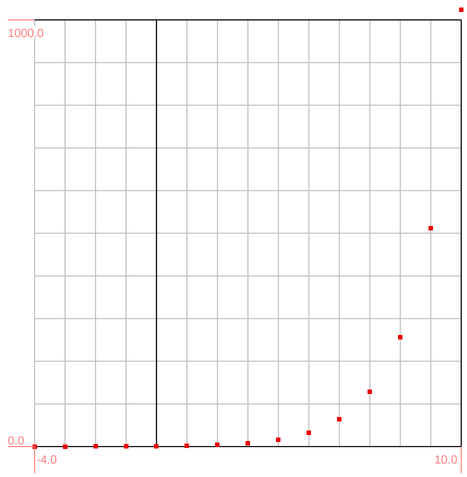
## 8.3 Growth of E. coli

- Mass of 1 E. coli cell : 1 nanogram = $10^{-9}$gm = $10^{-12}$kg.
- Mass of Planet Earth : $6 \cdot 10^{24}$ kg
- Size of E. coli colony equal in mass to Planet Earth:

$$m = \frac{6 \cdot 10^{24}}{10^{-12}} = 6 \cdot 10^{36}$$

In a period of 24 hours, there are many 20-minute generations. To be exact, there are $24 \times 3 = 72$ generations, with each one producing a doubling. This means that there would be, after 1 day, a number of cells equal to

$$2^{72}.$$




| $n$ | $2^n$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |
| 6 | 64 |
| 7 | 128 |
| 8 | 256 |
| 9 | 512 |
| 10 | 1024 |

**Table 8.1.** *Powers of 2 including both negative and positive integers: Here we show* $2^n$ *for* $-4 < n < 10$. *Note that* $2^{10} \approx 1000 = 10^3$.

We can estimate it using the approximate decimal form as follows:

$$2^{72} = 2^2 \cdot 2^{70} = 4 \cdot (2^{10})^7 \approx 4 \cdot (10^3)^7 = 4 \cdot 10^{21}.$$

The actual value is found to be $4.7 \cdot 10^{21}$, so the approximation is relatively good.

Apparently, the estimate made by Crichton is not quite accurate. However it can be shown that it takes less than 2 days to produce a number far in excess of the desired size. (The exact number of generations is left as an exercise for the reader.. but we will return to this in due time.)

## 8.4 The function $2^x$

From previous familiarity with power functions such as $y = x^2$ (not to be confused with $2^x$), we know the value of

$$2^{1/2} = \sqrt{2} \approx 1.41421\ldots$$

We can use this value to compute

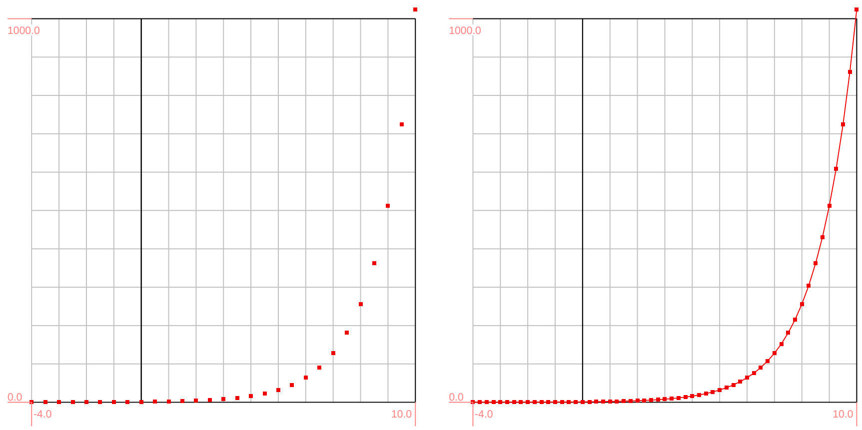
$$2^{3/2} = (\sqrt{2})^3$$

$$2^{5/2} = (\sqrt{2})^5$$

and all other fractional exponents that are multiples of $1/2$. We can add these to the graph of our previous powers of 2 to fill in additional points. This is shown on Figure 8.1(a).

In this way, we could also calculate exponents that are multiples of $1/4$ since

$$2^{1/4} = \sqrt{\sqrt{2}}$$




**(a)** **(b)**

**Figure 8.1.** *(a) Values of the function $2^x$ for $x = 0, 0.5, 1, 1.5$, etc. (b) The function $2^x$ is shown extended to negative values of $x$ and connected smoothly to form a continuous curve.*

is a value that we can obtain. We show how adding these values leads to an even finer set of points. By continuing in the same way, we fill in the graph of the emerging function. Connecting the dots smoothly allows us to define a value for any real $x$, of the new function,

$$y = f(x) = 2^x$$

This function is shown in Figure 8.1(b) as the smooth curve superimposed on the points we have gathered.

We can generalize this idea to defining an exponential function with an arbitrary base. Given some positive constant $a$, we will define the new function $f(x) = a^x$ as the exponential function with base $a$. Shown in Figure 8.2 are the functions $y = 2^x$, $y = 3^x$, $y = 4^x$ and $y = 10^x$.

## 8.5 Derivative of an exponential function

In this section we show how to compute the derivative of the new exponential function just defined. We first consider an arbitrary positive constant $a$ that will be used for the base of the function. Then for $a > 0$ let

$$y = f(x) = a^x.$$

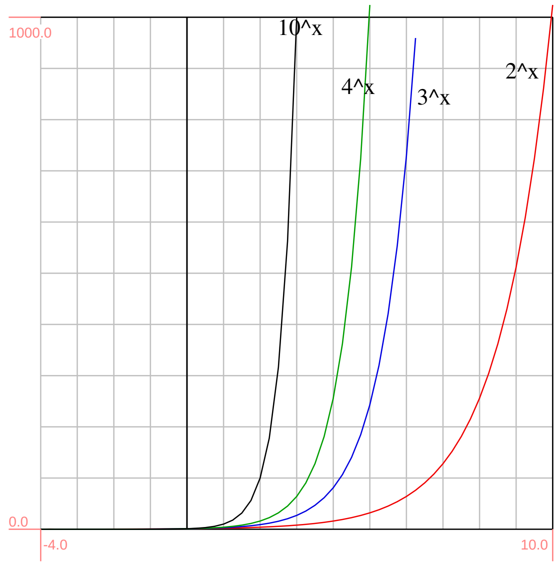



**Figure 8.2.** *The function $y = f(x) = a^x$ is shown here for a variety of bases, $a =$ 2, 3, 4, and 10.*

Then

$$\begin{aligned}\frac{da^x}{dx} &= \lim_{h\to 0}\frac{(a^{x+h} - a^x)}{h}\\ &= \lim_{h\to 0}\frac{(a^x a^h - a^x)}{h}\\ &= \lim_{h\to 0} a^x\frac{(a^h - 1)}{h}\\ &= a^x\left[\lim_{h\to 0}\frac{a^h - 1}{h}\right]\end{aligned}$$

Notice that the variable $x$ appears only in the form of $a^x$. Everything inside the limit does not depend on $x$ at all, but does depend on the base we used.

**Example 8.1 (Derivative of $2^x$)** Compute the derivative for the base $a = 2$ using the above result. ■

**Solution:** For base $a = 2$, we have

$$\frac{d2^x}{dx} = 2^x\left(\lim_{h\to 0}\frac{2^h - 1}{h}\right)$$

Let

$$C_2(h) = \lim_{h\to 0}\frac{2^h - 1}{h}.$$

We can calculate this quantity, or at least find a good estimate, by taking small values of $h$ i.e. by the approximation

$$C_2(h) \approx \frac{2^h - 1}{h}$$

**Example 8.2 (The value of $C_2$)** Compute $C_2$ for $h = 1, 0.1, 0.01$, etc. Does this value approach a fixed real number? Use your calculation to find the derivative of the function $y = 2^x$. How would this result change if we computed the derivative of $y = 10^x$? ∎

**Solution:** We find that $h = 1$ leads to $C_2 = 1.0$, $h = 0.1$ leads to $C_2 = 0.7177$, $h = 0.001$ leads to $C_2 = 0.6934$, $h = 0.00001$ to $C_2 = 0.6931$. Using this result, we see that $C_2 \to 0.6931$, so that

$$\frac{d2^x}{dx} = C_2 2^x = (0.6931) \cdot 2^x.$$

**Example 8.3 (The base 10 and the derivative of $10^x$)** Determine the derivative of $y = f(x) = 10^x$. ∎

**Solution:** If we had chosen base 10 for our exponential function, we would have

$$C_{10}(h) = \frac{10^h - 1}{h}.$$

We find, by similar approximation, that

$$C_{10} = 2.3026,$$

so that

$$\frac{d10^x}{dx} = C_{10} 10^x = (2.3026) \cdot 10^x.$$

Thus, different bases come with different constant multipliers when derivatives are computed.

### 8.5.1 A convenient base for the exponential function

These are rather messy constants, and hard to remember. We ask whether we can find some more convenient base (call it "e") such that the constant is nice and simple, e.g. $C_e = 1$.

Such a base would have to have the property that

$$C_e = \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

i.e. that, for small $h$

$$\frac{e^h - 1}{h} \approx 1.$$

This means that

$$e^h - 1 \approx h \quad \Rightarrow \quad e^h \approx h + 1 \quad \Rightarrow \quad e \approx (1 + h)^{1/h}$$

More specifically,

$$e = \lim_{h \to 0} (1+h)^{1/h}$$

We can find an approximate value for this interesting new base by calculating the expression shown above for some very small (but finite value) of $h$, e.g. $h = 0.00001$. Using this value, we find that

$$e \approx (1.00001)^{100000} \approx 2.71826$$

To summarize, we have found that for this special base, $e$, we have the following property:

**The derivative of the function $e^x$ is the same function, $e^x$.**

Remark: In the above computation, we came up with a little "recipe" for calculating the value of the base $e$. The recipe involves shrinking some value $h$ and computing a limit. We can restate this recipe in another way. Let $n = 1/h$. Then as $h$ shrinks, $n$ will be a growing number, i.e $h \to 0$ implies $n \to \infty$. We find that

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

## 8.6 Properties of the function $e^x$

We list below some of the key features of this function:

1. $e^a e^b = e^{a+b}$ as with all similar exponent manipulations.
2. $(e^a)^b = e^{ab}$ also stems from simple rules for manipulating exponents
3. $e^x$ is a function that is defined, continuous, and differentiable for all real numbers $x$.
4. $e^x > 0$ for all values of $x$.
5. $e^0 = 1$, and $e^1 = e$.
6. $e^x \to 0$ for increasing negative values of $x$
7. $e^x \to \infty$ for increasing positive values of $x$
8. The derivative of $e^x$ is $e^x$.
9. By parts 3. and 6. above, the slope of the tangent to $e^x$ at $x = 0$ is $e^0$, which is 1. This is shown in Figure 8.3

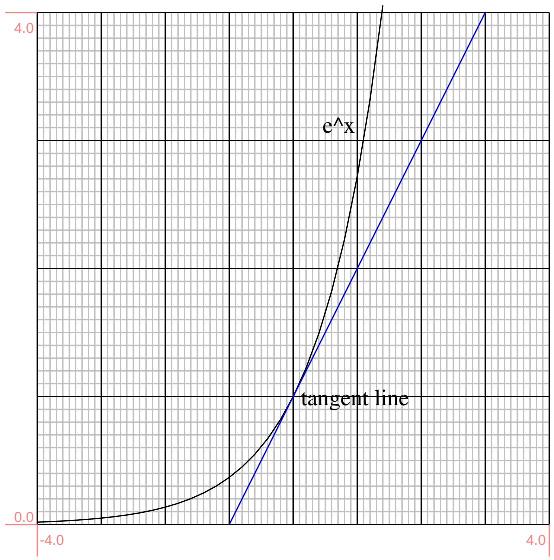



**Figure 8.3.** *The function $y = e^x$ has the property that its tangent line at $x = 0$ has slope 1. (Note that the horizontal scale on this graph is $-4 < x < 4$.)*

## 8.7 An interesting observation

We have seen that the function

$$y = f(x) = e^x$$

satisfies the relationship

$$\frac{dy}{dx} = f'(x) = f(x) = y$$

in other words, when differentiating, we get the same function back again.

The function $y = f(x) = e^x$ is equal to its own derivative and hence, it satisfies the equation

$$\frac{dy}{dx} = y.$$

An equation of this type, linking a function and its derivative(s) is called a **differential equation**.

This is a new type of equation, unlike ones seen before in this course. We will see later in this course that such equations have important applications..

## 8.8 The natural logarithm, an inverse function for $e^x$

We have defined a new function $y = f(x) = e^x$.

Here is it's inverse function, shown on Figure 8.4. We will call this function the logarithm (base $e$), and write it as

$$y = f^{-1}(x) = \ln(x).$$

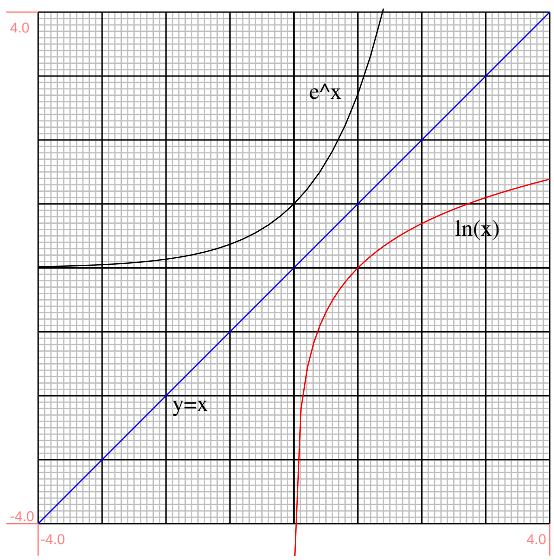



**Figure 8.4.** *The function $y = e^x$ is shown here with its inverse function, $y = \ln x$.*

We have the following connection: $y = e^x$ implies $x = \ln(y)$.
The fact that the functions are inverses also implies that

$$e^{\ln(x)} = x \quad \text{and} \quad \ln(e^x) = x.$$

Properties of the logarithm stem directly from properties of the exponential function, and include the following:

1. $\ln(ab) = \ln(a) + \ln(b)$
2. $\ln(a^b) = b\ln(a)$
3. $\ln(1/a) = \ln(a^{-1}) = -\ln(a)$

The inverse function can be quite helpful in changing from one base to another.

**Example 8.4** Rewrite $y = 2^x$ in terms of base $e$. ■

**Solution:**

$$y = 2^x \quad \Rightarrow \quad \ln(y) = \ln(2^x) = x\ln(2)$$

$$e^{\ln(y)} = e^{x\ln(2)} \quad \Rightarrow \quad y = e^{x\ln(2)}$$

We find (using a calculator) that $\ln(2) = 0.6931..$ so we have

$$y = e^{kx} \quad \text{where} \quad k = \ln(2) = 0.6931..$$

**Example 8.5** Find the derivative of $y = e^{kx}$. $\blacksquare$

**Solution:** The simple chain rule with $u = kx$ leads to

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

but

$$\frac{du}{dx} = k \quad \text{so} \quad \frac{dy}{dx} = e^u k = ke^{kx}.$$

This is a useful result, which we highlight for future use:

The derivative of $y = f(x) = e^{kx}$ is $f'(x) = ke^{kx}$

**Example 8.6** Find the derivative of $y = 2^x$. $\blacksquare$

**Solution:** We have expressed this function in the alternate form

$$y = 2^x = e^{kx}$$

with $k = \ln(2)$. From example 2 we have

$$\frac{dy}{dx} = ke^{kx} = \ln(2)e^{\ln(2)x} = \ln(2)2^x$$

Thus we now see that the constant obtained by computing this derivative from the definition is actually $C_2 = \ln(2)$.

## 8.9 How many bacteria

We can now return to our Andromeda strain and answer a question we had left unanswered: How long will it take for the population to attain a size of $6 \cdot 10^{36}$ cells, i.e. to grow to an Earth-sized colony.

We recall that the doubling time for the bacteria is $20$ min, so that one generation (or one doubling occurs for every multiple of $t/20$). However, it is not necessarily true that all cells will split in a synchronized way. This means that after $t$ minutes, we expect that the number, $B(t)$ of bacteria would be roughly given by the smooth function:

$$B(t) = 2^{t/20}.$$

(Note that this function agrees with our previous table and graph for powers of 2 at all integer multiples of the generation time, i.e. for $t = 20, 40, 60, 80..$ minutes.)

**Example 8.7** How long will it take to reach size $B = 6 \cdot 10^{36}$? ∎

**Solution:** We can compute this as follows:

$$6 \cdot 10^{36} = 2^{t/20} \quad \Rightarrow \quad \ln(6 \cdot 10^{36}) = \ln(2^{t/20})$$

$$\ln(6) + 36\ln(10) = \frac{t}{20}\ln(2)$$

so

$$t = 20\frac{\ln(6) + 36\ln(10)}{\ln(2)} = 20\frac{1.79 + 36(2.3)}{0.693} = 2441.27$$

This is the time in minutes. In hours, it would take 2441.27/60 = 40.68 hours for the colony to grow to such a size.

**Example 8.8 (Using base $e$)** Express the number of bacteria can in terms of base $e$ (for practice with base conversions). ∎

**Solution:** We would do this as follows:

$$B(t) = 2^{t/20} \quad \Rightarrow \quad \ln(B) = \frac{t}{20}\ln(2)$$

$$e^{\ln(B(t))} = e^{\frac{t}{20}\ln(2)} \quad \Rightarrow \quad B(t) = e^{kt} \quad \text{where} \quad k = \frac{\ln(2)}{20}.$$

The constant $k$ will be referred to as the growth rate of the bacteria. We observe that this constant can be written as:

$$k = \frac{\ln(2)}{\text{doubling time}}.$$

We will see the usefulness of this approach very soon.

## 8.10 Derivative of the natural logarithm

Here we use Implicit differentiation to find the derivative of the newly defined function, $y = \ln(x)$ as follows: First, restate the relationship in the inverse form, but consider $y$ as the dependent variable:

$$y = \ln(x) \quad \Rightarrow \quad e^y = x \quad \Rightarrow \quad \frac{d}{dx}e^y = \frac{d}{dx}x$$

Here we apply the chain rule:

$$\frac{de^y}{dy}\frac{dy}{dx} = 1 \quad \Rightarrow \quad e^y\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

We have thus shown the following:

The derivative of $\ln(x)$ is $1/x$:

$$\frac{d\ln(x)}{dx} = \frac{1}{x}$$

# 8.11 Additional problems

## 8.11.1 Chemical reactions

According to the collision theory of bimolecular gas reactions, a reaction between two molecules occurs when the molecules collide with energy greater than some activation energy, $E_a$, referred to as the Arrhenius activation energy. We will assume that $E_a > 0$ is constant for the given substance. The fraction of bimolecular reactions in which this collision energy is achieved is

$$F = e^{-(E_a/RT)}$$

where $T$ is temperature (in degrees Kelvin) and $R > 0$ is the gas constant.

**Example 8.9** Suppose that the temperature $T$ increases at some constant rate $C$ per unit time. Determine the rate of change of the fraction $F$ of collisions that result in a successful reaction. ■

**Solution:** We are given

$$F = e^{-(E_a/RT)}$$

and

$$\frac{dT}{dt} = C$$

Let $u = -E_a/RT$ then $F = e^u$, We use the chain rule to calculate:

$$\frac{dF}{dt} = \frac{dF}{du}\frac{du}{dT}\frac{dT}{dt}$$

Further, we have

$$\frac{dF}{du} = e^u$$

$$\frac{du}{dT} = \frac{E_a}{RT^2}$$

Assembling these parts, we have

$$\frac{dF}{dt} = e^u \frac{E_a}{RT^2} C = C\frac{E_a}{R} T^{-2} e^{-(E_a/RT)}.$$

## Exercises

8.1. Graph the following functions:

(a) $f(x) = x^2 e^{-x}$

(b) $f(x) = \ln(x^2 + 3)$

(c) $f(x) = \ln(e^{2x})$

8.2. Express the following in terms of base $e$:

(a) $y = 3^x$

(b) $y = \frac{1}{7^x}$

(c) $y = 15^{x^2+2}$

Express the following in terms of base 2:

(d) $y = 9^x$

(e) $y = 8^x$

(f) $y = -e^{x^2+3}$

Express the following in terms of base 10:

(g) $y = 21^x$

(h) $y = 1000^{-10x}$

(i) $y = 50^{x^2-1}$

8.3. Compare the values of each pair of numbers (i.e. indicate which is larger):

(a) $5^{0.75}, 5^{0.65}$

(b) $0.4^{-0.2}, 0.4^{0.2}$

(c) $1.001^2, 1.001^3$

(d) $0.999^{1.5}, 0.999^{2.3}$

8.4. Rewrite each of the following equations in logarithmic form:

(a) $3^4 = 81$

(b) $3^{-2} = \frac{1}{9}$

(c) $27^{-\frac{1}{3}} = \dfrac{1}{3}$

8.5. Solve the following equations for $x$:

(a) $\ln x = 2\ln a + 3\ln b$

(b) $\log_a x = \log_a b - \frac{2}{3}\log_a c$

8.6. **Reflections and transformations:** What is the relationship between the graph of $y = 3^x$ and the graph of each of the following functions?
(a) $y = -3^x$ (b) $y = 3^{-x}$ (c) $y = 3^{1-x}$
(d) $y = 3^{|x|}$ (e) $y = 2 \cdot 3^x$ (e) $y = \log_3 x$

8.7. Solve the following equations for $x$:

(a) $e^{3-2x} = 5$

(b) $\ln(3x - 1) = 4$

(c) $\ln(\ln(x)) = 2$

(d) $e^{ax} = Ce^{bx}$, where $a \neq b$ and $C > 0$.

8.8. Find the first derivative for each of the following functions:

(a) $y = \ln(2x + 3)^3$

(b) $y = \ln^3(2x + 3)$

(c) $y = \ln(\cos \frac{1}{2}x)$

(d) $y = \log_a(x^3 - 2x)$ (Hint : $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$ )

(e) $y = e^{3x^2}$

(f) $y = a^{-\frac{1}{2}x}$

(g) $y = x^3 \cdot 2^x$

(h) $y = e^{e^x}$

(i) $y = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$

8.9. Find the maximum and minimum points as well as all inflection points of the following functions:

(a) $f(x) = x(x^2 - 4)$

(b) $f(x) = x^3 - \ln(x), x > 0$

(c) $f(x) = xe^{-x}$

(d) $f(x) = \frac{1}{1-x} + \frac{1}{1+x}, -1 < x < 1$

(e) $f(x) = x - 3\sqrt[3]{x}$

(f) $f(x) = e^{-2x} - e^{-x}$

8.10. Shown in Figure 10 is the graph of $y = Ce^{kt}$ for some constants $C, k$, and a tangent line. Use data from the graph to determine $C$ and $k$.

8.11. Consider the two functions

(a) $y_1(t) = 10e^{-0.1t}$,

(b) $y_2(t) = 10e^{0.1t}$.

Which one is decreasing and which one is increasing? In each case, find the value of the function at $t = 0$. Find the time at which the increasing function has doubled from this initial value. Find the time at which the decreasing function has fallen to half of its initial value. [Remark: these values of $t$ are called, the doubling time, and the half-life, respectively]

8.12. **Shannon Entropy:** In a recent application of information theory to the field of genomics, a function called the Shannon entropy, $H$, was considered. A given gene is represented as a binary device: it can be either “on” or “off” (i.e. being expressed

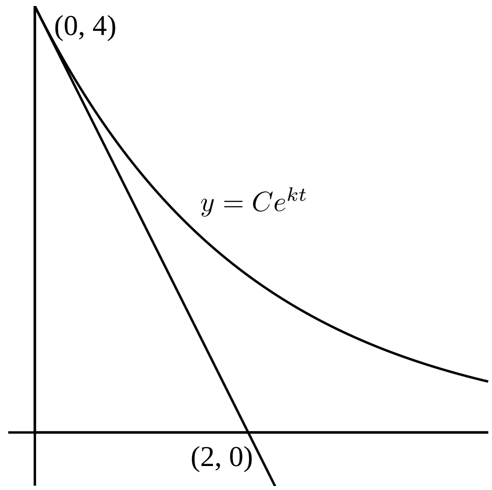



**Figure 8.5.** *Figure for Problem 10*

or not). If $x$ is the probability that the gene is "on" and $y$ is the probability that it is "off", the Shannon entropy function for the gene is defined as

$$H = -x\log(x) - y\log(y)$$

[Remark: the fact that $x$ and $y$ are probabilities, just means that they satisfy $0 < x \leq 1$, and $0 < y \leq 1$.] The gene can only be in one of these two states, so $x + y = 1$. Use these facts to show that the Shannon entropy for the gene is greatest when the two states are equally probable, i.e. for $x = y = 0.5$.

8.13. **A threshold function:** The response of a regulatory gene to inputs that affect it is not simply linear. Often, the following so called "squashing function" or "threshold function" is used to link the input $x$ to the output $y$ of the gene.

$$y = f(x) = \frac{1}{1 + e^{(ax+b)}}$$

where $a$, $b$ are constants.

(a) Show that $0 < y < 1$.

(b) For $b = 0$ and $a = 1$ sketch the shape of this function.

(c) How does the shape of the graph change as $a$ increases?

8.14. Sketch the graph of the function $y = e^{-t}\sin\pi t$.

8.15. **The Mexican Hat:** Find the critical points of the function

$$y = f(x) = 2e^{-x^2} - e^{-x^2/3}$$

and determine the value of $f$ at those critical points. Use these results and the fact that for very large $x$, $f \to 0$ to draw a rough sketch of the graph of this function.

Comment on why this function might be called “a Mexican Hat”. (Note: The second derivative is not very informative here, and we will not ask you to use it for determining concavity in this example. However, you may wish to calculate it just for practice with the chain rule.)

8.16. **The Ricker Equation:** In studying salmon populations, a model often used is the Ricker equation which relates the size of a fish population this year, $x$ to the expected size next year $y$. (Note that these populations do not change continuously, since all the parents die before the eggs are hatched.) The Ricker equation is

$$y = \alpha x e^{-\beta x}$$

where $\alpha, \beta > 0$. Find the value of the current population which maximizes the salmon population next year according to this model.

8.17. **Spacing in a fish school:** Life in a social group has advantages and disadvantages: protection from predators is one advantage. Disadvantages include competition with others for food or resources. Spacing of individuals in a school of fish or a flock of birds is determined by the mutual attraction and repulsion of neighbors from one another: each individual does not want to stray too far from others, nor get too close.

Suppose that when two fish are at distance $x > 0$ from one another, they are attracted with “force” $F_a$ and repelled with “force” $F_r$ given by:

$$F_a = Ae^{-x/a}$$

$$F_r = Re^{-x/r}$$

where $A, R, a, r$ are positive constants. $A, R$ are related to the magnitudes of the forces, and $a, r$ to the spatial range of these effects.

(a) Show that at the distance $x = a$ the first function has fallen to $(1/e)$ times its value at the origin. (Recall $e \approx 2.7$.) For what value of $x$ does the second function fall to $(1/e)$ times its value at the origin? Note that this is the reason why $a, r$ are called spatial ranges of the forces.

(b) It is generally assumed that $R > A$ and $r < a$. Interpret what this mean about the comparative effects of the forces and sketch a graph showing the two functions on the same set of axes.

(c) Find the distance at which the forces exactly balance. This is called the comfortable distance for the two individuals.

(d) If either $A$ or $R$ changes so that the ratio $R/A$ decreases, does the comfortable distance increase or decrease? (Give reason.)

(e) Similarly comment on what happens to the comfortable distance if $a$ increases or $r$ decreases.

8.18. **Seed distribution:** The density of seeds at a distance $x$ from a parent tree is observed to be

$$D(x) = D_0 e^{-x^2/a^2},$$

where $a > 0, D_0 > 0$ are positive constants. Insects that eat these seeds tend to congregate near the tree so that the fraction of seeds that get eaten is

$$F(x) = e^{-x^2/b^2}$$

where $b > 0$. (Remark: These functions are called Gaussian or Normal distributions. The parameters $a, b$ are related to the "width" of these bell-shaped curves.) The number of seeds that survive (i.e. are produced and not eaten by insects) is

$$S(x) = D(x)(1 - F(x))$$

Determine the distance $x$ from the tree at which the greatest number of seeds survive.

8.19. **Euler's "$e$":** In 1748, Euler wrote a classic book on calculus ("Introductio in Analysin Infinitorum") in which he showed that the function $e^x$ could be written in an expanded form similar to an (infinitely long) polynomial:

$$e^x = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + ...$$

Use as many terms as necessary to find an approximate value for the number $e$ and for $1/e$ to 5 decimal places. Remark: we will see later that such expansions, called power series, are central to approximations of many functions.

## Chapter 9

# Exponential Growth and Decay: Differential Equations

## 9.1 Observations about the exponential function

In a previous chapter we made an observation about a special property of the exponential function

$$y = f(x) = e^x$$

namely, that

$$\frac{dy}{dx} = e^x = y$$

so that this function satisfies the relationship

$$\frac{dy}{dx} = y.$$

We call this a **differential equation** because it connects one (or more) derivatives of a function with the function itself.

In this chapter we will study the implications of the above observation. Since most of the applications that we examine will be time-dependent processes, we will here use $t$ (for time) as the independent variable.

Then we can make the following observations:

1. Let $y$ be the function of time:

   $$y = f(t) = e^t$$

   Then

   $$\frac{dy}{dt} = e^t = y$$

   With this slight change of notation, we see that the function $y = e^t$ satisfies the differential equation

   $$\frac{dy}{dt} = y.$$

2. Now consider
$$y = e^{kt}.$$
Then, using the chain rule, and setting $u = kt$, and $y = e^u$ we find that
$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = e^u \cdot k = ke^{kt} = ky.$$
So we see that the function $y = e^{kt}$ satisfies the differential equation
$$\frac{dy}{dt} = ky.$$

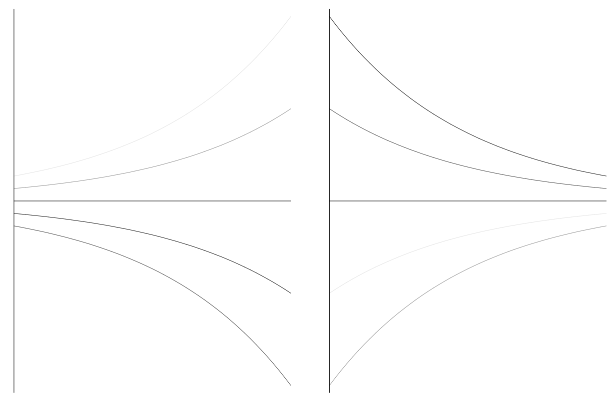
3. If instead we had the function
$$y = e^{-kt}$$
we could similarly show that the differential equation it satisfies is
$$\frac{dy}{dt} = -ky.$$

4. Now suppose we had a constant in front, e.g. we were interested in the function
$$y = 5e^{kt}.$$
Then, by simple differentiation and rearrangement we have
$$\frac{dy}{dt} = 5\frac{d}{dt}e^{kt} = 5(ke^{kt}) = k(5e^{kt}) = ky.$$
So we see that this function with the constant in front *also* satisfies the differential equation
$$\frac{dy}{dt} = ky.$$

5. The conclusion we reached in the previous step did not depend at all on the constant out front. Indeed, if we had started with a function of the form
$$y = Ce^{kt}$$
where $C$ is any constant, we would still have a function that satisfies the same differential equation.

6. While we will not prove this here, it turns out that these are the *only* functions that satisfy this equation.

The differential equation
$$\frac{dy}{dt} = ky \tag{9.1}$$
has as its solution, the function
$$y = Ce^{kt} \tag{9.2}$$



**(a)** **(b)**

**Figure 9.1.** *Functions of the form $y = Ce^{kt}$ (a) for $k > 0$ these represent exponentially growing solutions, whereas (b) for $k < 0$ they represent exponentially decaying solutions.*

A few comments are worth making: First, unlike *algebraic* equations, (whose solutions are numbers), **differential equations** have solutions that are *functions*. We have seen above that depending on the constant $k$, we get either functions with a positive or with a negative exponent (assuming that time $t > 0$). This leads to the two distinct types of behaviour, *exponential growth* or *exponential decay* shown in Figures 9.1(a) and (b). In each of these figures we see a *family* of curves, each of which represents a function that satisfies one of the differential equations we have discussed.

## 9.2 The solution to a differential equation

**Definition 9.1 (Solution to a differential equation).** *By a* **solution** *to a differential equation, we mean a function that satisfies that equation.*

In the previous section we have seen a collection of solutions to each of the differential equations we discussed. For example, each of the curves shown in Figure 9.1(a) share the property that they satisfy the equation

$$\frac{dy}{dt} = ky.$$

We now ask: what distinguishes one from the other? More specifically, how could we specify one particular member of this family as the one of interest to us? As we saw above, the differential equation does not distinguish these: we need some additional information. For example, if we had some coordinates, say $(a, b)$ that the function of interest should go through, this would select one function out of the collection. It is common practice (though not essential) to specify the starting value or **initial value** of the function i.e. its value at time $t = 0$.

**Definition 9.2 (Initial value).** *An* **initial value** *is the value at time $t = 0$ of the desired solution of a differential equation.*

**Example 9.3** Suppose we are given the differential equation (9.1) and the initial value

$$y(0) = y_0$$

where $y_0$ is some (known) fixed value. Find the value of the constant $C$ in the solution (9.2). $\blacksquare$

**Solution:** We proceed as follows:

$$y(t) = Ce^{kt}, \quad \text{so} \quad y(0) = Ce^{k\cdot 0} = Ce^0 = C \cdot 1 = C$$

but, by the initial condition, $y(0) = y_0$. So then,

$$C = y_0$$

and we have established that

$$y(t) = y_0 e^{kt},$$

where $y_0$ is the initial value.

## 9.3 Where do differential equations come from?

Figure 9.2 shows how differential equations arise in scientific investigations. The process of going from initial vague observations about a system of interest (such as planetary motion) to a mathematical model, often involves a great deal of speculation, at first, about what is happening, what causes the motion or the changes that take place, and what assumptions might be fruitful in trying to analyze and understand the system.

Once the cloud of doubt and vague ideas settles somewhat, and once the right simplifying assumptions are made, we often find that the mathematical model leads to a differential equation. In most scientific applications, it may then be a huge struggle to figure out which functions would be the appropriate class of solutions to that differential equation, but if we can find those functions, we are in position to make quantitative predictions about the system of interest.

In our case, we have stumbled on a simple differential equation by noticing a property of functions that we were already familiar with. This is a lucky accident, and we will exploit it in an application shortly.

In many cases, the process of modelling hardly stops when we have found the link between the differential equation and solutions. Usually, we would then compare the predictions to observations that may help us to refine the model, reject incorrect or inaccurate assumptions, or determine to what extent the model has limitations.

A simple example of population growth modelling is given as motivation for some of the ideas seen in this discussion.

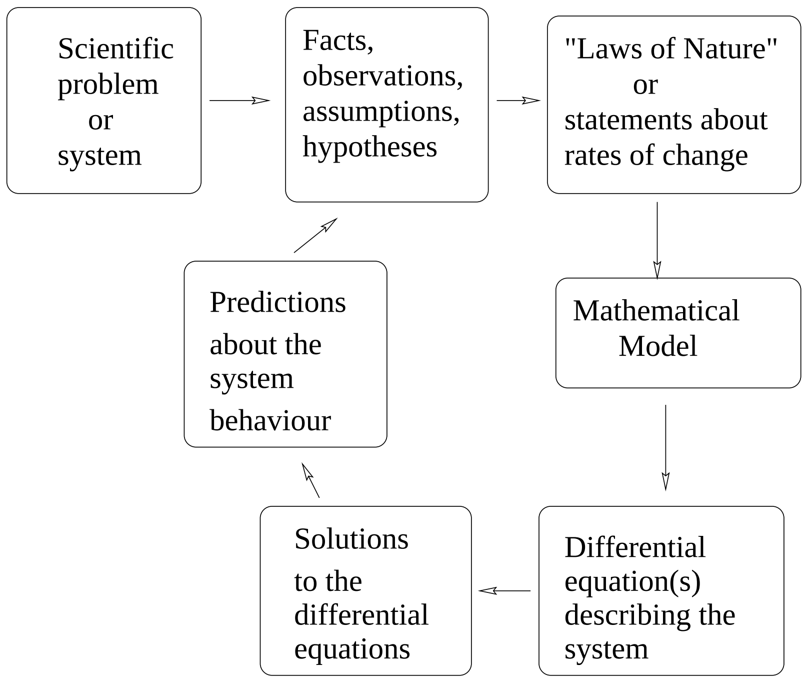



**Figure 9.2.** *A "flow chart" showing how differential equations originate from scientific problems.*

## 9.4 Population growth

In this section we will examine the way that a simple differential equation arises when we study the phenomenon of population growth.

We will let $N(t)$ be the number of individuals in a population at time $t$. The population will change with time. Indeed the rate of change of $N$ will be due to births (that increase $N$) and deaths (that decrease it).

$$\text{Rate of change of N} = \text{Rate of births} - \text{Rate of deaths}$$

We will assume that all individuals are identical in the population, and that the average **per capita birth rate**, $r$, and average **per capita mortality rate**, $m$ are some fixed positive constants. That is

$$r = \text{per capita birth rate} = \frac{\text{number births per year}}{\text{population size}},$$

$$m = \text{per capita mortality rate} = \frac{\text{number deaths per year}}{\text{population size}}.$$

We will refer to such constants as **parameters**. In general, for a given population, these would have certain numerical values that one could obtain by experiment, by observation, or by simple assumptions. In the next section, we will show how a set of assumptions would lead to such values.

Then the total number of births into the population in year $t$ is $rN$, and the total number of deaths out of the population in year $t$ is $mN$. The rate of change of the population as a whole is given by the derivative $dN/dt$. Thus we have arrived at:

$$\frac{dN}{dt} = rN - mN.$$

This is a differential equation: it links the derivative of $N(t)$ to the function $N(t)$. By solving the equation (i.e. identifying its solution), we will be able to make a projection about how fast the world population is growing.

We can first simplify the above by noting that

$$\frac{dN}{dt} = rN - mN = (r - m)N = kN.$$

where

$$k = (r - m).$$

This means that we have shown that the population satisfies a differential equation of the form

$$\frac{dN}{dt} = kN,$$

provided $k$ is the so-called “net growth rate”, i.e birth rate minus mortality rate. This leads us to the following conclusions:

- The function that describes population over time is (by previous results) simply

  $$N(t) = N_0 e^{kt}.$$

  (The result is identical to what we saw previously, but with $N$ rather than $y$ as the time-dependent function.)

- We are no longer interested in negative values of $N$ since it now represents a quantity that has to be positive to have biological relevance, i.e. population size.

- The population will grow provided $k > 0$ which happens when $r - m > 0$ i.e. when the per capita birth rate, $r$ exceeds the per capita mortality rate $m$.

- If $k < 0$, or equivalently, $r < m$ then more people die on average than are born, so that the population will shrink and (eventually) go extinct.

## 9.5 Human population growth: a simple model

We have seen how a statement about changes that take place over time can lead to the formulation of a differential equation. In this section, we will estimate the values of the parameters for the birth rate, $r$ and the mortality rate, $m$.

To do so, we must make some simplifying assumptions:

**Assumptions:**

- The age distribution of the population is "flat", i.e. there are as many 10 year-olds as 70 year olds. (This is quite inaccurate, but will be a good place to start, as it will be easy to estimate some of the quantities we need.) Figure 9.3 shows such a distribution.

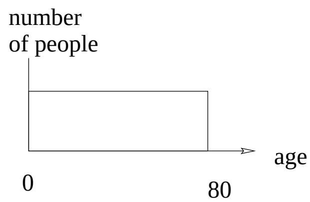



**Figure 9.3.** *We assume a "flat" age distribution to make it easy to determine the fraction of people who give birth or die.*

- The sex ratio is roughly 50%. This means that half of the population is female and half male.

- Women are fertile and can have babies only during part of their lives: We will assume that the fertile years are between age 15 and age 55, as shown in Figure 9.4.

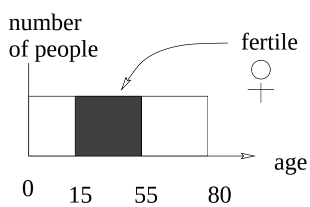



**Figure 9.4.** *Only fertile women (between the ages of 15 and 55 years old) give birth. This sketch shows that half of all women are between these ages.*

- A lifetime lasts 80 years. This means that for half of that time a given woman can contribute to the birth rate, or that (55-15)/80=50% of women alive at any time are able to give birth.

- During a woman's fertile years, we'll assume that on average, she has one baby every 10 years. (This is also a suspect assumption, since in the Western world, a woman has on average 2-2.3 children over her lifetime, while in the Developing nations, the number of children per woman is much higher. )

Based on the above assumptions, we can estimate the parameter $r$ as follows:

$$r = \frac{\text{number women}}{\text{population}} \cdot \frac{\text{years fertile}}{\text{years of life}} \cdot \frac{\text{number babies per woman}}{\text{number of years}}$$

Thus we compute that

$$r = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{10} = 0.025 \text{ babies per person per year.}$$

Thus, we have arrived at an approximate value for human per capita birth rate.

We can now estimate the mortality.

- We also assume that deaths occur only from old age (i.e. we ignore disease, war, famine, and child mortality.)
- We assume that everyone lives precisely to age 80, and then dies instantly. (Not an assumption our grandparents would happily live with!)

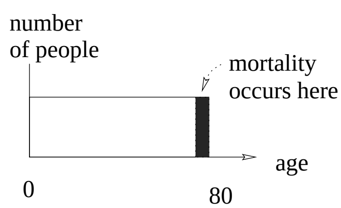



**Figure 9.5.** *We assume that the people in the age bracket 79-80 years old all die each year, and that those are the only deaths.*

But, with the flat age distribution shown in Figure 9.3, there would be a fraction of 1/80 of the population who are precisely removed by mortality every year (i.e. only those of age 80.) In this case, we can estimate that the per capita mortality is:

$$m = \frac{1}{80} = 0.0125$$

Putting our results together, we have the net growth rate $k = r - m = 0.025 - 0.0125 = 0.0125$ per person per year. In the context of such growth problems, we will often refer to the constant $k$ as the **rate constant**, or the **growth rate** of the population.

**Example 9.4** Using the results of this section, find a prediction for the population size $N(t)$ as a function of time $t$. ■

**Solution:** We have found that our population satisfies the equation

$$\frac{dN}{dt} = 0.0125N$$

so that

$$N(t) = N_0 e^{0.0125t}$$

where $N_0$ is the starting population size. Figure 9.6 illustrates how this function behaves, using a starting value of $N(0) = N_0 = 6$ billion.

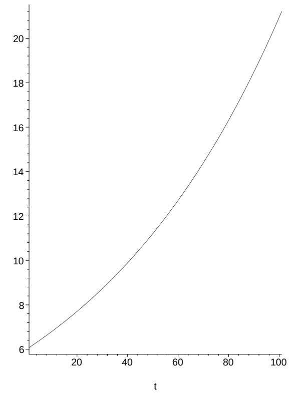



**Figure 9.6.** *Projected world population over the next 100 years or so.*

## 9.6 Growth and doubling

We ask how long it would take for a population to double given that it is growing exponentially, with growth rate $k$, as described above. That is, we ask at what time $t$ it would be true that $n$ reaches twice its starting value, i.e. $N(t) = 2N_0$. We determine this time as follows:

$$N(t) = 2N_0$$

but

$$N(t) = N_0 e^{kt}$$

so the population has doubled when $t$ satisfies

$$2N_0 = N_0 e^{kt}, \quad \Rightarrow \quad 2 = e^{kt}$$

Taking the natural log of both sides leads to

$$\ln(2) = \ln(e^{kt}) = kt.$$

Thus, the **doubling time**, which we'll call $\tau$ is:

$$\tau = \frac{\ln(2)}{k}.$$

**Example 9.5 (Human population doubling time)** Determine the doubling time for the human population based on the results of our approximate growth model. ∎

**Solution:** We have found a growth rate of roughly $k = 0.0125$ for the human population. Based on this, it would take

$$\tau = \frac{\ln(2)}{0.0125} = 55.45 \text{ years}$$

for the population to double.

In general, an equation of the form

$$\frac{dy}{dt} = ky$$

that represents an exponential growth will have a doubling time of

$$\tau = \frac{\ln(2)}{k}.$$

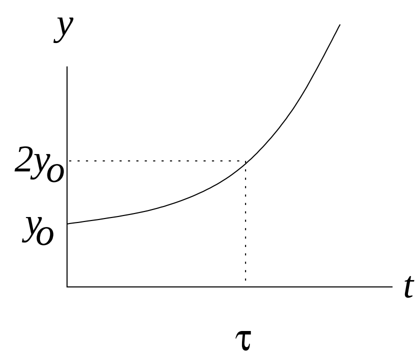



**Figure 9.7.** *Doubling time for exponential growth.*

This is shown in Figure 9.7. The interesting thing that we discovered is that the population doubles *every 55 years*! So that, for example, after 110 years there have been two doublings, or a quadrupling of the population.

**Example 9.6 (Human population in 100 years)** Determine what the population growth model predicts will be the human population level in 100 years? ■

**Solution:** Suppose that currently $N(0) = 6$ billion. Then in billions,

$$N(t) = 6e^{0.0125t}$$

so that when $t = 100$ we would have

$$N(100) = 6e^{0.0125 \cdot 100} = 6e^{1.25} = 6 \cdot 3.49 = 20.94$$

Thus, with population around the 6 billion now, we should see about 21 billion people on Earth in 100 years.

**Example 9.7 (A ten year doubling time)** Suppose we are told that some animal population doubles every 10 years. What growth rate would lead to such a trend? ■

**Solution:** Rearranging

$$t_2 = \frac{\ln(2)}{k}$$

we obtain

$$k = \frac{\ln(2)}{t_2} = \frac{0.6931}{10} \approx 0.07.$$

Thus, we may say that a growth rate of 7% leads to doubling roughly every 10 years.

### 9.6.1 A critique

Before leaving our population model, we should remember that our projections hold only so long as some rather restrictive assumptions are made. We have made many simplifications, and ignored many features that would seriously affect these results.

These include variations in the birth and mortality rates that stem from competition for the Earth's resources, epidemics that take hold when crowding occurs, uneven distributions of resources or space, and other factors. We have also assumed that the age distribution is uniform (flat), but that is clearly wrong: the population grows only by adding new infants, and this would skew the distribution even if it starts out uniform. All these factors would lead us to be skeptical, and to eventually think about more advanced ways of describing the population growth.

## 9.7 Exponential decay and radioactivity

A radioactive material consists of atoms that undergo a spontaneous change. Every so often, an atom will emit a particle, and change to another form. We call this a process of **radioactive decay**.

For any one atom, it is impossible to predict when this event would occur. However, if we have very many atoms, on average some fraction, $k$, will undergo this decay during any given unit time. (This fraction will depend on the material.) This means that $ky$ of the amount will be lost per unit time.

We will define $y(t)$ to be the amount of radioactivity remaining at time $t$. This quantity can be measured with Geiger counters, and will depend on time. In the decay process, radioactivity will be continually lost. Thus

$$[\text{rate of change of y}] = -[\text{amount lost per unit time}] \quad \Rightarrow \quad \frac{dy}{dt} = -ky.$$

We see again, a (by now) familiar differential equation.

Suppose that initially, there was an amount $y_0$. Then the initial condition that comes with this differential equation is

$$y(0) = y_0.$$

From familiarity with the differential equation, we know that the function that satisfies it will be

$$y(t) = Ce^{-kt}$$

and using the initial condition will specify that

$$y(t) = y_0 e^{-kt}.$$

For $k > 0$ a constant, this is a decreasing function of time that we refer to as exponential decay.

### 9.7.1 The half life

Given a process of exponential decay, we can ask how long it would take for half of the original amount to remain. That is, we look for $t$ such that

$$y(t) = \frac{y_0}{2}.$$

We will refer to the value of $t$ that satisfies this as the **half life**.

**Example 9.8 (Half life)** Determine the half life in the exponential decay described above ■

**Solution:** We compute:

$$\frac{y_0}{2} = y_0 e^{-kt} \quad \Rightarrow \quad \frac{1}{2} = e^{-kt}$$

Now taking reciprocals:

$$2 = \frac{1}{e^{-kt}} = e^{kt}.$$

Thus we find the same result as in our calculation for doubling times, namely,

$$\ln(2) = \ln(e^{kt}) = kt$$

so that the half life is

$$\tau = \frac{\ln(2)}{k}.$$

This is shown in Figure 9.8.

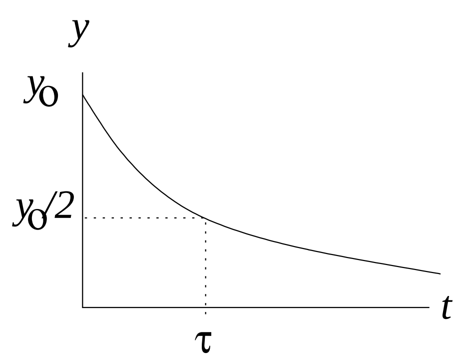



**Figure 9.8.** *Half-life in an exponentially decreasing process.*

**Example 9.9 (Chernobyl: April 1986)** In 1986 the Chernobyl nuclear power plant exploded, and scattered radioactive material over Europe. Of particular note were the two radioactive elements iodine-131 ($I^{131}$) whose half-life is 8 days and cesium-137 ($Cs^{137}$) whose half life is 30 years. Use the model for radioactive decay to predict how much of this material would remain over time. ■

**Solution:** We first determine the decay constants for each of these two elements, by noting that

$$k = \frac{\ln(2)}{\tau}$$

and recalling that $\ln(2) \approx 0.693$. Then for $I^{131}$ we have

$$k = \frac{\ln(2)}{\tau} = \frac{\ln(2)}{8} = 0.0866 \text{ per day}.$$

This means that for $t$ measured in days, the amount of $I^{131}$ left at time $t$ would be

$$y_I(t) = y_0 e^{-0.0866t}.$$

For $Cs^{137}$

$$k = \frac{\ln(2)}{30} = 0.023 \text{ per year}.$$

so that for $T$ in years,

$$y_C(t) = y_0 e^{-0.023T}.$$

(We have used $T$ rather than $t$ to emphasize that units are different in the two calculations done in this example.

**Example 9.10 (Decay to $0.1\%$ of the initial level)** How long it would take for $I^{131}$ to decay to 0.1 % of its initial level, just after the explosion at Chernobyl? ■

**Solution:** We must calculate the time $t$ such that $y_I = 0.001 y_0$:

$$0.001 y_0 = y_0 e^{-0.0866t} \quad \Rightarrow \quad 0.001 = e^{-0.0866t} \quad \Rightarrow \quad \ln(0.001) = -0.0866t.$$

Therefore,

$$t = \frac{\ln(0.001)}{-0.0866} = \frac{-6.9}{-0.0866} = 79.7 \text{days}$$

Thus it would take about 80 days for the level of Iodine-131 to decay to 0.1 % of its initial level.

## 9.8 Checking (analytic) solutions to a differential equation

By **analytic** solution, we mean a "formula" in the form $y = f(x)$. We have seen a number of examples of simple differential equations in this chapter, and our main purpose was to show how these arise in the context of a physical or biological process of growth or decay. Most of these examples led to the differential equation

$$\frac{dy}{dt} = ky$$

and therefore, by our observations, to its analytic solution, the exponential function

$$y = f(t) = Ce^{kt}.$$

However, as we will see, there can be many distinct types of differential equations, and it may not always be clear which function is a solution. Finding the correct solution can be quite challenging, even to professional mathematicians. We mention two ideas that are sometimes helpful.

In some cases, we encounter a new differential equation, and we are given a function that is believed to satisfy that equation. We can always check and verify that this claim is correct (or find it incorrect) by simple differentiation.

### Newtons Law of Cooling

Newton's Law of Cooling states that the rate of change of the temperature of an object $T$, is proportional to the difference between the ambient (environment) temperature, $E$, and the temperature of the object, $T$. If the temperature of the environment is constant, and the objects starts out at temperature $T_0$ initially, then the differential equation and initial condition describing this process is

$$\frac{dT}{dt} = k(E - T), \quad T(0) = T_0. \tag{9.3}$$

The parameter $k$ is a constant that represents the properties of the material. (Some objects conduct heat better than others, and thus cool off or heat up more quickly. The reader should be able to figure out that these types of objects have higher values of $k$, as this implies larger rates of change per unit time.) We study properties of this equation later, but here we show how to check which of two possible functions are its solutions.

**Example 9.11 (Candidate 1:)** Consider the function $T(t) = T_0 e^{-kt}$. Show that this simple exponential function **is NOT** usually a solution to the differential equation (9.3) for Newton's Law of Cooling. ■

**Solution:** We observe, first that this function does satisfy the initial condition, $T(0) = T_0$ by plugging $t = 0$ into the function:

$$T(0) = T_0 e^{-k \cdot 0} = T_0.$$

Next, by simply differentiating the above "candidate function", we find that its derivative is

$$\frac{dT}{dt} = -T_0 k e^{-kt}.$$

To satisfy the differential equation, we must have

$$\frac{dT}{dt} = k(E - T).$$

The term on the right hand side would lead to the expression

$$k(E - T_0 e^{-kt})$$

once the candidate function $T(t)$ is substituted for $T$. However, in general,

$$\frac{dT}{dt} = -T_0 k e^{-kt} \neq k(E - T_0 e^{-kt}).$$

(Only in the case that $E = 0$ do the two sides match, but for arbitrary ambient temperature, this is not the case.) Thus the simple exponential function $T(t) = T_0 e^{-kt}$ **is NOT** a solution to this differential equation when $E \neq 0$.

**Example 9.12 (Candidate 2:)** Show that the modified exponential function

$$T(t) = E + (T_0 - E)e^{-kt}$$

is a solution to the differential equation and initial value. ∎

**Solution:** Note, first, that by plugging in the initial time, $t = 0$, we have

$$T(0) = E + (T_0 - E)e^{-k\cdot 0} = E + (T_0 - E)\cdot 1 = E + (T_0 - E) = T_0.$$

Thus the initial condition is satisfied.

Second, note that the derivative of this function is

$$\frac{dT}{dt} = \frac{d}{dt}\left(E + (T_0 - E)e^{-kt}\right) = -k(T_0 - E)e^{-kt}.$$

(This follows from the fact that $E$ is a constant, $(T_0 - E)$ is constant, and from the chain rule applied to the exponent $-kt$.) The term on the right hand side of the differential equation leads to

$$k(E - T) = k(E - [E + (T_0 - E)e^{-kt}]) = -k(T_0 - E)e^{-kt}.$$

We now observe agreement between the terms obtained from each of the right and left hand sides of the differential equation, applied to the above function. We conclude that the differential equation is satisfied, so that indeed this candidate function is a solution, as claimed.

As shown in Example 9.12, if we are told that a function is a solution to a differential equation, we can check the assertion and verify that it is correct or incorrect. A much more difficult task is to find the solution of a new differential equation from first principles. In some cases, the technique of integration, learned in second semester calculus, can be used. In other cases, some transformation that changes the problem to a more familiar one is helpful. (An example of this type is presented in Chapter 13). In many cases, particularly those of so-called non-linear differential equations, it requires great expertise and familiarity with advanced mathematical methods to find the solution to such problems in an analytic form, i.e. as an explicit formula. In such cases, approximation and numerical methods are helpful.

## 9.9 Finding (numerical) solutions to a differential equation

In cases where it is difficult or impossible to find the desired solution with guesses, integration methods, or from previous experience, we can use approximation methods and numerical computations to do the job. Most of these methods rely on the fact that derivatives can be approximated by finite differences. For example, suppose we are given a differential equation of the form

$$\frac{dy}{dt} = f(y)$$

with initial value $y(0) = y_0$, can be approximated by selecting a set of time points $t_1, t_2, \ldots,$ which are spaced apart by time steps of size $\Delta t$, and replacing the differential equation by the approximate *finite difference* equation

$$\frac{y_1 - y_0}{\Delta t} = f(y_0).$$

This relies on the approximation

$$\frac{dy}{dt} \approx \frac{\Delta y}{\Delta t},$$

which is a relatively good approximation for small step size $\Delta t$. Then by rearranging this approximation, we find that

$$y_1 = y_0 + f(y_0)\Delta t.$$

Knowing the quantities on the right allows us to compute the value of $y_1$, i.e. the value of the approximate "solution" at the time point $t_1$. We can then continue to generate the value at the next time point in the same way, by approximating the derivative again as a secant slope. This leads to

$$y_2 = y_1 + f(y_1)\Delta t.$$

The approximation so generated, leading to values $y_1, y_2, \ldots$ is called **Euler's method**. We explore an application of this method to Newton's law of cooling in chapter 13. In lab 5, the reader is invited to try out this method on the simple differential equation for exponential growth that was discussed in this chapter.

## Exercises

9.1. A differential equation is an equation in which some function is related to its own derivative(s). For each of the following functions, calculate the appropriate derivative, and show that the function satisfies the indicated *differential equation*

(a) $f(x) = 2e^{-3x}, \quad f'(x) = -3f(x)$

(b) $f(t) = Ce^{kt}, \quad f'(t) = kf(t)$

(c) $f(t) = 1 - e^{-t}, \quad f'(t) = 1 - f(t)$

9.2. Consider the function $y = f(t) = Ce^{kt}$ where $C$ and $k$ are constants. For what value(s) of these constants does this function satisfy the equation

(a) $\frac{dy}{dt} = -5y$,

(b) $\frac{dy}{dt} = 3y$.

[Remark: an equation which involves a function and its derivative is called a differential equation.]

9.3. Find a function that satisfies each of the following *differential equations*. [Remark: all your answers will be exponential functions, but they may have different dependent and independent variables.]

(a) $\dfrac{dy}{dt} = -y$,

(b) $\dfrac{dc}{dx} = -0.1c$ and $c(0) = 20$,

(c) $\dfrac{dz}{dt} = 3z$ and $z(0) = 5$.

9.4. If $70\%$ of a radioactive substance remains after one year, find its half-life.

9.5. **Carbon 14:** Carbon 14 has a half-life of 5730 years. This means that after 5730 years, a sample of Carbon 14, which is a radioactive isotope of carbon will have lost one half of its original radioactivity.

(a) Estimate how long it takes for the sample to fall to roughly 0.001 of its original level of radioactivity.

(b) Each gram of $^{14}C$ has an activity given here in units of 12 decays per minute. After some time, the amount of radioactivity decreases. For example, a sample 5730 years old has only one half the original activity level, i.e. 6 decays per minute. If a 1 gm sample of material is found to have 45 decays per hour, approximately how old is it? (Note: $^{14}C$ is used in radiocarbon dating, a process by which the age of materials containing carbon can be estimated. W. Libby received the Nobel prize in chemistry in 1960 for developing this technique.)

9.6. **Strontium-90:** Strontium-90 is a radioactive isotope with a half-life of 29 years. If you begin with a sample of 800 units, how long will it take for the amount of radioactivity of the strontium sample to be reduced to

(a) 400 units

(b) 200 units

(c) 1 unit

9.7. **More radioactivity:** The half-life of a radioactive material is 1620 years.

(a) What percentage of the radioactivity will remain after 500 years?

(b) Cobalt 60 is a radioactive substance with half life 5.3 years. It is used in medical application (radiology). How long does it take for 80% of a sample of this substance to decay?

9.8. Assume the atmospheric pressure $y$ at a height $x$ meters above the sea level satisfies the relation $\frac{dy}{dx} = kx$. If one day at a certain location the atmospheric pressures are $760$ and $675$ torr (unit for pressure) at sea level and at $1000$ meters above sea level, respectively, find the value of the atmospheric pressure at $600$ meters above sea level.

9.9. **Population growth and doubling:** A population of animals has a per-capita birth rate of $b = 0.08$ per year and a per-capita death rate of $m = 0.01$ per year. The population density, $P(t)$ is found to satisfy the differential equation

$$\frac{dP(t)}{dt} = bP(t) - mP(t)$$

(a) If the population is initially $P(0) = 1000$, find how big the population will be in 5 years.

(b) When will the population double?

9.10. **Rodent population:** The per capita birthrate of one species of rodent is 0.05 newborns per day. (This means that, on average, each member of the population will result in 5 newborn rodents every 100 days.) Suppose that over the period of 1000 days there are no deaths, and that the initial population of rodents is 250. Write a differential equation for the population size $N(t)$ at time $t$ (in days). Write down the initial condition that $N$ satisfies. Find the solution, i.e. express $N$ as some function of time $t$ that satisfies your differential equation and initial condition. How many rodents will there be after 1 year ?

9.11. **Growth and extinction of microorganisms:**

(a) The population $y(t)$ of a certain microorganism grows continuously and follows an exponential behaviour over time. Its doubling time is found to be 0.27 hours. What differential equation would you use to describe its growth ? (Note: you will have to find the value of the rate constant, $k$, using the doubling time.)

(b) With exposure to ultra-violet radiation, the population ceases to grow, and the microorganisms continuously die off. It is found that the half-life is then 0.1 hours. What differential equation would now describe the population?

9.12. **A bacterial population:** A bacterial population grows at a rate proportional to the population size at time $t$. Let $y(t)$ be the population size at time $t$. By experiment it is determined that the population at $t = 10\ min$ is $15,000$ and at $t = 30\ min$ it is $20,000$.

(a) What was the initial population?

(b) What will the population be at time $t = 60\ min$?

9.13. **Antibiotic treatment:** A colony of bacteria is treated with a mild antibiotic agent so that the bacteria start to die. It is observed that the density of bacteria as a function of time follows the approximate relationship $b(t) = 85e^{-0.5t}$ where $t$ is time in hours. Determine the time it takes for half of the bacteria to disappear (This is called the *half life*.) Find how long it takes for 99% of the bacteria to die.

9.14. **Chemical breakdown:** In a chemical reaction, a substance S is broken down. The concentration of the substance is observed to change at a rate proportional to the current concentration. It was observed that 1 Mole/liter of S decreased to 0.5 Moles/liter in 10 minutes. How long will it take until only 0.25 Moles per liter remain? Until only 1% of the original concentration remains?

9.15. **Two populations:** Two populations are studied. Population **1** is found to obey the differential equation

$$dy_1/dt = 0.2y_1$$

and population **2** obeys

$$dy_2/dt = -0.3y_2$$

where $t$ is time in years.

(a) Which population is growing and which is declining?

(b) Find the doubling time (respectively half-life) associated with the given population.

(c) If the initial levels of the two populations were $y_1(0) = 100$ and $y_2(0) = 10,000$, how big would each population be at time $t$ ?

(d) At what time would the two populations be exactly equal?

9.16. **The human population:** The human population on Earth doubles roughly every 50 years. In October 2000 there were 6.1 billion humans on earth. Determine what the human population would be 500 years later under the uncontrolled growth scenario. How many people would have to inhabit each square kilometer of the planet for this population to fit on earth? (Take the circumference of the earth to be 40,000 km for the purpose of computing its surface area.)

9.17. **First order chemical kinetics:** When chemists say that a chemical reaction follows “first order kinetics”, they mean that the concentration of the reactant at time $t$, i.e. $c(t)$, satisfies an equation of the form $\frac{dc}{dt} = -rc$ where $r$ is a rate constant, here assumed to be positive. Suppose the reaction mixture initially has concentration 1M (“1 molar”) and that after 1 hour there is half this amount.

(a) Find the “half life” of the reactant.

(b) Find the value of the rate constant $r$.

(c) Determine how much will be left after 2 hours.

(d) When will only 10% of the initial amount be left?

9.18. **Fish in two lakes:** Two lakes have populations of fish, but the conditions are quite different in these lakes. In the first lake, the fish population is growing and satisfies the differential equation

$$\frac{dy}{dt} = 0.2y$$

where $t$ is time in years. At time $t = 0$ there were 500 fish in this lake. In the second lake, the population is dying due to pollution. Its population satisfies the differential equation

$$\frac{dy}{dt} = -0.1y,$$

and initially there were 4000 fish in this lake. At what time will the fish populations in the two lakes be identical?

9.19. A barrel initially contains $2\ kg$ of salt dissolved in $20\ L$ of water. If water flows in the rate of $0.4\ L$ per minute and the well-mixed salt water solution flows out at the same rate. How much salt is present after $8$ minutes?

9.20. **A savings account:** You deposit a sum $P$ ("the Principal") in a savings account with an annual **interest rate**, $r$ and make no withdrawals over the first year. If the interest is **compounded annually**, after one year the amount in this account will be

$$A(1) = P + rP = P(1 + r).$$

If the interest is compounded semi-annually (once every 1/2 year), then every 6 months half of the interest is added to your account, i.e.

$$A\left(\frac{1}{2}\right) = P + \frac{r}{2}P = P\left(1 + \frac{r}{2}\right)$$

$$A(1) = A\left(\frac{1}{2}\right)\left(1 + \frac{r}{2}\right) = P\left(1 + \frac{r}{2}\right)\left(1 + \frac{r}{2}\right) = P\left(1 + \frac{r}{2}\right)^2$$

(a) Suppose that you invest $500 in an account with interest rate 4% compounded semi-annually. How much money would you have after 6 months? After 1 year ? After 10 years ? Roughly how long does it take to double your money in this way? How would it differ if the interest was 8% ?

(b) Interest can also be compounded more frequently, for example monthly (i.e. 12 times per year, each time with an increment of $r/12$). Answer the questions posed in part (a) in this case

(c) Is it better to save your money in a bank with 4% interest compounded monthly, or 5% interest compounded annually?

# Chapter 10

# Trigonometric functions

In this chapter we will explore periodic and oscillatory phenomena. The trigonometric functions will be the basis for much of what we construct, and hence, we first introduce these and familiarize ourselves with their properties.

## 10.1 Introduction: angles and circles

Angles can be measured in a number of ways. One way is to assign a value in degrees, with the convention that one complete revolution is represented by $360°$. Why 360? And what is a degree exactly? Is this some universal measure that any intelligent being (say on Mars or elsewhere) would find appealing? Actually, 360 is a rather arbitrary convention that arose historically, and has no particular meaning. We could as easily have had mathematical ancestors that decided to divide circles into 1000 “equal pieces” or 240 or some other subdivision. It turns out that this measure is not particularly convenient, and we will replace it by a more universal quantity.

The universal quantity stems from the fact that circles of all sizes have one common geometric feature: they have the same ratio of circumference to diameter, no matter what their size (or where in the universe they occur). We call that ratio $\pi$, that is

$$\pi = \frac{\text{Circumference of circle}}{\text{Diameter of circle}}$$

The diameter $D$ of a circle is just

$$D = 2r$$

so this naturally leads to the familiar relationship of circumference, $C$, to radius, $r$,

$$C = 2\pi r$$

(But we should not forget that this is merely a *definition* of the constant $\pi$. The more interesting conclusion that develops from this definition is that the area of the circle is $A = \pi r^2$, but we shall see the reason for this later, in the context of areas and integration.)

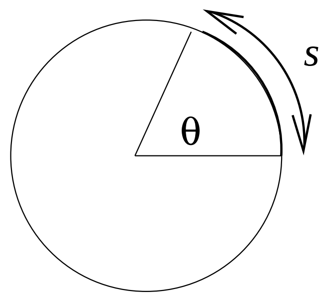



**Figure 10.1.** *The angle $\theta$ in radians is related in a simple way to the radius $R$ of the circle, and the length of the arc $S$ shown.*

From Figure 10.1 we see that there is a correspondence between the angle ($\theta$) subtended in a circle of given radius and the length of arc along the edge of the circle. For a circle of radius R and angle $\theta$ we will define the arclength, $S$ by the relation

$$S = R\theta$$

where $\theta$ is measured in a convenient unit that we will now select. We now consider a circle of radius $R = 1$ (called a *unit circle*) and denote by $s$ a length of arc around the perimeter of this unit circle. In this case, the arc length is

$$S = R\theta = \theta$$

We note that when $S = 2\pi$, the arc consists of the entire perimeter of the circle. This leads us to define the unit called a *radian*: we will identify an angle of $2\pi$ radians with one complete revolution around the circle. In other words, we use the length of the arc in the unit circle to assign a numerical value to the angle that it subtends.

We can now use this choice of unit for angles to assign values to any fraction of a revolution, and thus, to any angle. For example, an angle of $90°$ corresponds to one quarter of a revolution around the perimeter of a unit circle, so we identify the angle $\pi/2$ radians with it. One degree is $1/360$ of a revolution, corresponding to $2\pi/360$ radians, and so on.

To summarize our choice of units we have the following two points:

1. **The length of an arc along the perimeter of a circle of radius $R$ subtended by an angle $\theta$ is $S = R\theta$ where $\theta$ is measured in radians.**
2. **One complete revolution, or one full cycle corresponds to an angle of $2\pi$ radians**.

It is easy to convert between degrees and radians if we remember that $360°$ corresponds to $2\pi$ radians. For example, $180°$ then corresponds to $\pi$ radians, $90°$ to $\pi/2$ radians, etc.

## 10.2 Defining the basic trigonometric functions

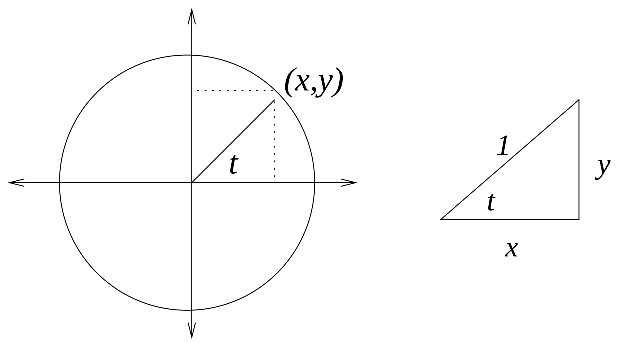



**Figure 10.2.** *Shown above is the circle of radius 1, $x^2 + y^2 = 1$. The radius vector that ends at the point $(x, y)$ subtends an angle $t$ (radians) with the $x$ axis. The triangle is also shown enlarged to the right, where the lengths of all three sides is labeled. The trigonometric functions are just ratios of two sides of this triangle.*

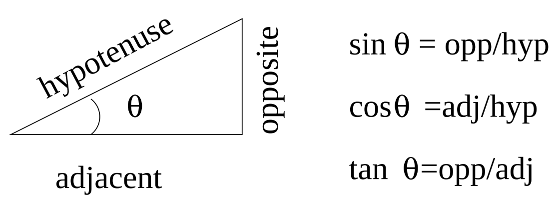



**Figure 10.3.** *Review of the relation between ratios of side lengths (in a right triangle) and trigonometric functions of the associated angle.*

Consider a point $(x, y)$ on a circle of radius 1, and let $t$ be some angle (measured in radians) formed by the $x$ axis and the radius vector to the point $(x, y)$ as shown in Figure 10.2.

We will define two new functions, sine and cosine (abbreviated $\sin$ and $\cos$) as follows:

$$\sin(t) = \frac{y}{1} = y, \quad \cos(t) = \frac{x}{1} = x$$

That is, the function sine tracks the $y$ coordinate of the point as it moves around the unit circle, and the function cosine tracks its $x$ coordinate. (Remark: this agrees with previous definitions of these trigonometric quantities as shown in Figure 10.3 as the opposite over hypotenuse and adjacent over hypotenuse in a right angle triangle that you may have encountered in high school. The hypotenuse in our diagram is simply the radius of the circle, which is 1 by assumption.)

| degrees | radians | $\sin(t)$ | $\cos(t)$ | $\tan(t)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 |
| 30 | $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| 45 | $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 |
| 60 | $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| 90 | $\frac{\pi}{2}$ | 1 | 0 | $\infty$ |

**Table 10.1.** *Values of the sines, cosines, and tangent for the standard angles.*

## 10.3 Properties of the trigonometric functions

We now explore the consequences of these definitions:

### Values of sine and cosine

- The radius of the circle is 1. This means that the $x$ coordinate cannot be larger than 1 or smaller than -1. Same holds for the $y$ coordinate. Thus the functions $\sin(t)$ and $\cos(t)$ are always swinging between -1 and 1. ($-1 \leq \sin(t) \leq 1$ and $-1 \leq \cos(t) \leq 1$ for all $t$). The peak (maximum) value of each function is 1, the minimum is -1, and the average value is 0.

- When the radius vector points along the $x$ axis, the angle is $t = 0$ and we have $y = 0, x = 1$. This means that $\cos(0) = 1, \sin(0) = 0$.

- When the radius vector points up the $y$ axis, the angle is $\pi/2$ (corresponding to one quarter of a complete revolution), and here $x = 0, y = 1$ so that $\cos(\pi/2) = 0, \sin(\pi/2) = 1$.

- Using simple geometry, we can also determine the lengths of all sides, and hence the ratios of the sides in a few particularly simple triangles, namely triangles (in which all angles are $60°$), and right triangles with two equal angles of $45°$. These types of calculations (omitted here) lead to some easily determined values for the sine and cosine of such special angles. These values are shown in the Table 10.1.

### Connection between sine and cosine

- The two functions, sine and cosine depict the same underlying motion, viewed from two perspectives: $\cos(t)$ represents the projection of the circularly moving point onto the $x$ axis, while $\sin(t)$ is the projection of that point onto the $y$ axis. In this sense, the functions are a pair of twins, and we can expect many relationships to hold between them.

- The cosine has its largest value at the beginning of the cycle, when $t = 0$ (since $\cos(0) = 1$), while the other the sine its peak value a little later, ($\sin(\pi/2) = 1$).

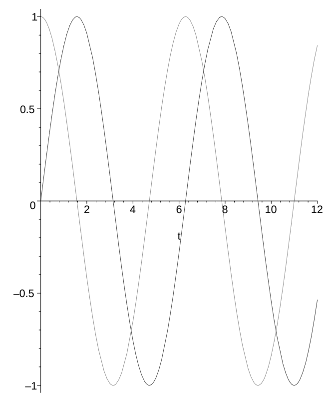



**Figure 10.4.** *The functions* $\sin(t)$ *and* $\cos(t)$ *are periodic, that is, they have a basic pattern that repeats. The two functions are also related, since one is just a copy of the other, shifted along the* $x$ *axis.*

Throughout their circular race, the sine function is $\pi/2$ radians ahead of the cosine i.e.

$$\cos(t) = \sin(t + \frac{\pi}{2}).$$

- The point $(x, y)$ is on a circle of radius 1, and, thus, its coordinates satisfy

$$x^2 + y^2 = 1$$

This implies that

$$\sin^2(t) + \cos^2(t) = 1$$

for any angle $t$. This is an important relation, (also called an *identity* between the two trigonometric functions, and one that we will use quite often.

### Periodicity: the pattern repeats

- A function is said to be **periodic** if its graph is repeated over and over again. For example, if the basic shape of the graph occurs in an interval of length $T$ on the $t$ axis, and this shape is repeated, then it would be true that

$$f(t) = f(t + T).$$

In this case we call $T$ the **period** of the function. All the trigonometric functions are periodic.

- The point $(x, y)$ in Figure 10.2 will repeat its trajectory every time a revolution around the circle is complete. This happens when the angle $t$ completes one full cycle of $2\pi$ radians. Thus, as expected, the trigonometric functions are periodic, that is

$$\sin(t) = \sin(t + 2\pi),$$

$$\cos(t) = \cos(t + 2\pi).$$

We say that the period is $T = 2\pi$ radians.

We can make other observations about the same two functions. For example, by noting the symmetry of the functions relative to the origin, we can see that $\sin(t)$ is an odd function and the $\cos(t)$ is an even function. This follows from the fact that for a negative angle (i.e. an angle clockwise from the x axis) the sine flips sign while the cosine does not.

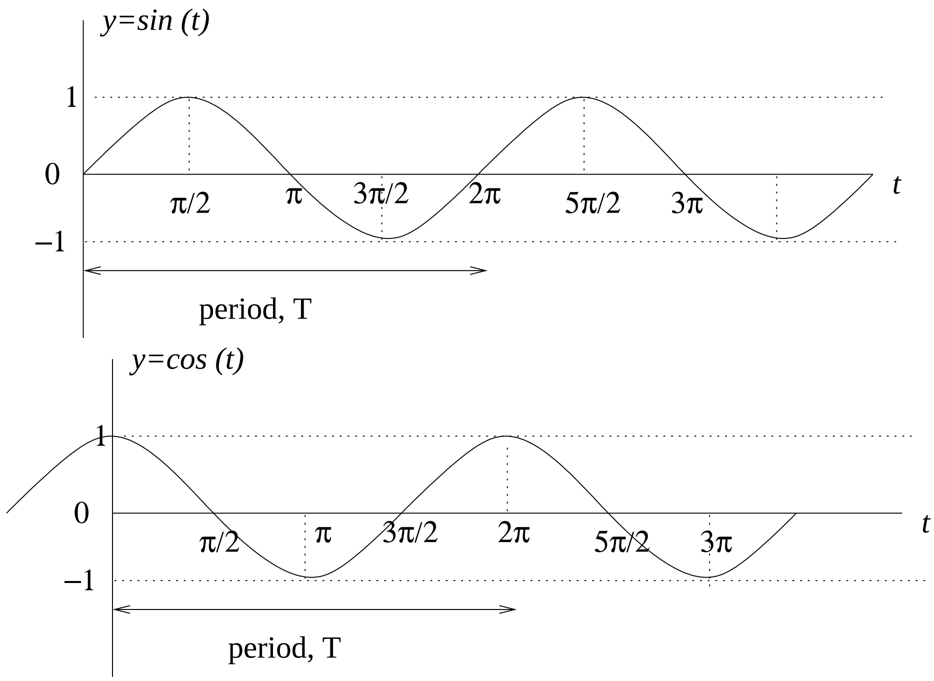



**Figure 10.5.** *Periodicity of the sine and cosine. Note that the two curves are just shifted versions of one another.*

## 10.4 Phase, amplitude, and frequency

We have already learned how the appearance of functions changes when we shift their graph in one direction or another, scale one of the axes, and so on. Thus it will be easy to follow the basic changes in shape of a typical trigonometric function.

A function of the form

$$y = f(t) = A\sin(\omega t)$$

has both its $t$ and $y$ axes scaled. The constant $A$, referred to as the *amplitude* of the graph, scales the $y$ axis so that the oscillation swings between a low value of $-A$ and a high value of $A$. The constant $\omega$, called the **frequency**, scales the $t$ axis. This results in crowding

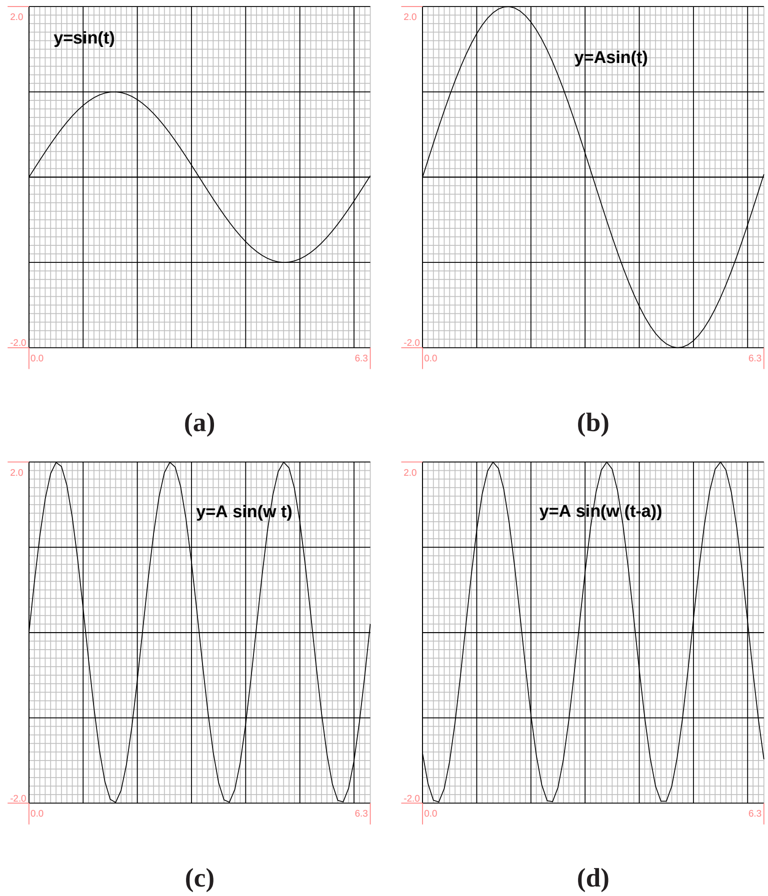



**Figure 10.6.** *Graphs of the functions (a)* $y = \sin(t)$, *(b)* $y = A\sin(t)$ *for* $A > 1$, *(c)* $y = A\sin(\omega t)$ *for* $\omega > 1$, *(d)* $y = A\sin(\omega(t-a))$.

together of the peaks and valleys (if $\omega > 1$) or stretching them out (if $\omega < 1$). One full cycle is completed when

$$\omega t = 2\pi$$

and this occurs at time

$$t = \frac{2\pi}{\omega}.$$

We will use the symbol $T$, to denote this special time, and we refer to $T$ as the *period*. We note the connection between frequency and period:

$$\omega = \frac{2\pi}{T},$$

$$T = \frac{2\pi}{\omega}.$$

If we examine a graph of function

$$y = f(t) = A\sin(\omega(t-a))$$

we find that the graph has been shifted in the positive $t$ direction by $a$. We note that at time $t = a$, the value of the function is

$$y = f(t) = A\sin(\omega(a-a)) = A\sin(0) = 0.$$

This tells us that the cycle "starts" with a delay, i.e. the value of $y$ goes through zero when when $t = a$.

Another common variant of the same function can be written in the form

$$y = f(t) = A\sin(\omega t - \phi).$$

Here $\phi$ is called the *phase shift* of the oscillation. Comparing the above two related forms, we see that they are the same if we identify $\phi$ with $\omega a$. The phase shift, $\phi$ is considered to be a quantity without units, whereas the quantity $a$ has units of time, same as $t$. When $\phi = 2\pi$, (which is the same as the case that $a = 2\pi/\omega$, the graph has been moved over to the right by one full period. (Naturally, when the graph is so moved, it looks the same as it did originally, since each cycle is the same as the one before, and same as the one after.)

Some of the scaled, shifted, sine functions described here are shown in Figure 10.6.

## 10.5 Rhythmic processes

Many natural phenomena are cyclic. It is often convenient to represent such phenomena with one or another simple periodic functions, and sine and cosine can be adapted for the purpose. The idea is to pick the right function , the right frequency (or period), the amplitude, and possibly the phase shift, so as to represent the desired behaviour.

To select one or another of these functions, it helps to remember that cosine starts a cycle (at $t = 0$) at its peak value, while sine starts the cycle at $0$, i.e., at its average value. A function that starts at the lowest point of the cycle is $-\cos(t)$. In most cases, the choice of function to use is somewhat arbitrary, since a phase shift can correct for the phase at which the oscillation starts.

Next, we pick a constant $\omega$ such that the trigonometric function $\sin(\omega t)$ (or $\cos(\omega t)$) has the correct period. Given a period for the oscillation, $T$, recall that the corresponding frequency is simply $\omega = \frac{2\pi}{T}$. We then select the amplitude, and horizontal and vertical shifts to complete the mission. The examples below illustrate this process.

**Example 10.1 (Daylight hours:)** In Vancouver, the shortest day (8 hours of light) occurs around December 22, and the longest day (16 hours of light) is around June 21. Approximate the cyclic changes of daylight through the season using the sine function. ■

**Solution:** On Sept 21 and March 21 the lengths of day and night are equal, and then there are 12 hours of daylight. (Each of these days is called an *equinox*). Suppose we call identify March 21 as the beginning of a yearly day-night length cycle. Let $t$ be time in days beginning on March 21. One full cycle takes a year, i.e. 365 days. The period of the function we want is thus

$$T = 365$$

and its frequency is

$$\omega = 2\pi/365.$$

Daylight shifts between the two extremes of 8 and 16 hours: i.e. $12 \pm 4$ hours. This means that the amplitude of the cycle is 4 hours. The oscillation take place about the average value of 12 hours. We have decided to start a cycle on a day for which the number of daylight hours is the average value (12). This means that the sine would be most appropriate, so the function that best describes the number of hours of daylight at different times of the year is:

$$D(t) = 12 + 4\sin\left(\frac{2\pi}{365}t\right)$$

where $t$ is time in days and $D$ the number of hours of light.

**Example 10.2 (Hormone levels:)** The level of a certain hormone in the bloodstream fluctuates between undetectable concentration at 7:00 and 100 ng/ml at 19:00 hours. Approximate the cyclic variations in this hormone level with the appropriate periodic trigonometric function. Let $t$ represent time in hours from 0:00 hrs through the day. ■

**Solution:** We first note that it takes one day (24 hours) to complete a cycle. This means that the period of oscillation is 24 hours, so that the frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24} = \frac{\pi}{12}.$$

The variation in the level of hormone is between 0 and 100 ng/ml, which can be expressed as $50 \pm 50$ ng/ml. (The trigonometric functions are symmetric cycles, and we are here finding both the average value about which cycles occur and the amplitude of the cycles.) We could consider the time midway between the low and high points, namely 13:00 hours as the point corresponding to the upswing at the start of a cycle of the sine function. (See Figure 10.7 for the sketch.) Thus, if we use a sine to represent the oscillation, we should shift it by 13 hrs to the left.

Assembling these observations, we obtain the level of hormone, $H$ at time $t$ in hours:

$$H(t) = 50 + 50\sin\left(\frac{\pi}{12}(t-13)\right).$$

In the expression above, the number 13 represents a shift along the time axis, and carries units of time. We can express this same function in the form

$$H(t) = 50 + 50\sin\left(\frac{\pi t}{12} - \frac{13\pi}{12}\right).$$

In this version, the quantity

$$\phi = \frac{13\pi}{12}$$

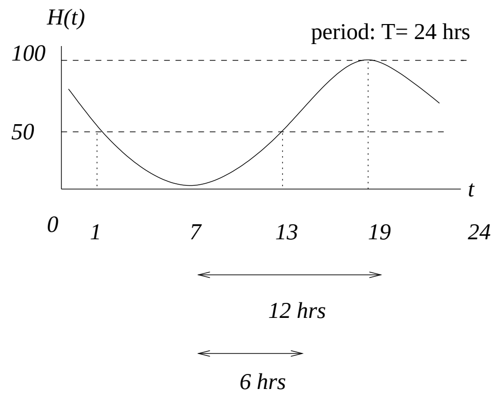



**Figure 10.7.** *Hormonal cycles. The full cycle is 24 hrs. The level $H(t)$ swings between 0 and 100 ng. From the given information, we see that the average level is 50 ng, and that the origin of a representative sine curve should be at $t = 13$ (i.e. 1/4 of the cycle which is 6 hrs past the time point $t = 7$) to depict this cycle.*

is what we have referred to as a phase shift. (This represents the point on the $2\pi$ cycle at which the function begins when we plug in $t = 0$.)

In selecting the periodic function to use for this example, we could have made other choices. For example, the same periodic can be represented by any of the functions listed below:

$$H(t) = 50 - 50\sin\left(\frac{\pi}{12}(t-1)\right),$$

$$H(t) = 50 + 50\cos\left(\frac{\pi}{12}(t-19)\right),$$

$$H(t) = 50 - 50\cos\left(\frac{\pi}{12}(t-7)\right).$$

All these functions have the same values, the same amplitudes, and the same periods.

**Example 10.3 (Phases of the moon:)** A cycle of waxing and waning moon takes 29.5 days approximately. Construct a periodic function to describe the changing phases, starting with a "new moon" (totally dark) and ending one cycle later. ■

**Solution:** The period of the cycle is $T = 29.5$ days, so

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{29.5}.$$

For this example, we will use the cosine function, for practice. Let $P(t)$ be the fraction of the moon showing on day $t$ in the cycle. Then we should construct the function so that $0 < P < 1$, with $P = 1$ in mid cycle (see Figure 10.8). The cosine function swings between the values -1 and 1. To obtain a positive function in the desired range for $P(t)$, we will add a constant and scale the cosine as follows:

$$\frac{1}{2}[1 + \cos(\omega t)].$$

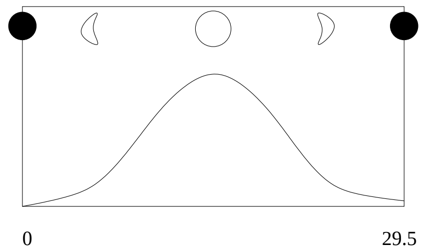



**Figure 10.8.** *Periodic moon phases*

This is not quite right, though because at $t = 0$ this function takes the value 1, rather than 0, as shown in Figure 10.8. To correct this we can either introduce a phase shift, i.e. set

$$P(t) = \frac{1}{2}[1 + \cos(\omega t + \pi)].$$

(Then when $t = 0$, we get $P(t) = 0.5[1 + \cos\pi] = 0.5[1 - 1] = 0$.) or we can write

$$P(t) = \frac{1}{2}[1 - \cos(\omega t + \pi)],$$

which achieves the same result.

## 10.6 Other trigonometric functions

Although we shall mostly be concerned with the two basic functions described above, several others are historically important and are encountered frequently in integral calculus. These include the following:

$$\tan(t) = \frac{\sin(t)}{\cos(t)}, \quad \cot(t) = \frac{1}{\tan(t)},$$

$$\sec(t) = \frac{1}{\cos(t)}, \quad \csc(t) = \frac{1}{\sin(t)}.$$

The identity

$$\sin^2(t) + \cos^2(t) = 1$$

then leads to two others of similar form. Dividing each side of the above relation by $\cos^2(t)$ yields

$$\tan^2(t) + 1 = \sec^2(t)$$

whereas division by $\sin^2(t)$ gives us

$$1 + \cot^2(t) = \csc^2(t).$$

These will be important for simplifying expressions involving the trigonometric functions, as we shall see.

### Law of cosines

This law relates the cosine of an angle to the lengths of sides formed in a triangle. (See figure 10.9.)

$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$

where the side of length $c$ is opposite the angle $\theta$.

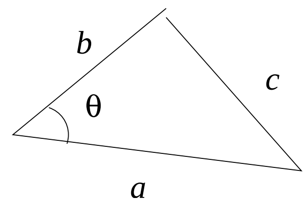



**Figure 10.9.** *Law of cosines states that* $c^2 = a^2 + b^2 - 2ab\cos(\theta)$.

Here are other important relations between the trigonometric functions that should be remembered. These are called trigonometric identities:

### Angle sum identities

The trigonometric functions are nonlinear. This means that, for example, the sine of the sum of two angles is *not* just the sum of the two sines. One can use the law of cosines and other geometric ideas to establish the following two relationships:

$$\boxed{\begin{aligned} \sin(A+B) &= \sin(A)\cos(B) + \sin(B)\cos(A) \\ \cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B) \end{aligned}}$$

These two identities appear in many calculations, and will be important for computing derivatives of the basic trigonometric formulae.

### Related identities

The identities for the sum of angles can be used to derive a number of related formulae. For example, by replacing $B$ by $-B$ we get the angle difference identities:

$$\boxed{\begin{aligned} \sin(A-B) &= \sin(A)\cos(B) - \sin(B)\cos(A) \\ \cos(A-B) &= \cos(A)\cos(B) + \sin(A)\sin(B) \end{aligned}}$$

By setting $\theta = A = B$ in these we find the subsidiary double angle formulae:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

and these can also be written in the form

$$2\cos^2(\theta) = 1 + \cos(2\theta)$$

$$2\sin^2(\theta) = 1 - \cos(2\theta).$$

(The latter four are quite useful in integration methods.)

## 10.7 Limits involving the trigonometric functions

Before we compute derivatives of the sine and cosine functions using the definition of the derivative, we will need to specify two limits that will be needed in those calculations.

If we zoom in on the graph of the sine function close to the origin, we will see a curve resembling a straight line with slope 1, i.e. the function $y = \sin(t)$ will look quite similar to the graph of $y = t$ close to $t = 0$. This is shown in the sequence of graphs in Figure 3.2. This means that, for small $t$

$$\sin(t) \approx t.$$

We can restate this as

$$\sin(h) \approx h$$

or as

$$\frac{\sin(h)}{h} \approx 1.$$

It turns out that this approximation becomes finer as $h$ decreases, i.e

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1.$$

This is a very important limit, and will be used in many applications.

A similar analysis of the graph of the cosine function, shown in Figure 10.10, will lead us to conclude that the related limit is

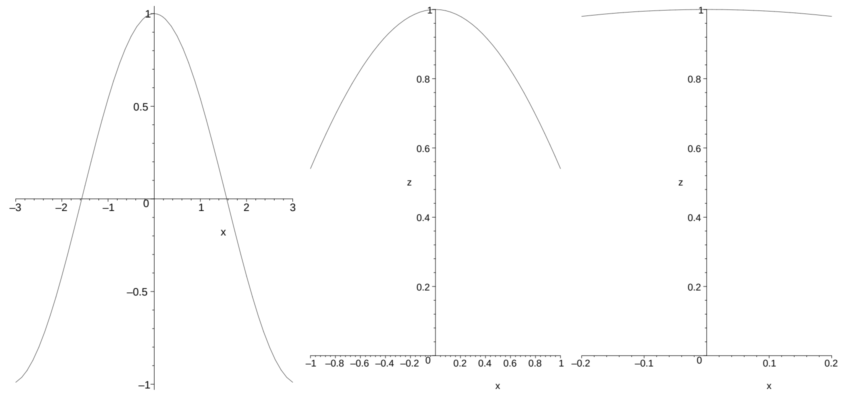
$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0.$$

We can now apply these to computing derivatives.

### 10.7.1 Derivatives of the trigonometric functions

Let $y = f(x) = \sin(x)$ be the function to differentiate, where $x$ is now the independent variable (previously called $t$). Below, we use the definition of the derivative to compute the derivative of this function.

**Example 10.4 (Derivative of $\sin(x)$:)** Compute the derivative of $y = \sin(x)$ using the definition of the derivative. ∎



**(a)** **(b)** **(c)**

**Figure 10.10.** *Zooming in on the graph of* $y = \cos(x)$ *at* $x = 0$.

**Solution:**

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
\frac{d \sin(x)}{dx} &= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \\
&= \lim_{h \to 0} \left( \sin(x) \frac{\cos(h) - 1}{h} + \cos(x) \frac{\sin(h)}{h} \right) \\
&= \sin(x) \left( \lim_{h \to 0} \frac{\cos(h) - 1}{h} \right) + \cos(x) \left( \lim_{h \to 0} \frac{\sin(h)}{h} \right) \\
&= \cos(x)
\end{aligned}
$$

Observe that the limits described in the preceding section were used in getting to our final result.

A similar calculation using the function $\cos(x)$ leads to the result

$$\frac{d \cos(x)}{dx} = -\sin(x).$$

(The same two limits appear in this calculation as well.)

We can now calculate the derivative of the any of the other trigonometric functions using the quotient rule.

**Example 10.5 (Derivative of the function $\tan(x)$:)** Compute the derivative of $y = \tan(x)$. ■

| $y = f(x)$ | $f'(x)$ |
|---|---|
| $\sin(x)$ | $\cos(x)$ |
| $\cos(x)$ | $-\sin(x)$ |
| $\tan(x)$ | $\sec^2(x)$ |
| $\csc(x)$ | $-\csc(x)\cot(x)$ |
| $\sec(x)$ | $\sec(x)\tan(x)$ |
| $\cot(x)$ | $-\csc^2(x)$ |

**Table 10.2.** *Derivatives of the trigonometric functions*

**Solution:**

$$\frac{d\tan(x)}{dx} = \frac{[\sin(x)]'\cos(x) - [\cos(x)]'\sin(x)}{\cos^2(x)}$$

Using the recently found derivatives for the sine and cosine, we have

$$\frac{d\tan(x)}{dx} = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)}.$$

But the numerator of the above can be simplified using the trigonometric identity, leading to

$$\frac{d\tan(x)}{dx} = \frac{1}{\cos^2(x)} = \sec^2(x).$$

The derivatives of the six trigonometric functions are given in the table below. The reader may wish to practice the use of the quotient rule by verifying one or more of the derivatives of the relatives $\csc(x)$ or $\sec(x)$. In practice, the most important functions are the first three, and their derivatives should be remembered, as they are frequently encountered in practical applications.

## 10.8 Trigonometric related rates

The examples in this section will allow us to practice chain rule applications using the trigonometric functions. We will discuss a number of problems, and show how the basic facts described in this chapter appear in various combinations to arrive at desired results.

**Example 10.6 (A point on a circle:)** A point moves around the rim of a circle of radius 1 so that the angle $\theta$ subtended by the radius vector to that point changes at a constant rate,

$$\theta = \omega t,$$

where $t$ is time. Determine the rate of change of the $x$ and $y$ coordinates of that point. ∎

**Solution:** We have $\theta(t), x(t)$, and $y(t)$ all functions of $t$. The fact that $\theta$ is proportional to $t$ means that

$$\frac{d\theta}{dt} = \omega.$$

The $x$ and $y$ coordinates of the point are related to the angle by

$$x(t) = \cos(\theta(t)) = \cos(\omega t),$$

$$y(t) = \sin(\theta(t)) = \sin(\omega t).$$

This implies (by the chain rule) that

$$\frac{dx}{dt} = \frac{d\cos(\theta)}{d\theta}\frac{d\theta}{dt},$$

$$\frac{dy}{dt} = \frac{d\sin(\theta)}{d\theta}\frac{d\theta}{dt}.$$

Performing the required calculations, we have

$$\frac{dx}{dt} = -\sin(\theta)\omega,$$

$$\frac{dy}{dt} = \cos(\theta)\omega.$$

We will see some interesting consequences of this in a later section.

**Example 10.7 (Runners on a circular track:)** Two runners start at the same position (call it $x = 0$) on a circular race track of length 400 meters. Joe Runner takes 50 sec, while Michael Johnson takes 43.18 sec to complete the 400 meter race. Determine the rate of change of the angle formed between the two runners and the center of the track, assuming that the runners are running at a constant rate. ■

**Solution:** The track is 400 meters in length (total). Joe completes one cycle around the track ($2\pi$ radians) in 50 sec, while Michael completes a cycle in 43.18 sec. (This means that Joe has period of $T = 50$ sec, and a frequency of $\omega_1 = 2\pi/T = 2\pi/50$ radians per sec. Similarly, Michael's period is $T = 43.18$ sec and his frequency is $\omega_2 = 2\pi/T = 2\pi/43.18$ radians per sec. From this, we find that

$$\frac{d\theta_J}{dt} = \frac{2\pi}{50} = 0.125 \text{ radians per sec}$$

$$\frac{d\theta_M}{dt} = \frac{2\pi}{43.18} = 0.145 \text{ radians per sec}$$

Thus the angle between the runners, $\theta_M - \theta_J$ changes at the rate

$$\frac{d(\theta_M - \theta_J)}{dt} = 0.145 - 0.125 = 0.02 \text{ radians per sec}$$

**Example 10.8 (Simple law of cosines:)** Consider the triangle as shown in Figure 10.9. Suppose that the angle $\theta$ increases at a constant rate, i.e. $d\theta/dt = k$. If the sides $a = 3$, $b = 4$, are of constant length, determine the rate of change of the length $c$ opposite this angle at the instant that $c = 5$. ■

**Solution:** Let $a, b, c$ be the lengths of the three sides, with $c$ the length of the side opposite angle $\theta$. The law of cosines states that

$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

We identify the changing quantities by writing this relation in the form

$$c^2(t) = a^2 + b^2 - 2ab\cos(\theta(t))$$

so it is evident that only $c$ and $\theta$ will vary with time, while $a, b$ remain constant. We are also told that

$$\frac{d\theta}{dt} = k.$$

Differentiating and using the chain rule leads to:

$$2c\frac{dc}{dt} = -2ab\frac{d\cos(\theta)}{d\theta}\frac{d\theta}{dt}$$

But $d\cos(\theta)/d\theta = -\sin(\theta)$ so that

$$\frac{dc}{dt} = -\frac{ab}{c}(-\sin(\theta))\frac{d\theta}{dt} = \frac{ab}{c}k\sin(\theta).$$

We now note that at the instant in question, $a = 3, b = 4, c = 5$, forming a Pythagorean triangle in which the angle opposite $c$ is $\theta = \pi/2$. We can see this fact using the law of cosines, and noting that

$$c^2 = a^2 + b^2 - 2ab\cos(\theta), \quad 25 = 9 + 16 - 24\cos(\theta).$$

This implies that $0 = -24\cos(\theta)$, $\cos(\theta) = 0$ so that $\theta = \pi/2$. Substituting these into our result for the rate of change of the length $c$ leads to

$$\frac{dc}{dt} = \frac{ab}{c}k = \frac{3 \cdot 4}{5}k.$$

**Example 10.9 (Clocks:)** Find the rate of change of the angle between the minute hand and hour hand on a clock. $\blacksquare$

**Solution:** We will call $\theta_1$ the angle that the minute hand subtends with the $x$ axis (horizontal direction) and $\theta_2$ the angle that the hour hand makes with the same axis.

If our clock is working properly, each hand will move around at a constant rate. The hour hand will trace out one complete revolution ($2\pi$ radians) every 12 hours, while the minute hand will complete a revolution every hour. Both hands move in a clockwise direction, which (by convention) is towards negative angles. This means that

$$\frac{d\theta_1}{dt} = -2\pi \text{ radians per hour},$$

$$\frac{d\theta_2}{dt} = -\frac{2\pi}{12} \text{ radians per hour}.$$

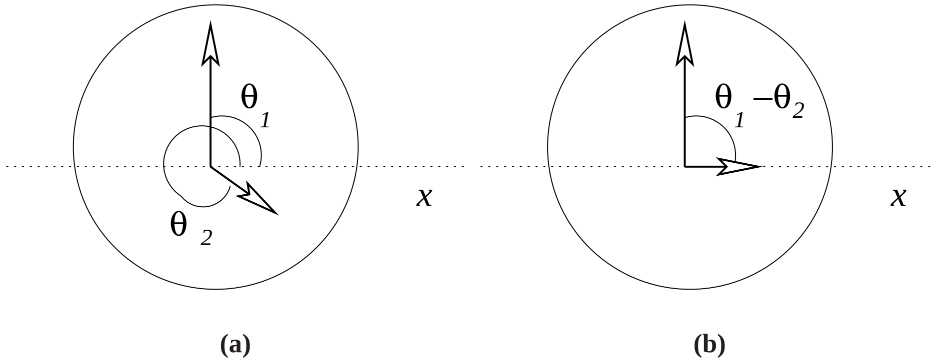



**Figure 10.11.** *Figure for Examples 10.9 and 10.10.*

The angle between the two hands is the difference of the two angles, i.e.

$$\theta = \theta_1 - \theta_2$$

Thus,

$$\frac{d\theta}{dt} = \frac{d}{dt}(\theta_1 - \theta_2) = \frac{d\theta_1}{dt} - \frac{d\theta_2}{dt} = -2\pi + \frac{2\pi}{12}$$

Thus, we find that the rate of change of the angle between the hands is

$$\frac{d\theta}{dt} = -2\pi\frac{11}{12} = -\pi\frac{11}{6}.$$

**Example 10.10 (Clocks, continued:)** Suppose that the length of the minute hand is 4 cm and the length of the hour hand is 3 cm. At what rate is the *distance* between the hands changing when it is 3:00 o'clock? ■

**Solution:** We use the law of cosines to give us the rate of change of the desired distance. We have the triangle shown in figure 10.11 in which side lengths are $a = 3$, $b = 4$, and $c(t)$ opposite the angle $\theta(t)$. From the previous example, we have

$$\frac{dc}{dt} = \frac{ab}{c}\sin(\theta)\frac{d\theta}{dt}.$$

At precisely 3:00 o'clock, the angle in question is $\theta = \pi/2$ and it can also be seen that the Pythagorean triangle $abc$ leads to

$$c^2 = a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25$$

so that $c = 5$. We found from our previous analysis that $d\theta/dt = \frac{11}{6}\pi$. Using this information leads to:

$$\frac{dc}{dt} = \frac{3 \cdot 4}{5}\sin(\pi/2)(-\frac{11}{6}\pi) = -\frac{22}{5}\pi \text{ cm per hr}$$

The negative sign indicates that at this time, the distance between the two hands is decreasing.

**Example 10.11 (Visual angles:)** In the triangle shown in Figure 10.12, an object of height $s$ is moving towards an observer. Its distance from the observer at some instant is labeled $x(t)$ and it approaches at some constant speed, $v$. Determine the rate of change of the angle $\theta(t)$and how it depends on speed, size, and distance of the object. Often $\theta$ is called a visual angle, since it represents the angle that an image subtends on the retina of the observer. A more detailed example of this type is discussed in the next chapter. ■

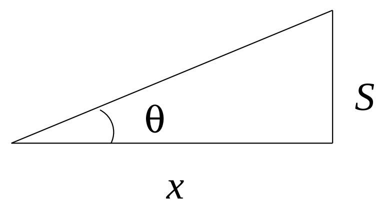



**Figure 10.12.** *A visual angle.*

**Solution:** We are given the information that the object approaches at some constant speed, $v$. This means that

$$\frac{dx}{dt} = -v.$$

(The minus sign means that the distance $x$ is decreasing.) Using the trigonometric relations, we see that

$$\tan(\theta) = \frac{s}{x}.$$

If the size, $s$, of the object is constant, then the changes with time imply that

$$\tan(\theta(t)) = \frac{s}{x(t)}.$$

We differentiate both sides of this equation with respect to $t$, and obtain

$$\frac{d\tan(\theta)}{d\theta}\frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{s}{x(t)}\right)$$

$$\sec^2(\theta)\frac{d\theta}{dt} = -s\frac{1}{x^2}\frac{dx}{dt}$$

so that

$$\frac{d\theta}{dt} = -s\frac{1}{\sec^2(\theta)}\frac{1}{x^2}\frac{dx}{dt}$$

We can use the trigonometric identity

$$\sec^2(\theta) = 1 + \tan^2(\theta)$$

to express our answer in terms only of the size, $s$, the distance of the object, $x$ and the speed:

$$\sec^2(\theta) = 1 + \left(\frac{s}{x}\right)^2 = \frac{x^2+s^2}{x^2}$$

so

$$\frac{d\theta}{dt} = -s\frac{x^2}{x^2+s^2}\frac{1}{x^2}\frac{dx}{dt} = \frac{S}{x^2+s^2}v.$$

(Two minus signs cancelled above.) Thus, the rate of change of the visual angle is $sv/(x^2+s^2)$. This calculation has some interesting implications for reactions to visual stimuli. We will explore some of these implications later on.

## 10.9 Trigonometric functions and differential equations

In this section, we will show the following relationship between trigonometric functions and differential equations:

> The functions
>
> $$x(t) = \cos(\omega t), \quad y(t) = \sin(\omega t)$$
>
> satisfy a pair of differential equations,
>
> $$\frac{dx}{dt} = -\omega y, \quad \frac{dy}{dt} = \omega x.$$

> The functions
>
> $$x(t) = \cos(\omega t), \quad y(t) = \sin(\omega t)$$
>
> also satisfy a related differential equation with a second derivative
>
> $$\frac{d^2x}{dt^2} = -\omega^2 x.$$

To show that these statements are true, we return to an example explored in the previous section: we considered a point moving around a unit circle at a constant angular rate, $\omega$, so that

$$\frac{d\theta}{dt} = \omega.$$

We then considered the $x$ and $y$ coordinates of the point,

$$x(t) = \cos(\theta(t)) = \cos(\omega t), \quad y(t) = \sin(\theta(t)) = \sin(\omega t),$$

and showed (using the chain rule) that these satisfy

$$\frac{dx}{dt} = -\sin(\theta)\omega,$$

$$\frac{dy}{dt} = \cos(\theta)\omega.$$

These relationships can also be written in the form

$$\frac{dx}{dt} = -\omega y,$$

$$\frac{dy}{dt} = \omega x,$$

where we have used the definitions of sine and cosine in terms of $x$ and $y$.

The above pair of equations describe the fact that the derivative of each of these trig functions, $x(t)$ and $y(t)$, is related to the other function. These two equations fall into a class we have already explored, namely differential equations.

We have just shown that the functions $x(t) = \cos(\omega t)$ and $y(t) = \sin(\omega t)$ also have a special connection to differential equations. In fact they are linked by the pair of interconnected equations displayed here as our result. Each equation involves the derivative of one or the other of the trig functions, and says that this derivative is just a constant multiple of the other function. (In a way, we already knew this relationship holds, since our table of derivatives illustrates the connection between $\sin$ and $\cos$. However, we here see the idea in a setting that reminds us of similar observations made for exponential functions. (Such interdependent differential equations are also referred to as a set of coupled equations, since each one contains variables that appear in the other.)

By differentiating both sides of the first equation, we find that

$$\frac{d^2x}{dt^2} = -\omega \frac{dy}{dt},$$

and now using the second equation, we simplify to

$$\frac{d^2x}{dt^2} = -\omega(\omega x),$$

finally obtaining

$$\frac{d^2x}{dt^2} = -\omega^2 x.$$

The reader can show that $y$ satisfies the same type of equation, namely that

$$\frac{d^2y}{dt^2} = -\omega^2 y.$$

This means that each of the above trigonometric functions satisfy a new type of differential equation containing a second derivative.

Students of physics will here recognize the equation that governs the behaviour of a **harmonic oscillator**, and will see the connection between the circular motion of our point on the circle, and the differential equation for periodic motion.

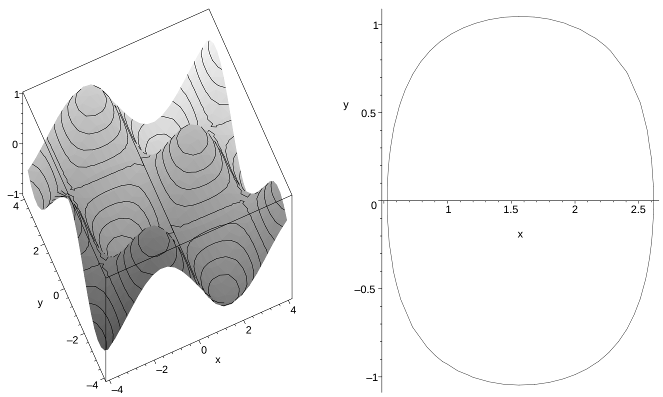
## 10.10 Additional examples

This section is dedicated to practicing implicit differentiation in the context of trigonometric functions.

A surface that looks like an "egg carton" can be described by the function

$$z = \sin(x)\cos(y)$$

See Figure 10.13(a) for the shape of this surface.



**(a)** **(b)**

**Figure 10.13.** *(a) The surface* $\sin(x)\cos(y) = \frac{1}{2}$ *(b) One level curve for this surface. Note that the scales are not the same for parts (a) and (b).*

Suppose we slice though the surface at various levels. We would then see a collection of circular contours, as found on a topographical map of a mountain range. Such contours are called "level curves", and some of these can be seen in Figure 10.13. We will here be interested in the contours formed at some specific height, e.g. at height $z = 1/2$. This *set* of curves can be described by the equation:

$$\sin(x)\cos(y) = \frac{1}{2}.$$

Let us look at one of these, e.g. the curve shown in Figure 10.13(b). This is just one of the contours, namely the one located in the portion of the graph for $-1 < y < 1$, $0 < x < 3$. We practice implicit differentiation for this curve, i.e. we find the slope of tangent lines to this curve.

**Example 10.12 (Implicit differentiation:)** Find the slope of the tangent line to a point on the curve shown in Figure 10.13(b). ■

**Solution:** Differentiating, we obtain:

$$\frac{d}{dx}\left(\sin(x)\cos(y)\right) = \frac{d}{dx}\left(\frac{1}{2}\right)$$

$$\frac{d\sin(x)}{dx}\cos(y) + \sin(x)\frac{d\cos(y)}{dx} = 0$$

$$\cos(x)\cos(y) + \sin(x)(-\sin(y))\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{\cos(x)\cos(y)}{\sin(x)\sin(y)} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\tan(x)\tan(y)}.$$

We can now determine the slope of the tangent lines to the curve at points of interest.

**Example 10.13** Find the slope of the tangent line to the same level curve at the point $x = \frac{\pi}{2}$. $\blacksquare$

**Solution:** At this point, $\sin(x) = \sin(\pi/2) = 1$ which means that the corresponding $y$ coordinate of a point on the graph satisfies $\cos(y) = 1/2$ so one value of $y$ is $y = \pi/3$. (There are other values, for example at $-\pi/3$ and at $2\pi n \pm \pi/3$, but we will not consider these here.) Then we find that

$$\frac{dy}{dx} = \frac{1}{\tan(\pi/2)\tan(\pi/3)}.$$

But $\tan(\pi/2) = \infty$ so that the ratio above leads to $\frac{dy}{dx} = 0$. The tangent line is horizontal as it goes though the point $(\pi/2, \pi/3)$ on the graph.

**Example 10.14** Find the slope of the tangent line to the same level curve at the point $x = \frac{\pi}{4}$. $\blacksquare$

**Solution:** Here we have $\sin(x) = \sin(\pi/4) = \sqrt{2}/2$, and we find that the y coordinate satisfies

$$\frac{\sqrt{2}}{2}\cos(y) = \frac{1}{2}$$

This means that $\cos(y) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ so that $y = \pi/4$. Thus

$$\frac{dy}{dx} = \frac{1}{\tan(\pi/4)\tan(\pi/4)} = \frac{1}{1} = 1$$

so that the tangent line at the point $(\pi/4, \pi/4)$ has slope 1.

## Exercises

10.1. Calculate the first derivative for the following functions.

(a) $y = \sin x^2$
(b) $y = \sin^2 x$
(c) $y = \cot^2 \sqrt[3]{x}$
(d) $y = \sec(x - 3x^2)$
(e) $y = 2x^3 \tan x$
(f) $y = \frac{x}{\cos x}$
(g) $y = x \cos x$
(h) $y = e^{-\sin^2 \frac{1}{x}}$
(i) $y = (2 \tan 3x + 3 \cos x)^2$
(j) $y = \cos(\sin x) + \cos x \sin x$

10.2. Take the derivative of the following functions.

(a) $f(x) = \cos(\ln(x^4 + 5x^2 + 3))$
(b) $f(x) = \sin(\sqrt{\cos^2(x) + x^3})$
(c) $f(x) = 2x^3 + \log_3(x)$
(d) $f(x) = (x^2 e^x + \tan(3x))^4$
(e) $f(x) = x^2 \sqrt{\sin^3(x) + \cos^3(x)}$

10.3. Convert the following expressions in radians to degrees:
(a) $\pi$ (b) $5\pi/3$ (c) $21\pi/23$ (d) $24\pi$
Convert the following expressions in degrees to radians:
(e) $100^o$ (f) $8^o$ (g) $450^o$ (h) $90^o$
Using a Pythagorean triangle, evaluate each of the following:
(i) $\cos(\pi/3)$ (j) $\sin(\pi/4)$ (k) $\tan(\pi/6)$

10.4. Graph the following functions over the indicated ranges:

(a) $y = x \sin(x)$ for $-2\pi < x < 2\pi$
(b) $y = e^x \cos(x)$ for $0 < x < 4\pi$.

10.5. Sketch the graph for each of the following functions:

(a) $y = \dfrac{1}{2} \sin 3(x - \dfrac{\pi}{4})$
(b) $y = 2 - \sin x$
(c) $y = 3 \cos 2x$
(d) $y = 2 \cos(\dfrac{1}{2}x + \dfrac{\pi}{4})$

10.6. The Radian is an important unit associated with angles. One revolution about a circle is equivalent to $360$ degrees or $2\pi$ radians. Convert the following angles (in degrees) to angles in radians. (Express these as multiples of $\pi$, not as decimal expansions):

(a) 45 degrees

(b) 30 degrees

(c) 60 degrees

(d) 270 degrees.

Find the sine and the cosine of each of these angles.

10.7. A point is moving on the perimeter of a circle of radius 1 at the rate of 0.1 radians per second. How fast is its $x$ coordinate changing when $x = 0.5$? How fast is its $y$ coordinate changing at that time?

10.8. The derivatives of the two important trig functions are $[\sin(x)]' = \cos(x)$ and $[\cos(x)]' = -\sin(x)$. Use these derivatives to answer the following questions. Let $f(x) = \sin(x) + \cos(x), \ \ 0 \le x \le 2\pi$

(a) Find all intervals where $f(x)$ is increasing.

(b) Find all intervals where $f(x)$ is concave up.

(c) Locate all inflection points.

(d) Graph $f(x)$.

10.9. Find the appropriate trigonometric function to describe the following rhythmic processes:

(a) Daily variations in the body temperature $T(t)$ of an individual over a single day, with the maximum of $37.5^o$C at 8:00 am and a minimum of $36.7^o$ C 12 hours later.

(b) Sleep-wake cycles with peak wakefulness ($W = 1$) at 8:00 am and 8:00pm and peak sleepiness ($W = 0$) at 2:00pm and 2:00 am.

(For parts (a) and (b) express $t$ as time in hours with $t = 0$ taken at 0:00 am.)

10.10. Find the appropriate trigonometric function to describe the following rhythmic processes:

(a) The displacement $S$ cm of a block on a spring from its equilibrium position, with a maximum displacement 3 cm and minimum displacement $-3$ cm, a period of $\frac{2\pi}{\sqrt{\frac{g}{l}}}$ and at $t = 0$, $S = 3$.

(b) The vertical displacement $y$ of a boat that is rocking up and down on a lake. $y$ was measured relative to the bottom of the lake. It has a maximum displacement of $12$ meters and a minimum of $8$ meters, a period of $3$ seconds, and an initial displacement of $11$ meters when measurement was first started (i.e., $t = 0$).

10.11. Find all points on the graph of $y = \tan(2x), \ -\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$, where the slope of the tangent line is $4$.

10.12. A "V" shaped formation of birds forms a symmetric structure in which the distance from the leader to the last birds in the V is $r = 10m$, the distance between those trailing birds is $D = 6m$ and the angle formed by the V is $\theta$, as shown in Figure 10.14 below. Suppose that the shape is gradually changing: the trailing birds start to get closer so that their distance apart shrinks at a constant rate $dD/dt = -0.2m/min$

while maintaining the same distance from the leader. (Assume that the structure is always in the shape of a V as the other birds adjust their positions to stay aligned in the flock.) What is the rate of change of the angle $\theta$?

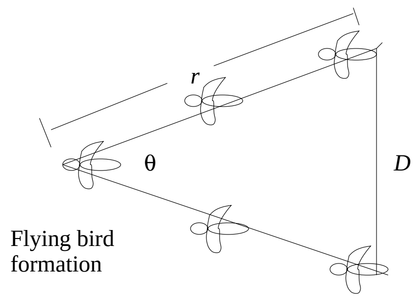



**Figure 10.14.** *Figure for Problem 12*

10.13. A hot air balloon on the ground is $200$ meters away from an observer. It starts rising vertically at a rate of $50$ meters per minute. Find the rate of change of the angle of elevation of the observer when the balloon is $200$ meters above the ground.

10.14. Match the differential equations given in parts (i-iv) with the functions in (a-f) which are solutions for them. (Note: each differential equation may have more than one solution)

*Differential equations:*

(i) $d^2y/dt^2 = 4y$

(ii) $d^2y/dt^2 = -4y$

(iii) $dy/dt = 4y$

(iv) $dy/dt = -4y$

*Solutions:*

(a) $y(t) = 4\cos(t)$

(b) $y(t) = 2\cos(2t)$

(c) $y(t) = 4e^{-2t}$

(d) $y(t) = 5e^{2t}$

(e) $y(t) = \sin(2t) - \cos(2t)$,

(f) $y(t) = 2e^{-4t}$.

10.15. Jack and Jill have an on-again off-again love affair. The sum of their love for one another is given by the function $y(t) = \sin(2t) + \cos(2t)$.

(a) Find the times when their total love is at a maximum.

(b) Find the times when they dislike each other the most.

10.16. A ladder of length $L$ is leaning against a wall so that its point of contact with the ground is a distance $x$ from the wall, and its point of contact with the wall is at height $y$. The ladder slips away from the wall at a constant rate $C$.

(a) Find an expression for the rate of change of the height $y$.

(b) Find an expression for the rate of change of the angle $\theta$ formed between the ladder and the wall.

10.17. A cannon-ball fired by a cannon at ground level at angle $\theta$ to the horizon ($0 \leq \theta \leq \pi/2$) will travel a horizontal distance (called the **range**, $R$) given by the formula below:

$$R = \frac{1}{16}\, v_0^2 \sin\theta \cos\theta.$$

Here $v_0 > 0$, the initial velocity of the cannon-ball, is a fixed constant and air resistance is neglected. (See Figure 10.15.) What is the maximum possible range?

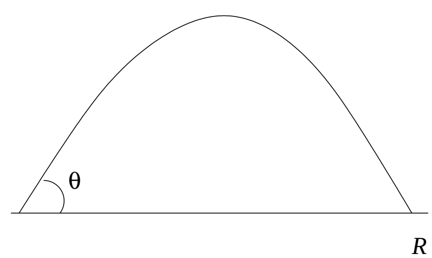



**Figure 10.15.** *Figure for problem 17*

10.18. A wheel of radius 1 meter rolls on a flat surface without slipping. The wheel moves from left to right, rotating clockwise at a constant rate of 2 revolutions per second. Stuck to the rim of the wheel is a piece of gum, (labeled $G$); as the wheel rolls along, the gum follows a path shown by the wide arc (called a "cycloid curve") in Figure 10.16. The $(x, y)$ coordinates of the gum ($G$) are related to the wheel's angle of rotation $\theta$ by the formulae

$$x = \theta - \sin\theta,$$

$$y = 1 - \cos\theta,$$

where $0 \leq \theta \leq 2\pi$. How fast is the gum moving horizontally at the instant that it reaches its highest point? How fast is it moving vertically at that same instant?

10.19. In Figure 10.17, the point P is connected to the point O by a rod 3 cm long. The wheel rotates around O in the clockwise direction at a constant speed, making 5 revolutions per second. The point Q, which is connected to the point P by a rod 5 cm long, moves along the horizontal line through O. How fast and in what direction is Q moving when P lies directly above O? (Remember the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$.)

10.20. A ship sails away from a harbor at a constant speed $v$. The total height of the ship including its mast is $h$. See Figure 10.18.

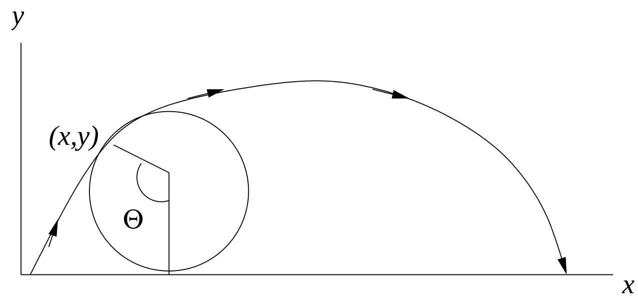



**Figure 10.16.** *Figure for Problem 18*

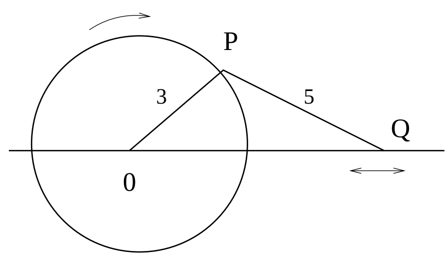



**Figure 10.17.** *Figure for Problem 19*

(a) At what distance away will the ship disappear below the horizon?

(b) At what rate does the top of the mast appear to drop toward the horizon just before this? (Note: In ancient times this effect lead people to conjecture that the earth is round (radius $R$), a fact which you need to take into account in solving the problem.)

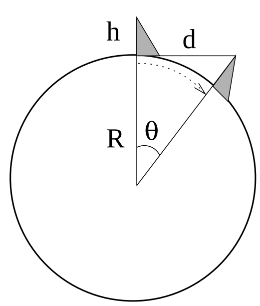



**Figure 10.18.** *Figure for Problem 20*

10.21. Find $\frac{dy}{dx}$ using implicit differentiation.

(a) $y = 2\tan(2x + y)$

(b) $\sin y = -2\cos x$

(c) $x\sin y + y\sin x = 1$

10.22. Use implicit differentiation to find the equation of the tangent line to the following curve at the point $(1, 1)$:

$$x\sin(xy - y^2) = x^2 - 1$$

## Chapter 11

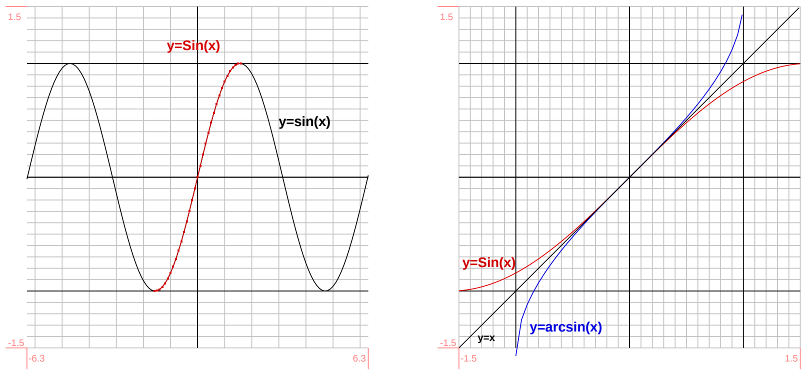
# Inverse Trigonometric functions

In this chapter, we investigate inverse trigonometric functions. As in other examples, the inverse of a given function leads to exchange of the roles of the dependent and independent variables, as well as the the roles of the domain and range. Geometrically, an inverse function is obtained by reflecting the original function about the line $y = x$. However, we must take care that the resulting graph represents a true function, i.e. satisfies all the properties required of a function.

The domains of $\sin(x)$ and $\cos(x)$ are both $-\infty < x < \infty$ while their ranges are $-1 \le y \le 1$. In the case of the function $\tan(x)$, the domain excludes values $\pm\pi/2$ as well as angles $2n\pi \pm \pi/2$ at which the function is undefined. The range of $\tan(x)$ is $-\infty < y < \infty$.

There is one difficulty in defining inverses for trigonometric functions: the fact that these functions repeat their values in a cyclic pattern means that a given $y$ value is obtained from many possible values of $x$. For example, all of the values $x = \pi/2, 5\pi/2, 7\pi/2$, etc all have identical sine values $\sin(x) = 1$. We say that these functions are not **one-to-one**. Geometrically, this is just saying that the graphs of the trig functions intersect a horizontal line in numerous places. When these graphs are reflected about the line $y = x$, they would intersect a *vertical* line in many places, and would fail to be functions: the function would have multiple $y$ values corresponding to the same value of $x$, which is not allowed. The reader may recall that a similar difficulty was encountered in an earlier chapter with the inverse function for $y = x^2$.

We can avoid this difficulty by restricting the domains of the trigonometric functions to a portion of their graphs that does not repeat. To do so, we select an interval over which the given trigonometric function is one-to -one, i.e. over which there is a unique correspondence between values of $x$ and values of $y$. (This just mean that we keep a portion of the graph of the function in which the y values are not repeated.) We then define the corresponding inverse function, as described below.




**(a)** **(b)**

**Figure 11.1.** *(a) The original trigonometric function,* $\sin(x)$*, in black, as well as the portion restricted to a smaller domain,* $Sin(x)$*, in red. The red curve is shown again in part b. (b) Relationship between the functions* $Sin(x)$*, defined on* $-\pi/2 < x < \pi/2$ *(in red) and* $\arcsin(x)$ *defined on* $-1 < x < 1$ *(in blue). Note that one is the reflection of the other about the line* $y = x$*. The graphs in parts (a) and (b) are not on the same scale.*

### Arcsine is the inverse of sine

The function $y = \sin(x)$ is one-to-one on the interval $-\pi/2 < x < \pi/2$. We will define the associated function $y = Sin(x)$ (shown in red on Figures 11.1(a) and (b) by restricting the domain of the sine function to $-\pi/2 < x < \pi/2$. On the given interval, we have $-1 < Sin(x) < 1$. We define the inverse function, called arcsine

$$y = \arcsin(x) \quad -1 < x < 1$$

in the usual way, by reflection of $Sin(x)$ through the line $y = x$ as shown in Figure 11.6(a).

To interpret this function, we note that $\arcsin(x)$ is "the angle whose sine is $x$". In Figure 11.2, we show a triangle in which $\theta = \arcsin(x)$. This follows from the observation that the sine of theta, opposite over hypotenuse, is $x/1$ which is simply $x$. The length of the other side of the triangle is then $\sqrt{1-x^2}$ by the Pythagorean theorem.

For example $\arcsin(\sqrt{2}/2)$ is the angle whose sine is $\sqrt{2}/2$, namely $\pi/4$. (We see this by checking the values of trig functions of standard angles shown in Table 1.) A few other inter-conversions are given by the examples below.

The functions $\sin(x)$ and $\arcsin(x)$, reverse (or "invert") each other's effect, that is:

$$\arcsin(\sin(x)) = x \quad \text{for} \quad -\pi/2 < x < \pi/2,$$

$$\sin(\arcsin(x)) = x \quad \text{for} \quad -1 < x < 1.$$

There is a subtle point that the allowable values of $x$ that can be "plugged in" are not exactly the same for the two cases. In the first case, $x$ is an angle whose sine we compute first, and then reverse the procedure. In the second case, $x$ is a number whose arc-sine is an angle.

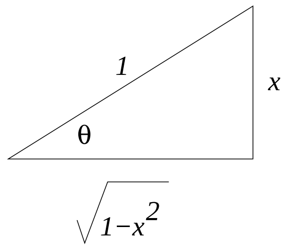
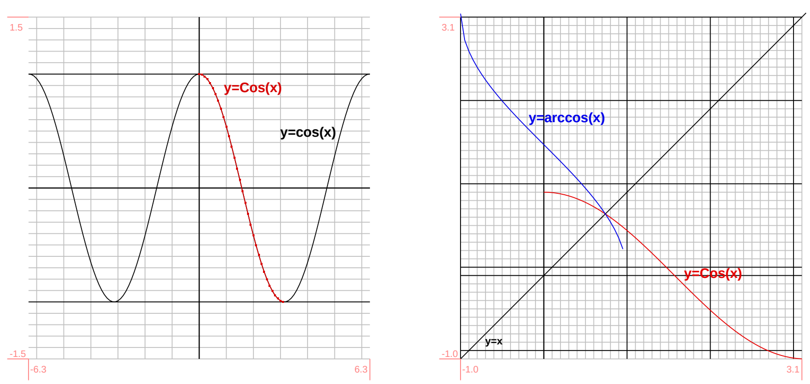



**Figure 11.2.** *This triangle has been constructed so that $\theta$ is an angle whose sine is $x/1 = x$. This means that $\theta = \arcsin(x)$*

We can evaluate $\arcsin(\sin(x))$ for any value of $x$, but the result may not agree with the original value of $x$ unless we restrict attention to the interval $-\pi/2 < x < \pi/2$. For example, if $x = \pi$, then $\sin(x) = 0$ and $\arcsin(\sin(x)) = \arcsin(0) = 0$, which is not the same as $x = \pi$. For the other case, i.e. for $\sin(\arcsin(x))$, we cannot plug in any value of $x$ outside of $-1 < x < 1$, since $\arcsin(x)$ is simply not define at all, outside this interval. This demonstrates that care must be taken in handling the inverse trigonometric functions.

### Inverse cosine




**(a)** **(b)**

**Figure 11.3.** *(a) The original function $\cos(x)$, is shown in black; the restricted domain version, $Cos(x)$ is shown in red. The same red curve appears in part (b) on a slightly different scale. (b) Relationship between the functions $Cos(x)$ (in red) and $\arccos(x)$ (in blue). Note that one is the reflection of the other about the line $y = x$.*

We cannot use the same interval to restrict the cosine function, because it has the same y values to the right and left of the origin. If we pick the interval $0 < x < \pi$, this difficulty is avoided, since we arrive at a one-to-one function. We will call the restricted-domain version of cosine by the name $y = Cos(x) = \cos(x)$ for$0 < x < \pi$. (See red curve in Figure 11.3(a). On the interval $0 < x < \pi$, we have $1 > Cos(x) > -1$ and we define the corresponding inverse function

$$y = \arccos(x) \quad -1 < x < 1$$

as shown in blue in Figure 11.3(b).

We understand the meaning of the expression $y = \arccos(x)$ as " the angle (in radians) whose cosine is $x$. For example, $\arccos(0.5) = \pi/3$ because $\pi/3$ is an angle whose cosine is 1/2. In Figure 11.4, we show a triangle constructed specifically so that $\theta = \arccos(x)$. Again, this follows from the fact that $\cos(\theta)$ is adjacent over hypotenuse. The length of the third side of the triangle is obtained using the Pythagorian theorem.

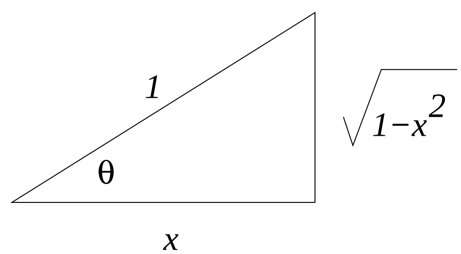



**Figure 11.4.** *This triangle has been constructed so that $\theta$ is an angle whose cosine is $x/1 = x$. This means that $\theta = arccos(x)$*

The inverse relationship between the functions mean that

$$\arccos(\cos(x)) = x \quad \text{for} \quad 0 < x < \pi,$$

$$\cos(\arccos(x)) = x \quad \text{for} \quad -1 < x < 1.$$

The same subtleties apply as in the previous case discussed for arc-sine.

### Inverse tangent

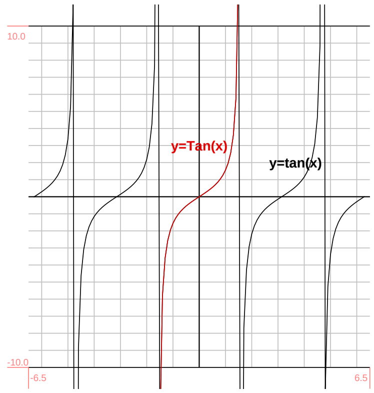
The function $y = \tan(x)$ is one-to -one on an interval $\pi/2 < x < \pi/2$, which is similar to the case for $Sin(x)$. We therefore restrict the domain to $\pi/2 < x < \pi/2$, that is, we define,

$$y = Tan(x) = \tan(x) \quad \pi/2 < x < \pi/2.$$

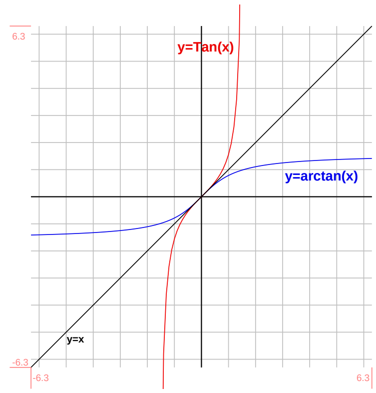
Unlike sine, as $x$ approaches either endpoint of this interval, the value of $Tan(x)$ approaches $\pm\infty$, i.e. $-\infty < Tan(x) < \infty$. This means that the domain of the inverse function will be from $-\infty$ to $\infty$, i.e. will be defined for all values of $x$ . We define the inverse tan function:

$$y = \arctan(x) \quad -\infty < x < \infty.$$

as before, we can understand the meaning of the inverse tan function, by constructing a




**(a)**




**(b)**

**Figure 11.5.** *(a) The function* $\tan(x)$*, is shown in black, and* $Tan(x)$ *in red. The same red curve is repeated in part b (b) Relationship between the functions* $Tan(x)$ *(in red) and* $\arctan(x)$ *(in blue). Note that one is the reflection of the other about the line* $y = x$*.*

| $x$ | $\arcsin(x)$ | $\arccos(x)$ |
|---|---|---|
| $-1$ | $-\pi/2$ | $\pi$ |
| $-\sqrt{3}/2$ | $-\pi/3$ | $5\pi/6$ |
| $-\sqrt{2}/2$ | $-\pi/4$ | $3\pi/4$ |
| $-1/2$ | $-\pi/6$ | $2\pi/3$ |
| $0$ | $0$ | $\pi/2$ |
| $1/2$ | $\pi/6$ | $\pi/3$ |
| $\sqrt{2}/2$ | $\pi/4$ | $\pi/4$ |
| $\sqrt{3}/2$ | $\pi/3$ | $\pi/6$ |
| $1$ | $\pi/2$ | $0$ |

**Table 11.1.** *Standard values of the inverse trigonometric functions.*

triangle in which $\theta = \arctan(x)$, shown in Figure 11.7.

The inverse tangent "inverts" the effect of the tangent on the relevant interval:

$$\arctan(\tan(x)) = x \quad \text{for} \quad -\pi/2 < x < \pi/2$$

$$\tan(\arctan(x)) = x \quad \text{for} \quad -\infty < x < \infty$$

The same comments hold in this case.

Some of the standard angles allow us to define precise values for the inverse trig functions. For other values of $x$, one has to calculate the decimal approximation of the function using a scientific calculator.

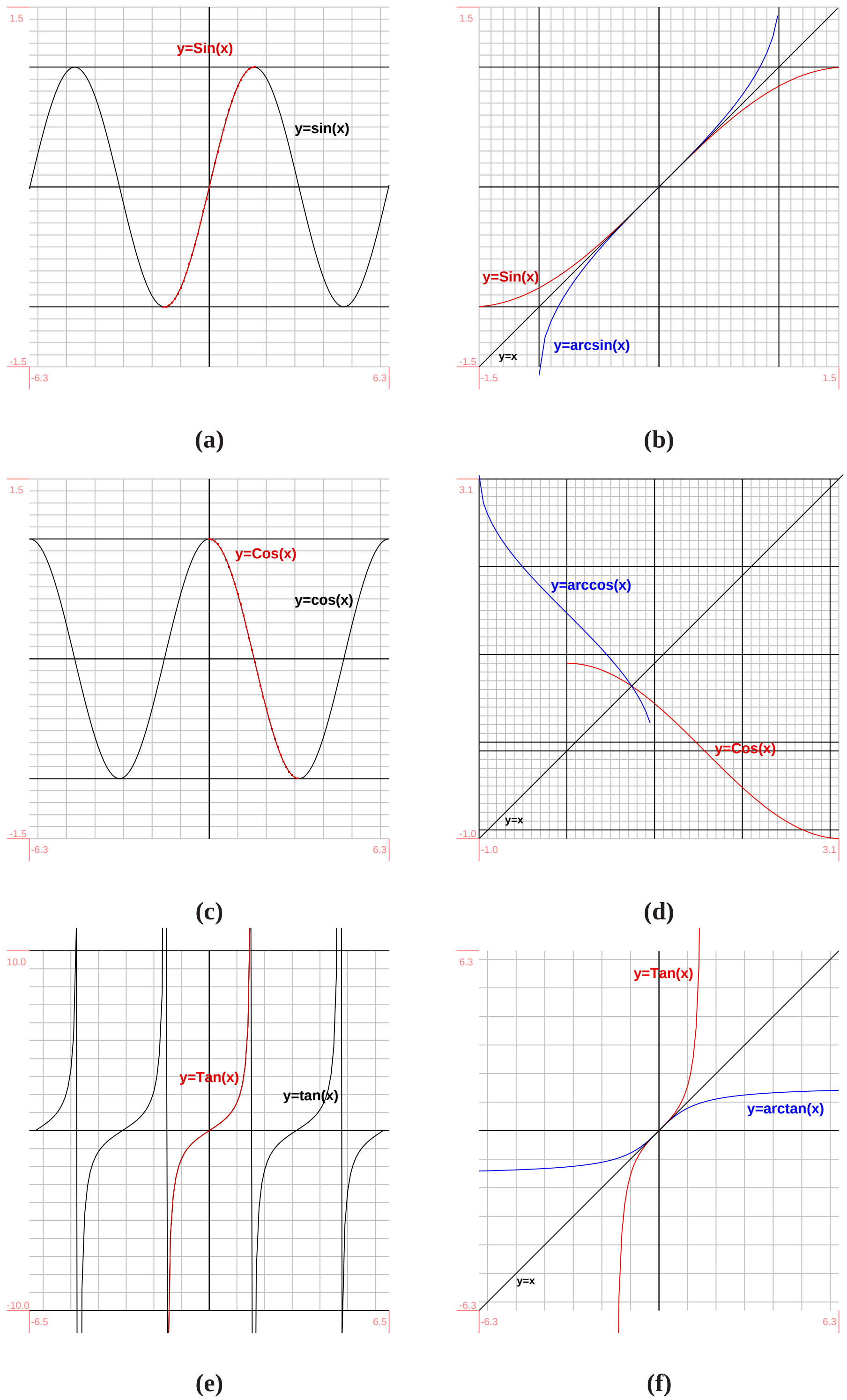



**Figure 11.6.** *A summary of the trigonometric functions and their inverses. (a)* $Sin(x)$ *(b)* $\arcsin(x)$*, (b)* $Cos(x)$ *(d)* $\arccos(x)$*, (e)* $Tan(x)$ *(f)* $\arctan(x)$*. The red curves are the restricted domain portions of the original trig functions. The blue curves are the inverse functions.*

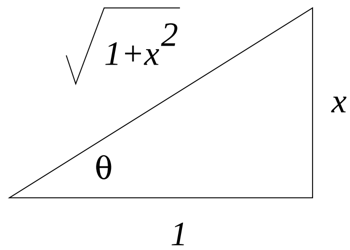



**Figure 11.7.** *This triangle has been constructed so that $\theta$ is an angle whose tan is $x/1 = x$. This means that $\theta = \arctan(x)$*

**Example 11.1** Simplify the following expressions: (a) $\arcsin(\sin(\pi/4)$, (b) $\arccos(\sin(-\pi/6))$ ■

**Solution:** (a) $\arcsin(\sin(\pi/4) = \pi/4$ since the functions are simple inverses of one another on the domain $-\pi/2 < x < \pi/2$.

(b) We evaluate this expression piece by piece: First, note that $\sin(-\pi/6) = -1/2$. Then $\arccos(\sin(-\pi/6)) = \arccos(-1/2) = 2\pi/3$. The last equality is obtained from the table of values prepared above.

**Example 11.2** Simplify the expressions: (a) $\tan(\arcsin(x)$, (b)$\cos(\arctan(x))$. ■

**Solution:** (a) Consider first the expression $\arcsin(x)$, and note that this represents an angle (call it $\theta$) whose sine is $x$, i.e. $\sin(\theta) = x$. Refer to Figure 11.2 for a sketch of a triangle in which this relationship holds. Now note that $\tan(\theta)$ in this same triangle is the ratio of the opposite side to the adjacent side, i.e.

$$\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$$

(b) Figure 11.7 shows a triangle that captures the relationship $\tan(\theta) = x$ or $\theta = \arctan(x)$. The cosine of this angle is the ratio of the adjacent side to the hypotenuse, so that

$$\cos(\arctan(x)) = \frac{1}{\sqrt{x^2+1}}$$

## 11.1 Derivatives of the inverse trigonometric functions

Implicit differentiation can be used to determine all derivatives of the new functions we have just defined. As an example, we demonstrate how to compute the derivative of $\arctan(x)$. To do so, we will need to recall that the derivative of the function $\tan(x)$ is $\sec^2(x)$. We will also use the identity $\tan^2(x) + 1 = \sec^2(x)$.

| $y = f(x)$ | $f'(x)$ |
|---|---|
| $\arcsin(x)$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\arccos(x)$ | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\arctan(x)$ | $\frac{1}{x^2+1}$. |

**Table 11.2.** *Derivatives of the inverse trigonometric functions.*

Let

$$y = \arctan(x).$$

Then on the appropriate interval, we can replace this relationship with the equivalent one:

$$\tan(y) = x.$$

Differentiating implicitly with respect to $x$ on both sides, we obtain

$$\sec^2(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{\tan^2(y)+1}$$

Now using again the relationship $\tan(y) = x$, we obtain

$$\frac{d\arctan(x)}{dx} = \frac{1}{x^2+1}.$$

This will form an important expression used frequently in integral calculus.

The derivatives of the important inverse trigonometric functions are shown in the table below.

## 11.2 The zebra danio and its escape response

In this section, we investigate an application of the trigonometric, and inverse trigonometric functions. This example is motivated by a problem in biology, studied by Larry Dill, a biologist at Simon Fraser University in Burnaby, BC.

The Zebra danio is a small tropical fish, which has many predators (larger fish) eager to have it for dinner. Surviving through the day means being able to sense danger quickly enough to escape from a hungry pair of jaws. However, the danio cannot spend all its time escaping. It too, must find food, mates, and carry on activities that sustain it. Thus, a finely tuned mechanism which allows it to react to danger but avoid over-reacting would be advantageous.

We investigate the visual basis of an escape response, based on a hypothesis formulated by Dill.

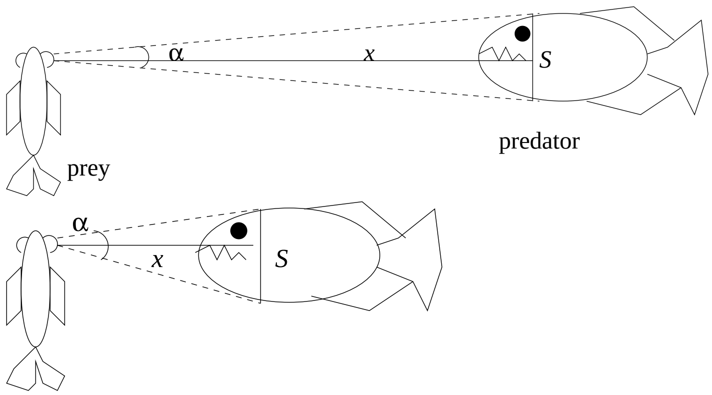



**Figure 11.8.** *A cartoon showing the visual angle,* $\alpha(t)$ *and how it changes as a predator approaches its prey, the zebra danio.*

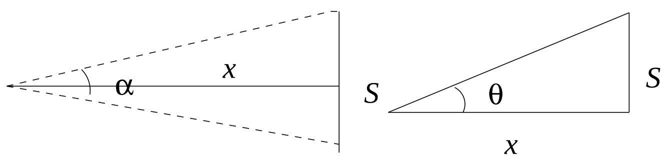



**Figure 11.9.** *The geometry of the escape response problem.*

Figure 11.9 shows the relation between the angle subtended at the Danio's eye and the size $S$ of an approaching predator, currently located at distance $x$ away. We will assume that the predator has a profile of size $S$ and that it is approaching the prey at a constant speed $v$. This means that the distance $x$ satisfies

$$\frac{dx}{dt} = -v.$$

If we consider the top half of the triangle shown in Figure 11.9 we find a Pythagorean triangle similar to the one we have seen before in our discussion of visual angles in Chapter 10. The connection between our previous calculation is $\theta = \alpha/2$, $s = S/2$ and $x$ identical in both pictures. Thus, the trigonometric relation that holds is:

$$\tan\left(\frac{\alpha}{2}\right) = \frac{(S/2)}{x}.$$

We can restate this relationship using the inverse trigonometric function $\arctan$ as follows:

$$\frac{\alpha}{2} = \arctan(\frac{S}{2x}).$$

Our experience with the derivative of this function will be useful below. Since both the angle $\alpha$ and the distance from the predator $x$ change with time, we indicate so by writing

$$\alpha(t) = 2\arctan\left(\frac{S}{2x(t)}\right).$$

We apply the chain rule to this expression to calculate the rate of change of the angle $\alpha$ with respect to time. Letting $u = S/2x$ and using the derivative of the inverse trigonometric function,

$$\frac{d\arctan(u)}{du} = \frac{1}{u^2+1}$$

and the chain rule, we would find, similarly that

$$\frac{d\alpha(t)}{dt} = \frac{d\arctan(u)}{du}\frac{du}{dx}\frac{dx}{dt} = \frac{1}{u^2+1}(-\frac{S}{2x^2(t)}(-v).$$

By simplifying, we arrive at the same result, namely that

$$\frac{d\alpha}{dt} = \frac{Sv}{x^2+(S^2/4)}.$$

This result should not be too surprising. Indeed, we have already derived it using implicit differentiation in a previous section. Recall a similar result derived for the rate of change of a visual angle, $\theta$ in Chapter 10.

$$\frac{d\theta}{dt} = \frac{sv}{x^2+s^2}.$$

In order to show this relation, we had to use the fact that $d\tan(\theta)/d\theta = \sec^2(\theta)$ and the trigonometric identity $\tan^2(\theta)+1 = \sec^2(\theta)$. By substituting the relations $\theta = \alpha/2$, $s = S/2$ into the equation for the rate of change of $\theta$ we find that

$$\frac{d\alpha}{dt} = \frac{Sv}{x^2+(S^2/4)}.$$

This is the rate of change of the visual angle. Now we consider the implications of this result.

We first observe that $d\alpha/dt$ depends on the size of the predator, $S$, its speed, $v$, and its distance away at the given instant. In fact, we can plot the way that this expression depends on the distance $x$ by noting the following:

- When $x = 0$, i.e., when the predator has reached its prey,

$$\frac{d\alpha}{dt} = \frac{Sv}{0+(S^2/4)} = \frac{4v}{S}.$$

- For $x \to \infty$, when the predator is very far away, we have a large value $x^2$ in the denominator, so

$$\frac{d\alpha}{dt} \to 0.$$

A rough sketch of the way that the rate of change of the visual angle depends on the current distance to the predator is shown in the curve on Figure 11.10.

### 11.2.1 Linking the visual angle to the escape response

What sort of visual input should the danio respond to, if it is to be efficient at avoiding the predator? In principle, we would like to consider a response that has the following features

- If the predator is too far away, if it is moving slowly, or if it is moving in the opposite direction, it should appear harmless and should not cause undue panic and inappropriate escape response, since this uses up the prey's energy to no good purpose.

- If the predator is coming quickly towards the danio, and approaching directly, it should be perceived as a threat and should trigger the escape response.

In keeping with these reasonable expectations, the hypothesis proposed by Dill is that:

**The escape response is triggered when the predator approaches so quickly, that the rate of change of the visual angle is greater than some critical value.**

We will call that critical value $K_{\text{crit}}$. This constant would depend on how "skittish" the Danio is given factors such as perceived risks of its environment. This means that the escape response is triggered in the Danio when

$$\frac{d\alpha}{dt} = K_{\text{crit}}$$

i.e. when

$$K_{\text{crit}} = \frac{Sv}{x^2 + (S^2/4)}.$$

Figure 11.10(a) illustrates geometrically a solution to this equation. We show the line $y = K_{\text{crit}}$ and the curve $y = Sv/(x^2 + (S^2/4))$ superimposed on the same coordinate system. The value of $x$, labeled $x_{\text{react}}$ will be the distance of the predator at the instant that the Danio realizes that it is under threat and should escape. We can determine the value of this distance, referred to as the *reaction distance*, by solving for $x$.

However, before doing so, we notice that another possibility, shown in Figure 11.10(b) has no intersection and will result in no distance for which $K_{\text{crit}} = Sv/(x^2 + (S^2/4))$. This may happen if either the Danio has a very high threshold of alert, so that it fails to react to threats, or if the curve depicting $d\alpha/dt$ is too low. That happens either if $S$ is very large (big predator) or if $v$ is small (slow moving predator "sneaking up" on its prey). From this scenario, we find that in some situations, the fate of the Danio would be sealed in the jaws of its pursuer.

To determine how far away the predator is detected in the happier scenario of Figure 11.10(a), we solve for the reaction distance, $x_{\text{react}}$:

$$x^2 + (S^2/4) = \frac{Sv}{K_{\text{crit}}} \quad \Rightarrow \quad x = \sqrt{\frac{Sv}{K_{\text{crit}}} - \frac{S^2}{4}},$$

$$x_{\text{react}} = \sqrt{S\left(\frac{v}{K_{\text{crit}}} - \frac{S}{4}\right)}.$$

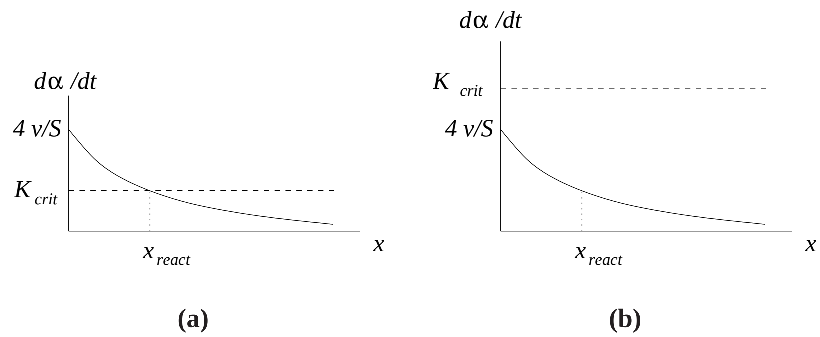
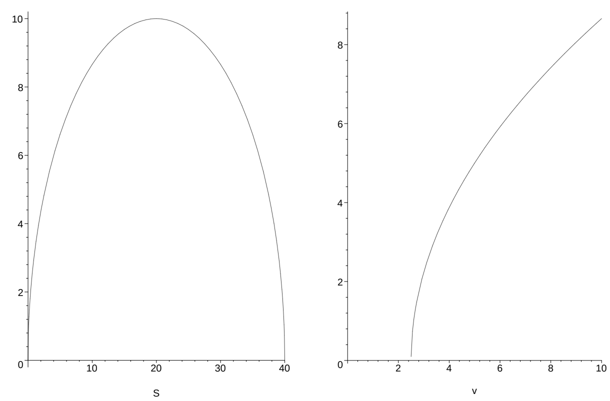



**Figure 11.10.** *The rate of change of the visual angle $d\alpha/dt$ in two cases, when the quantity $4v/S$ is above (a) and below (b) some critical value.*

It is clear that the reaction distance of the Danio with reaction threshold $K_{\text{crit}}$ would be greatest for certain sizes of predators. In Figure 11.11, we plot the reaction distance $x_{\text{react}}$ (on the vertical axis) versus the predator size $S$ (horizontal axis). We see that very small predators $S \approx 0$ or large predators $S \approx 4v/K_{\text{crit}}$ the distance at which escape response is triggered is very small. This means that the Danio may miss noticing such predators until they are too close for a comfortable escape, resulting in calamity. Some predators will be detected when they are very far away (large $x_{\text{react}}$). (We can find the most detectable size by finding the value of $S$ corresponding to a maximal $x_{\text{react}}$. The reader may show as an exercise that this occurs for size $S = 2v/K_{\text{crit}}$.) At sizes $S > 4v/K_{\text{crit}}$, the reaction distance is not defined at all: we have already seen this fact from Figure 11.10(b): when $K_{\text{crit}} > 4v/S$, the straight line and the curve fail to intersect, and there is no solution.

Figure 11.11(b) illustrates the dependence of the reaction distance $x_{\text{react}}$ on the speed $v$ of the predator. We find that for small values of $v$, i.e. $v < K_{\text{crit}}S/4$, $x_{\text{react}}$ is not defined: the Danio would not notice the threat posed by predators that swim very slowly.




**(a)** **(b)**

**Figure 11.11.** *(a) The reaction distance $x_{react}$ (on the vertical axis) is shown for various predator sizes $S$ (on the horizontal axis). (b) The reaction distance is shown as a function of the velocity of the predator.*

## Exercises

11.1. The function $y = \arcsin(ax)$ is a so-called *inverse trigonometric function*. It expresses the same relationship as does the equation $ax = \sin(y)$. (However, this function is defined only for values of $x$ between $1/a$ and $-1/a$.) Use implicit differentiation to find $y'$.

11.2. The inverse trigonometric function $\arctan(x)$ (also written $\arctan(x)$) means the angle $\theta$ where $-\pi/2 < \theta < \pi/2$ whose tan is $x$. Thus $\cos(\arctan(x)$ (or $\cos(\arctan(x))$ is the cosine of that same angle. By using a right triangle whose sides have length $1, x$ and $\sqrt{1+x^2}$ we can verify that

$$\cos(\arctan(x)) = 1/\sqrt{1+x^2}.$$

Use a similar geometric argument to arrive at a simplification of the following functions:

(a) $\sin(\arcsin(x))$,

(b) $\tan(\arcsin(x)$,

(c) $\sin(\arccos(x)$.

11.3. Find the first derivative of the following functions.

(a) $y = \arcsin x^{\frac{1}{3}}$

(b) $y = (\arcsin x)^{\frac{1}{3}}$

(c) $\theta = \arctan(2r+1)$

(d) $y = x$ arcsec$\frac{1}{x}$

(e) $y = \frac{x}{a}\sqrt{a^2-x^2} - \arcsin\frac{x}{a}$, $a > 0$.

(f) $y = \arccos\frac{2t}{1+t^2}$

11.4. Your room has a window whose height is 1.5 meters. The bottom edge of the window is 10 cm above your eye level. (See Figure 11.12.) How far away from the window should you stand to get the best view? ("Best view" means the largest visual angle, i.e. angle between the lines of sight to the bottom and to the top of the window.)

11.5. You are directly below English Bay during a summer fireworks event and looking straight up. A single fireworks explosion occurs directly overhead at a height of 500 meters. (See Figure 11.13.) The rate of change of the radius of the flare is 100 meters/sec. Assuming that the flare is a circular disk parallel to the ground, (with its center right overhead) what is the rate of change of the visual angle at the eye of an observer on the ground at the instant that the radius of the disk is $r = 100$ meters? (Note: the visual angle will be the angle between the vertical direction and the line between the edge of the disk and the observer).

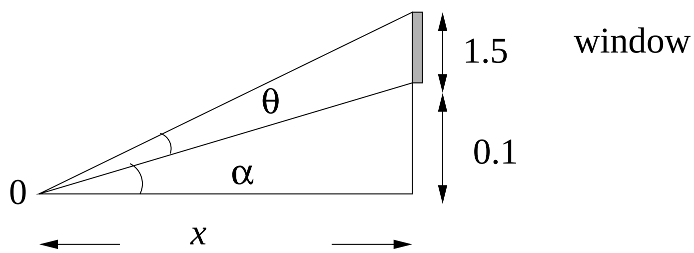



**Figure 11.12.** *Figure for Problem 4*

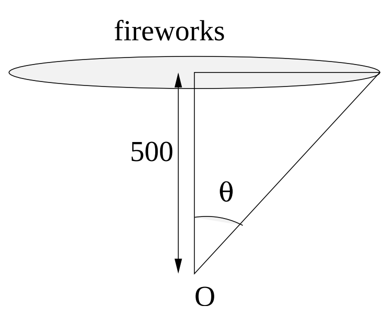



**Figure 11.13.** *Figure for Problem 5*

## Chapter 12

# Approximation methods

## 12.1 Introduction

In this chapter we explore a few techniques for finding approximate solutions to problems of great practical significance. The techniques here described are linked by a number of common features; most notably, all are based on exploiting the fact that a tangent line is a good (local) approximation to the behaviour of a function (at least close to the point of tangency).

The first method, that of linear approximation has been discussed before, and is a direct application of the tangent line as such an approximation. We will illustrate how this approximation can lead to simple one-step computation of rough values that a function of interest takes.

A second technique described here, **Newton's method**, is used to find precise decimal approximations to zeros of a function: recall these are places where a function crosses the $x$-axis, i.e. where $f(x) = 0$. The method gives an important example of an *iteration scheme*: that is, a recipe that is repeated (several times) to generate successively finer approximations.

A third technique is applied to calculating numerical solutions of a differential equation. This method, called **Euler's method**, uses the initial condition and the differential equation to compute approximate values of the solution step by step, starting with the initial time and incrementally computing the solution value for each of many small time steps.

While some of these techniques have been superseded by improved (graphics) calculators, or mathematical software, the concepts behind the methods are still fundamental. Also important is understanding the limitations of such methods, since each relies on certain assumptions and underlying concepts.

## 12.2 Linear approximation

We have already encountered the idea that the tangent line approximates the behaviour of a function. In this technique, the approximation is used to generate rough values of a function close to some point at which the value of the function and of its derivative are known, or easy to calculate.

Below we illustrate the idea of **linear approximation** to the function

$$y = f(x) = \sqrt{x}.$$

The exact value of this function is well known at a number of judiciously chosen values of $x$, e.g. $\sqrt{1} = 1, \sqrt{4} = 2, \sqrt{9} = 3$, etc. Suppose we want to approximate the value of the square root of 6. This is easily done with a scientific calculator, of course, but we can also use a rough approximation which uses only simple "known" values of the square root function and some elementary manipulations.

We use the following facts:

1. We know the value of the function at an adjoining point, i.e. at $x = 4$, since $f(4) = \sqrt{4} = 2$.

2. We also know that the tangent line can approximate the behaviour of a function close to the point of tangency

3. The derivative of $y = f(x) = \sqrt{x} = x^{1/2}$ is $dy/dx = (1/2)x^{-1/2}$, i.e. the slope of the tangent line to the curve $y = f(x) = \sqrt{x}$ is

$$f'(x) = \frac{1}{2\sqrt{(x)}}.$$

In particular, at $x = 4$, the derivative is $f'(4) = 1/(2\sqrt{4}) = 1/4 = 0.25$

4. The equation of the tangent line to a curve at a point $(x_0, f(x_0))$ is

$$\frac{y - f(x_0)}{x - x_0} = f'(x_0)$$

or simply

$$y = f(x_0) + f'(x_0)(x - x_0),$$

as we have seen earlier, when we first introduced the idea of a tangent line to a curve.

5. According to this *linear approximation*,

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

The approximation is exact at $x = x_0$, and holds well provided $x$ is close to $x_0$. (The expression on the right hand side is precisely the value of $y$ on the tangent line at $x = x_0$

Putting these facts together, we find that the equation of a tangent line to the curve $y = f(x) = \sqrt{x}$ at the point $x = 4$ is

$$y = f(4) + f'(4)(x - 4)$$

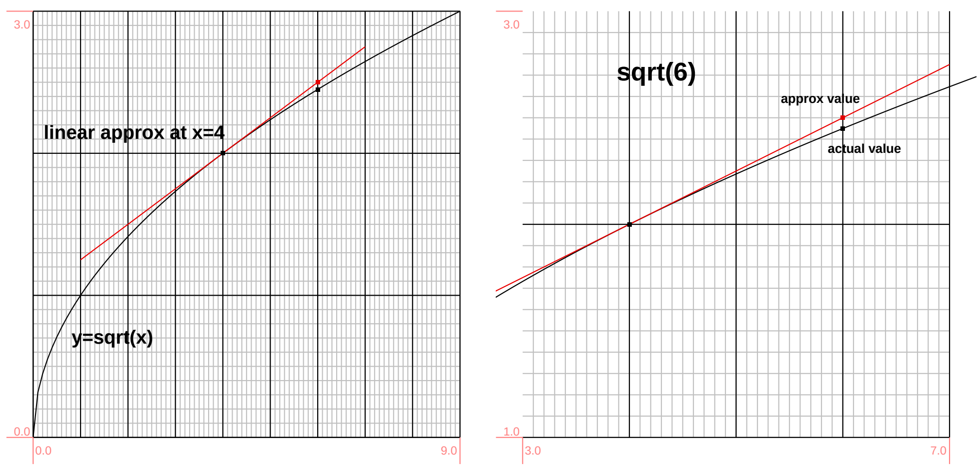
i.e. that

$$y = 2 + 0.25(x - 4).$$

In Figure 12.1(a), we show the original curve with tangent line superimposed. In Figure 12.1(b) we show a zoomed portion of the same graph, on which the true value of $\sqrt{6}$ (black dot) is compared to the value on the tangent line, which approximates it (red dot) i.e. to

$$y_{\text{approx}} = 2 + 0.25(6 - 4) = 2.5$$

It is evident from this picture that there is some error in the approximation, since the values are clearly different. However, if we do not stray too far from the point of tangency ($x = 4$), the error will not be too large.




**(a)** **(b)**

**Figure 12.1.** *Linear approximation based at $x = 4$ to the function $y = f(x) = \sqrt{(x)}$.*

In Table 12.1, we collect true exact values of the function $f(x) = \sqrt{x}$ (computed by the spreadsheet), values of its derivative, $f'(x)$, (note in particular the value at $x = 4$ which forms the slope of the tangent line of interest) and values on the tangent line through the point $(4, 2)$. (The third column corresponds to the linear approximation values that we are focusing on in this section.) At $x = 4$, the values of the function and of its approximation are identical (naturally - since we "rigged it" so). Close the $x = 4$, the values of the approximation are fairly close to the values of the function. Further away, however, the difference between these gets bigger, and the approximation is no longer very good at all.

These remarks illustrate two features: (1) the method is easy to use, and involves only determination of a derivative, and elementary arithmetic. (2) The method has limitations, and work well only close to the point at which the tangent line is based.

| $x$ | $f(x)=\sqrt{(x)}$ (exact value) | $f'(x)=1/(2\sqrt{(x)})$ | $y=f(x_0)+f'(x_0)(x-x_0)$ (approx value) |
|---|---|---|---|
| 0.0000 | 0.0000 | $\infty$ | 1.0000 |
| 2.0000 | 1.4142 | 0.3536 | 1.5000 |
| 4.0000 | 2.0000 | 0.2500 | 2.0000 |
| 6.0000 | 2.4495 | 0.2041 | 2.5000 |
| 8.0000 | 2.8284 | 0.1768 | 3.0000 |
| 10.0000 | 3.1623 | 0.1581 | 3.5000 |
| 12.0000 | 3.4641 | 0.1443 | 4.0000 |
| 14.0000 | 3.7417 | 0.1336 | 4.5000 |
| 16.0000 | 4.0000 | 0.1250 | 5.0000 |

**Table 12.1.** *Linear approximation to* $\sqrt{x}$.

## 12.3 Newton's method

Newton's method is a technique for finding approximate values for roots of an algebraic equation of the form

$$f(x)=0.$$

(These values are also called *zeros* of the function $f(x)$.) While this seems like a fairly restricted type of problem, actually, there are numerous applications in which this technique is useful, and many problems in applied and basic science that lead to such equations. We have seen that finding critical points of some function $G(x)$, is equivalent to solving $G'(x)=0$. (i.e. if define the function of interest to be $f(x)=G'(x)$, then we are solving precisely an equation of the form shown above.

We first distinguish between cases that do and do not require Newton's method, and then show how Newton's method is derived and how it is used.

### 12.3.1 When Newton's method is not needed

**Example 12.1** Find the value of $x$ that satisfies

$$f(x)=0$$

where $f(x)=x^2+4x+3$. $\blacksquare$

**Solution:** We are asked to solve the equation

$$x^2+4x+3=0.$$

This is a simple quadratic equation, and we have an exact formula(the quadratic formula) for the roots, i.e.

$$x=\frac{-4\pm\sqrt{4^2-4(3)}}{2}=\frac{-4\pm\sqrt{4}}{2}=3,1.$$

In this case, we do not need Newton's formula.

**Example 12.2** Find values of $x$ for which $\sin(x) = 1$.

**Solution:** If we set $f(x) = \sin(x) - 1$, then the problem of solving our equation reduces to the problem of solving $f(x) = 0$. (Many example of this sort occur.) However, we need no fancy techniques to solve this equation, since we know about that the function $\sin(x)$ takes on the value 1 when $x = \pi/2$, and all other values that correspond to this angle, to which multiples of $2\pi$ have been added (i.e. $\pi/2 \pm 2n\pi$ where $n$ is an integer).

**Example 12.3** Find critical points of the function

$$y = g(x) = 2x^3 - 9x^2 + 12x + 1$$

**Solution:** We would first compute the derivative $g'(x) = 6x^2 - 18x + 12$, and then set it equal zero. The problem reduces to solving an equation of the form $f(x) = 0$ where $f(x) = 6x^2 - 18x + 12$ is just the derivative of $g(x)$. Since this, too, is a quadratic, the solution can be found easily using the quadratic formula, i.e.

$$x = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = 1, 3$$

So far, most problems encountered in this course could be solved by such elementary algebraic simplification and rearrangement. However, for polynomials of degree higher than 3, this technique can lead to equations that are not easy to solve with elementary methods. In such cases, Newton's method can be indispensable.

### 12.3.2 Derivation of the recipe for Newton's method

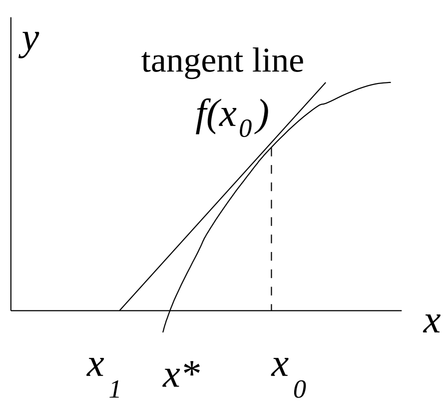



**Figure 12.2.** *Sketch showing the idea behing Newton's method.*

Consider the function $y = f(x)$ shown in Figure 12.2. We want to find the value $x$ such that

$$f(x) = 0.$$

In Figure 12.2, the desired point is indicated with the notation $x^*$. Usually, the decimal expansion for the coordinate $x^*$ is not known in advance: that is what we are trying to find. We will see that by applying Newton's method several times, we can generate such a decimal expansion to any desired level of accuracy.

Suppose we have some very rough idea of some initial guess for the value of this root. (how to find this initial guess will be discussed later.) Newton's method is a recipe for getting better and better approximations of the true value, $x^*$.

In the diagram shown in Figure 12.2, $x_0$ represents an initial starting guess. We observe that a tangent line to the graph of $f(x)$ at the point $x_0$ gives a rough indication of the behaviour of the function near that point. We will use the tangent line as an approximation of the actual function. We look for the point at which the tangent line intersects the $x$ axis. Let $x_1$ denote that point of intersection. Then as shown in Figure 12.2, $x_1$ is a better approximation of the root we want to find, i.e. $x_1$ is closer to $x^*$ than our initial guess. If we can find $x_1$ and repeat the same idea over and over again, we hope to find values that get closer and closer to the root $x^*$.

Our task is now to figure out a formula for the point $x_1$. We will use the following facts:

- The point on the graph of the function corresponding to the initial guess is $(x_0, y_0)$ where $y_0 = f(x_0)$.

- The slope of the tangent line at the point $x_0$ is $m = f'(x_0)$.

- The equation of a line through the point $(x_0, y_0)$ with slope $m$ is

$$\frac{y - y_0}{x - x_0} = m$$

- Using the above facts, and substituting $m = f'(x_0)$ and $y_0 = f(x_0)$ leads to the following equation for the tangent line shown in Figure 12.2:

$$\frac{y - f(x_0)}{x - x_0} = f'(x_0).$$

- We are interested in the place where the tangent line intersects the $x$ axis, i.e. in the point $(x_1, 0)$. We want to find $x_1$, since this will be the more accurate approximation for the root at $x^*$. so we have

$$\frac{0 - f(x_0)}{x_1 - x_0} = f'(x_0).$$

- Solving for $x_1$ we have

$$\frac{x_1 - x_0}{0 - f(x_0)} = \frac{1}{f'(x_0)}.$$

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)} \quad \Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This gives a recipe for obtaining an improved value, $x_1$ from the initial guess.

We have found that the initial guess, $x_0$, and Newton's method lead to the recipe for the improved guess

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

We can repeat this procedure to get a better value

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

$$\vdots$$

In general, we can refine the approximation using as many steps as it takes to get the accuracy we want. (We will see in upcoming examples how to recognize when this accuracy is attained.) In the step $k+1$ we find a value that improves on the approximation made in step $k$ as follows:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

**Example 12.4** Apply Newton's method to the same problem tackled earlier, namely determine the square root of 6. ■

**Solution:** It is first necessary to restate the problem in the form "Find a value of $x$ such that a certain function $f(x) = 0$." Clearly, a function that would accomplish this is

$$f(x) = x^2 - 6$$

since the value of $x$ for which $f(x) = 0$ is indeed $x^2 - 6 = 0$, i.e. $x = \sqrt{6}$. (We could also find other functions that have the same property, e.g. $f(x) = x^4 - 36$, but the above is one of the simplest such functions.

We compute the derivative for this function:

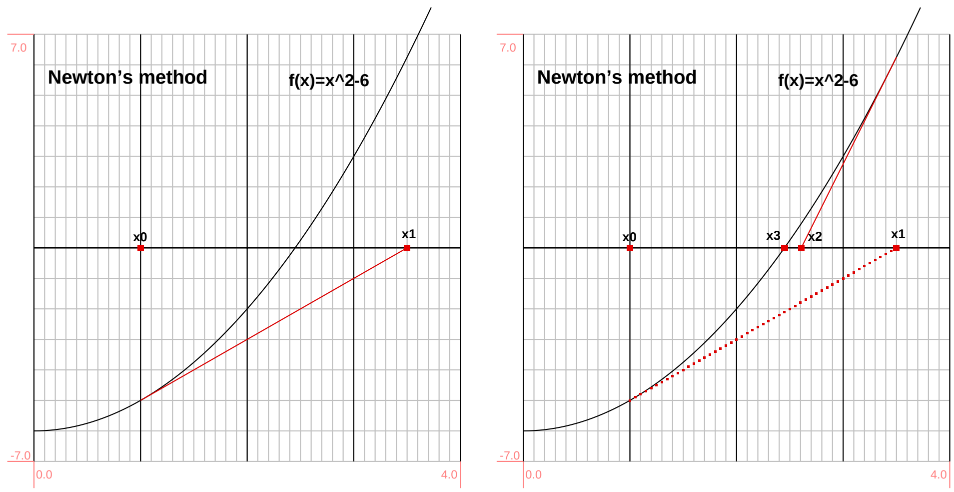
$$f'(x) = 2x$$

Thus the iteration for Newton's method is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

that is

$$x_1 = x_0 - \frac{x_0^2 - 6}{2x_0}.$$




**(a)** **(b)**

**Figure 12.3.** *Newton's method applied to solving* $y = f(x) = x^2 - 6 = 0$.

| $x_k$ | $f(x_k)$ | $f'(x_k)$ | $x_{k+1}$ |
|---|---|---|---|
| 1.00 | -5.00 | 2.00 | 3.5 |
| 3.5 | 6.250 | 7.00 | 2.6071 |
| 2.6071 | 0.7972 | 5.2143 | 2.4543 |
| 2.4543 | 0.0234 | 4.9085 | 2.4495 |
| 2.4495 | 0.000 | 4.8990 | 2.4495 |

**Table 12.2.** *Newton's method applied to Example 12.4.*

Suppose we start with the initial guess $x_1 = 1$ (which is actually not very close to the value of the root) and see how well Newton's method perform: This is shown in Figure 12.3. In Figure 12.3(a) we see the graph of the function, the position of our initial guess $x_0$, and the result of the improved Newton's method approximation $x_1$. In 12.3(b), we see how the value of $x_1$ is then used to obtain $x_2$ by applying a second iteration (i.e repeating the calculation with the new value used as initial guess.)

A spreadsheet is ideal for setting up the rather repetitive calculations involved, as shown in the table. For example, we compute the following set of values using our spreadsheet. Observe that the fourth column contains the computed (Newton's method) values, $x_1, x_2$, etc. These values are then copied onto the first column to be used as new "initial guesses". We also observe that after several repetitions, the numbers calculated *converge* (i.e. get closer and closer) to 2.4495, and no longer change to that level of accuracy. This is a signal that we need no longer repeat the iteration, if we are satisfied with 5 significant figures of accuracy.

In the next example, we show how Newton's method can be helpful in identifying critical points of a certain function.

**Example 12.5** Find the intersection point of the graph of $y = x^2$ with the graph of $y = \sin(x)$. ∎

**Solution:** The problem consists of finding a positive value of $x$ such that

$$x^2 = \sin(x).$$

Rewriting this in the form

$$x^2 - \sin(x) = 0,$$

we recognize a problem of the sort that Newton's method can solve. We let $f(x) = x^2 - \sin(x)$, and look for roots of $f(x) = 0$.

To start, we observe that for $x_0 = 1$ it is true that $x^2 = 1$, and $\sin(x) \approx x \approx 1$. This suggests that we could use the initial guess $x_0 = 1$.

We have

$$f(x) = x^2 - \sin(x),$$

so

$$f'(x) = 2x - \cos(x).$$

and thus

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - \sin(x_0)}{2x_0 - \cos(x_0)}.$$

So

$$x_1 = 1 - \frac{1 - \sin(1)}{2 - \cos(1)} = 1 - \frac{1 - 0.8414}{2 - 0.54} = 0.8914$$

we then evaluate

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^2 - \sin(x_1)}{2x_1 - \cos(x_1)}.$$

Plugging in the value of $x_1$ we had found, and calculating the value leads to

$$x_2 = 0.8770,$$

We similarly find that

$$x_3 = 0.8767,$$

$$x_4 = 0.8767.$$

Thus, the sequence of values converges easily to the value of the root after only three repetitions.

**Example 12.6** Find a critical point of the function

$$y = x^3 + e^{-x}.$$

∎

**Solution:** The function is shown in Figure 12.4. We must turn this into the appropriate problem to which Newton's method will apply. The critical points of this function are values of $x$ such that

$$\frac{dy}{dx} = 3x^2 - e^{-x} = 0.$$

It is not possible to devise a simple algebraic way to solve this equation for $x$. We must apply some approximation method. We will define the function

$$f(x) = 3x^2 - e^{-x}.$$

Clearly, the zeros of this function are the critical points we are looking for. We will find one of these points using Newton's method.

To do so, we must find the derivative of $f$ (which happens to be the second derivative of the original function)

$$f'(x) = 6x + e^{-x}.$$

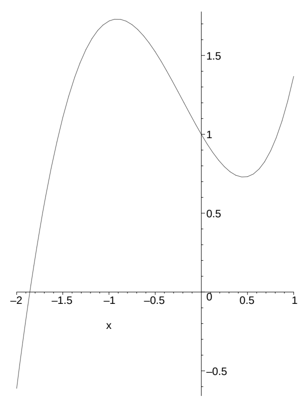



**Figure 12.4.** *We are asked to find the critical points of the function* $y = x^3 + e^{-x}$.

As an initial guess we note that $e \approx 3$ so that

$$f(1) \approx 3 - (1/3) \approx 2.6$$

We will use $x_0 = 1$ as the initial guess even though this is not a very accurate value. For this problem

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{3x_k^2 - e^{-x_k}}{6x_k + e^{-x_k}}$$

We show some calculations in Table 12.3.

Thus, we see that the values converge to the critical point $x = 0.4590$. Again, the iteration leads to convergence to a desired level of accuracy shown above. From the graph of the function shown above, it is apparent that we can expect a second critical point at around $x \approx -1$. It is left to the reader to find the exact decimal expansion of that second critical point, using similar steps with $x_0 = -1$.

| $x_k$ | $f(x_k) =$ $3x_k^2 - e^{-x_k}$ | $f'(x_k) =$ $6x_k + e^{x_k}$ | $x_{k+1} =$ |
|---|---|---|---|
| 1.0000 | 2.6321 | 6.3679 | 0.5867 |
| 0.5867 | 0.4763 | 4.0761 | 0.4698 |
| 0.4698 | 0.0370 | 3.4439 | 0.4591 |
| 0.4591 | 0.0003 | 3.3862 | 0.4590 |
| 0.4590 | 0.0000 | 3.3857 | 0.4590 |

**Table 12.3.** *Table for calculations in Example 12.6.*

**Example 12.7 (Effect of the initial guess:)** How does the initial guess we use in Newton's method affect the value of the root to which the method will converge? ■

**Solution:** We illustrate below the effect of starting out with several distinct guesses and note that the result will depend on the root closest to the guess. For this example we will solve the problem of finding roots of the equation

$$f(x) = \sin(x) = 0.$$

(In actual fact, we do not need Newton's method, since we know that the function $\sin(x)$ has zeros for integer multiples of $\pi$ (i.e., at $x = n\pi$ where $n = 0, 1, 2\ldots$ is any integer). However, we will use this example to check Newton's method, to see how well the method works for various starting guesses.)

Since $f'(x) = \cos(x)$, Newton's method will be based on the recipe

$$x_1 = x_0 - \frac{\sin(x_0)}{\cos(x_0)}.$$

In the graph below we show the sequence of iterates that Newton's method generates for two different starting values. In the first experiment, we use $x_0 = 0.5$. We'll then get the sequence of values

$$x_0 = 0.5, \ \ x_1 = -0.0463, \ \ x_2 = 0.0000$$

and so on. This experiment leads us to find the root at $x = 0$. (We say that the sequence of iterates *converges* to $x = 0$.)

In the second experiment we start with $x_0 = 4$. We then find that

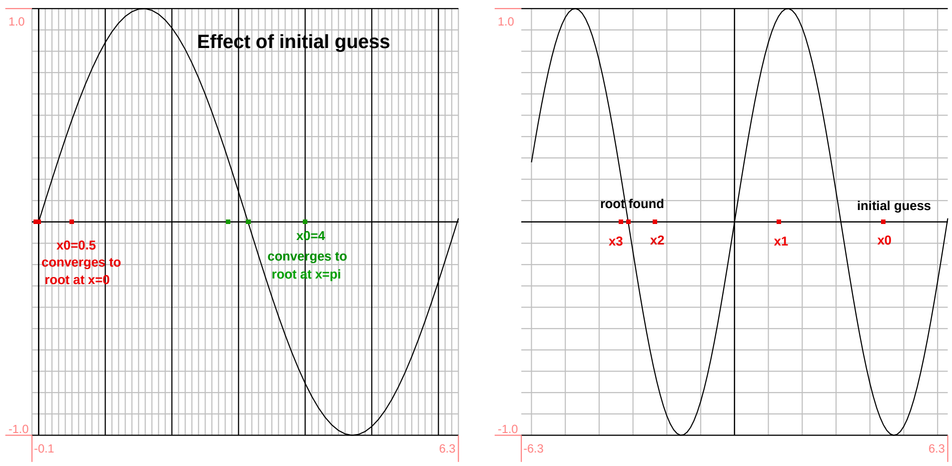
$$x_0 = 4, \quad x_1 = 2.8422, \quad x_2 = 3.1509, \quad x_3 = 3.1416$$

Thus, this sequence converges to the root at $x = \pi$.

In a third experiment, we try $x_0 = 4.4$. This guess is unfortunately rather close to a critical point on the function $y = \sin(x)$. The result is that the tangent line at $x_0 = 4.4$ has a very shallow slope (close to slope =0) and intersects the $x$ axis quite far away. We see that the value of $x_1$ and $x_2$ bounce around as follows:

$$x_0 = 4.4, \ \ x_1 = 1.3037, \ \ x_2 = -2.3505, \ \ x_3 = -3.3620, \ \ x_4 = -3.1380, \ \ x_5 = -3.1416$$

The result is shown in Figure 12.5(b). This illustrates the fact that an initial guess that is too close to a critical point of the function in question may misdirect us. In some cases, the result will be convergence to a root far away, while in other cases, the sequence may fail to converge altogether.




**(a)** **(b)**

**Figure 12.5.** *Figure for Example 12.7*

## 12.4 Euler's method

Euler's method is a technique used to find approximate numerical values for solutions to a differential equations. In general, we might have a problem in which $y = g(t)$ is some (unknown) function to be determined, where we are given information about the rate of change of $y$ such as

$$\frac{dy}{dt} = f(y)$$

and some initial condition

$$y(0) = y_0.$$

This type of problem is an *initial value problem* (i.e differential equation together with initial condition.)

Euler's method consists of replacing the differential equation by the approximation

$$\frac{y_{k+1} - y_k}{\Delta t} = f(y_k).$$

Clearly, this approximation is only good if the step size $\Delta t$ is quite small. (In that case, the derivative is well approximated by the term on the left which is actually the slope of a secant line)

This approximation leads to the recipe

$$y_1 = y_0 + f(y_0)\Delta t$$

$$y_2 = y_1 + f(y_1)\Delta t$$

$$\vdots$$

$$y_{k+1} = y_k + f(y_k)\Delta t.$$

We get from this iterated technique the approximate values of the function for as many time steps as desired starting from $t = 0$ in increments of $\Delta t$.

To set up the recipe for generating successive values of the desired solution, we first have to pick a "step size", $\Delta t$, and subdivide the $t$ axis into discrete steps of that size. This is shown in Figure 12.6. Our procedure will be to start with a known initial value of $y$, specified by the initial condition, and use it to generate the value at the next time point, then the next and so on.

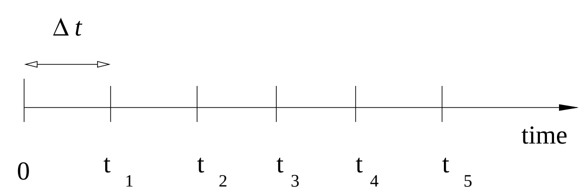



**Figure 12.6.** *The time axis is subdivided into steps of size* $\Delta t$.

It is customary to use the following notation to refer to the true ideal solution and the one that is actually produced by this approximation method:

- $t_0 =$ the initial time point, usually at $t = 0$.
- $h = \Delta t =$ common notations for the step size, i.e. the distance between the points along the $t$ axis.
- $t_k =$ the k'th time point. Since the points are just at multiples of the step size that we have picked, it follows that $t_k = k\Delta t = h\Delta t$.
- $y(t) =$ the actual value of the solution to the differential equation at time $t$. This is usually not known, but in the examples discussed here, we can solve the differential equation exactly, so we have a formula for the function $y(t)$. In most hard scientific problems, no such formula is known in advance.
- $y(t_k) =$ the actual value of the solution to the differential equation at one of the discrete time points, $t_k$. (Again, not usually known.)

- $y_k$ = the approximate value of the solution obtained by Euler's method. We hope that this approximate value is fairly close to the true value, i.e. that $y_k \approx y(t_k)$, but there is always some error in the approximation. More advanced methods that are specifically designed to reduce such errors are discussed in courses on numerical analysis.

**Example 12.8** Apply Euler's method to approximating solutions for the simple exponential growth model that was studied in Chapter 9,

$$\frac{dy}{dt} = ay,$$

(where $a$ is a constant) with initial condition

$$y(0) = y_0.$$

(See Eqn 9.1.) ∎

**Solution:** Let us subdivide the $t$ axis into steps of size $\Delta t$, starting with $t_0 = 0$, and $t_1 = \Delta t, t_2 = 2\Delta t, \dots$ From the above discussion, we note that the first value of $y$ is known to us exactly, namely,

$$y_0 = y(0) = y_0.$$

We replace the differential equation by the approximation

$$\frac{y_{k+1} - y_k}{\Delta t} = ay_k.$$

Then

$$y_{k+1} = y_k + a\Delta t y_k, \quad k = 1, 2, \dots$$

In particular,

$$y_1 = y_0 + a\Delta t y_0 = y_0(1 + a\Delta t),$$

$$y_2 = y_1(1 + a\Delta t),$$

$$y_3 = y_2(1 + a\Delta t),$$

and so on. At every stage, the quantity on the right hand side depends only on values of $y_k$ that are already known, so that this generates a recipe for moving from the initial value to successive values of the approximation for $y$.

**Example 12.9** Consider the specific problem in which

$$\frac{dy}{dt} = -0.5y, \quad y(0) = 100.$$

Use step size $\Delta t = 0.1$ and Euler's method to approximate the solution for two time steps. ∎

**Solution:** Euler's method applied to this example would lead to

$$
\begin{aligned}
y_0 =& 100.\\
y_1 =& y_0(1 + a\Delta t) = 100(1 + (-0.5)(0.1)) = 95,\\
y_2 =& y_1(1 + a\Delta t) = 95(1 + (-0.5)(0.1)) = 90.25,
\end{aligned}
$$

and so on.

Clearly, these kinds of repeated calculations are best handled on a spreadsheet or similar computer software. Later in this chapter we will compare the results of Euler's method applied to a differential equation with the actual solution known to us from studying exponential behaviour.

The next example is somewhat more complex, but deals also with a differential equation whose solution is available exactly. We use it to illustrate the comparison between the exact solution and the approximate solution generated by Euler's method.

### 12.4.1 Applying Euler's method to Newton's law of cooling

We apply Euler's method to the differential equation that describes the cooling of an object. (This differential equation is called Newton's Law of Cooling, though it is unrelated to the Newton's method we investigated above.) According to Newton,

**The rate of change of temperature $T$ of an object is proportional to the difference between its temperature and the ambient temperature, $E$.**

This hypothesis about the way that temperature $T(t)$ changes leads to the differential equation

$$\frac{dT}{dt} = k(E - T).$$

Suppose we are given an initial condition that prescribes the temperature at the beginning of the observation

$$T(0) = T_0.$$

This initial value problem has a solution that can be written in a simple formula, i.e.

$$T(t) = E + (T_0 - E)e^{-kt}.$$

(See chapter 13 for a derivation of this result.) We will refer to this formula, which holds exactly for all values of time $t$ as the *true solution*. Observe that it is some function of time that gives a full record of the behaviour of the temperature of the object as it cools off or heats up in its environment. However, the numerical values of temperature at a given time still need to be plotted or computed for this formula to be useful quantitatively. This motivates us to look for some approximate technique that would use the differential equation and the initial condition directly to plot quantitative behaviour, without the need for a formula for the solution, $T(t)$.

**Example 12.10 (Newton's law of cooling:)** For simplicity, consider the case that the ambient temperature is $E = 10$ degrees. Suppose that the constant $k$ is $k = 0.2$. Find the exact solution, that is, determine the (formula for) the temperature as a function of time $T(t)$. ■

**Solution:** In this case, the differential equation has the form

$$\frac{dT}{dt} = 0.2(10 - T),$$

and its solution (which is known exactly) is

$$T(t) = 10 + (T_0 - 10)e^{-0.2t}.$$

We will investigate the solutions from several initial conditions, $T = 0, 5, 15, 20$ degrees.

**Example 12.11 (Euler's method applied to Newton's law of cooling:)** Write down the Euler's method formula that can be used to find an approximation to the solution of Example 12.10 ■

**Solution:** The derivative $dT/dt$ is approximated by the secant line slope, that is

$$\frac{dT}{dt} \approx \frac{T(t + \Delta t) - T(t)}{\Delta t}.$$

This means that an approximation to the differential equation

$$\frac{dT}{dt} = 0.2(10 - T),$$

is

$$\frac{T(t + \Delta t) - T(t)}{\Delta t} = 0.2(10 - T(t)).$$

or, in simplified form,

$$T(t + \Delta t) \approx T(t) + 0.2(10 - T(t))\Delta t.$$

This is only an approximation, but for small $\Delta t$, the approximation should be relatively good.

Suppose we are given the initial temperature, e.g. $T(0) = T_0$. The important observation is that the above recipe gives us a way to compute the temperature at a slightly later time. Indeed, as we have already seen, we can use this recipe over and over again to generate a succession of values of the temperature $T$ each at the next time point. We will not get a smooth curve: just a collection of point values, but these can be connected to form a solution curve, i.e. a record of the temperature through time.

**Example 12.12** Use the formula from Example 12.11 and time steps of size $\Delta t = 1.0$ to find the first few values of temperature versus time. ■

**Solution:** Note that while $\Delta t = 1.0$ is not a "small step", we use it here only to illustrate the idea. Subdivide the horizontal ($t$) axis into steps of size $\Delta t$, and label the successive time values as $t_0, t_1, t_2, \ldots t_n$ where

$$t_0 = 0, \quad t_k = k\Delta t.$$

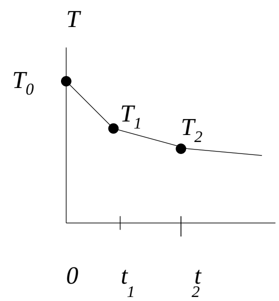



**Figure 12.7.** *Using Euler's method to approximate the temperature over time.*

This is shown in Figure 12.6. We will refer to the true solution (obtained with the exact formula) at a given time as $T(t_k)$. We will refer to the Euler's method value of the approximate solution at the time point $t_k$ as $T_k$. Then the initial condition will give us the value of $T_0 = T(0)$. We will find the temperatures at the successive times by

$$\begin{aligned} T_1 =& T_0 + 0.2(10 - T_0)\Delta t \\ T_2 =& T_1 + 0.2(10 - T_1)\Delta t \\ T_3 =& T_2 + 0.2(10 - T_2)\Delta t. \\ &\vdots \end{aligned}$$

By the time we get to the $k$'th step, we have:

$$T_{k+1} = T_k + 0.2(10 - T_k)\Delta t.$$

Again we note that at each step, the right hand side involves a calculation that depends only on known quantities.

In Table 12.4, we show a typical example of the method with initial value $T(0) = T_0 = 0$ and with a (large) step size $\Delta t = 1.0$. The true (red) and approximate (black) solutions are then shown in Figure 12.8. In this figure we illustrate four distinct solutions, each one representing an experiment with a different initial temperature. (For the approximate solution point values at are shown at each time step.)

The approximate solution is close to, but not identical to the true solution.

| time $t_k$ | approx solution $T_k$ | exact soln $T(t)$ |
|---|---|---|
| 0.0000 | 0.0000 | 0.0000 |
| 1.0000 | 2.0000 | 1.8127 |
| 2.0000 | 3.6000 | 3.2968 |
| 3.0000 | 4.8800 | 4.5119 |
| 4.0000 | 5.9040 | 5.5067 |
| 5.0000 | 6.7232 | 6.3212 |
| 6.0000 | 7.3786 | 6.9881 |
| 7.0000 | 7.9028 | 7.5340 |
| 8.0000 | 8.3223 | 7.9810 |

**Table 12.4.** *Euler's method applied to newton's law of cooling generates the values shown here.*

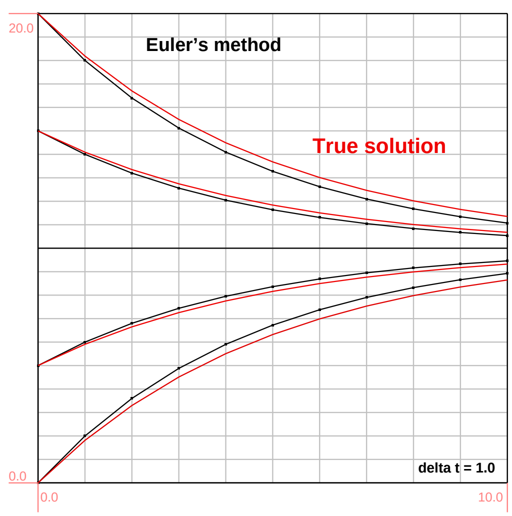



**Figure 12.8.** *A comparison of the true solution and the approximate solution provided by Euler's method.*

## Exercises

12.1. **An approximation for the square root:** Use a linear approximation to find a rough estimate of the following functions at the indicated points.

(a) $y = \sqrt{x}$ at $x = 10$. (Use the fact that $\sqrt{9} = 3$.)

(b) $y = 5x - 2$ at $x = 1$.

(c) $y = \sin(x)$ at $x = 0.1$ and at $x = \pi + 0.1$

12.2. Use the method of linear approximation to find the cube root of

(a) $0.065$ (Hint: $\sqrt[3]{0.064} = 0.4$)

(b) $215$ (Hint: $\sqrt[3]{216} = 6$)

12.3. Use the data in the graph in Figure 12.9 to make the best approximation you can to $f(2.01)$.

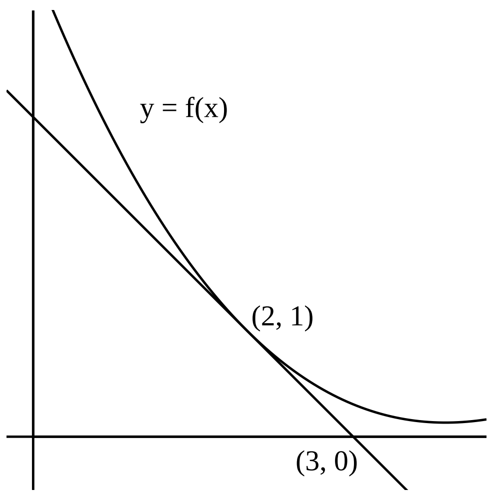



**Figure 12.9.** *Figure for Problem 3*

12.4. Using linear approximation, find the value of

(a) $\tan 44\,^\circ$, given $\tan 45\,^\circ = 1$, $\sec 45\,^\circ = \sqrt{2}$, and $1\,^\circ \approx 0.01745$ radians.

(b) $\sin 61\,^\circ$, given $\sin 60\,^\circ = \frac{\sqrt{3}}{2}$, $\cos 60\,^\circ = \frac{1}{2}$ and $1\,^\circ \approx 0.01745$ radians.

12.5. Approximate the value of $f(x) = x^3 - 2x^2 + 3x - 5$ at $x = 1.001$ using the method of linear approximation.

12.6. Use linear approximation to show that the each function below can be approximated by the given expression when $|x|$ is small (i.e. when $x$ is close to 0).

(a) $\sin x \approx x$

(b) $e^x \approx 1 + x$

(c) $\ln(1+x) \approx x$

12.7. Approximate the volume of a cube whose length of each side is $10.1$ cm.

12.8. **Finding critical points:** Use Newton's method to find critical points of the function $y = e^x - 2x^2$

12.9. **Estimating a square root:** Use Newton's method to find an approximate value for $\sqrt{8}$. (Hint: First think of a function, $f(x)$, such that $f(x) = 0$ has the solution $x = \sqrt{8}$).

12.10. **Finding points of intersection:** Find the point(s) of intersection of:

(a) $y_1 = 8x^3 - 10x^2 + x + 2$ and $y_2 = x^3 + 15x^2 - x - 4$ (Hint: an intersection point exists between $x = 3$ and $x = 4$).

(b) $y_1 = e^{-x}$ and $y_2 = \ln x$

12.11. **Roots of cubic equations:** Find the roots for each of the following cubic equations using Newton's method:

(a) $x^3 + 3x - 1 = 0$

(b) $x^3 + x^2 + x - 2 = 0$

(c) $x^3 + 5x^2 - 2 = 0$ (Hint: Find an approximation to a first root $a$ using Newton's method, then divide the left hand side of the equation by $(x - a)$ to obtain a quadratic equation, which can be solved by the quadratic formula.)

(d) $f(x) = x \ln x$ and $g(x) = 2$ (find the larger root only)

12.12. Use Newton's method to find an approximation (correct to $\pm 0.01$ ) for any roots of the equation

$$\sin(x) = \frac{1}{2}x$$

How many roots does this equation actually have ? Draw a diagram showing the functions $y = \sin(x)$ and $y = x/2$ on the same set of axes to help answer this question.

12.13. **More critical points:** Let $f(x) = (x^2 + x - 1)e^{-x^2}$.

(a) Find all critical points of $f(x)$ and indicate whether each one is a local maximum, local minimum, or neither.

(b) Graph $f(x)$. Indicate the regions where $f''(x)$ is positive.

12.14. Use Newton's method to find a value of $x$ that satisfies

$$e^x - x^2/2 = 0 \ .$$

Use the starting value $x_0 = 0$. Display your answer to $4$ significant figures.

12.15. We will use Newton's method to investigate zeros of the function

$$y = f(x) = \sin(x) - e^{-x}$$

(i.e. roots of the equation $f(x) = 0$.)

(a) Use the spreadsheet to graph the function

$$f(x) = \sin(x) - \exp(-x)$$

for values of $x$ from 0 up to 6.

(b) Find the equation of the tangent line to the curve at the point $x_0 = 1$ and plot it on the same graph.

(c) Use the value $x_0 = 1$ as an initial guess, and apply Newton's method. Plot the values $(x_1, 0), (x_2, 0)$, and $(x_3, 0)$ on the same graph produced by your spreadsheet to show how the points approach the zero of the function. Use this method to find the value of the root to four significant figures.

(d) Use a separate computation with the spreadsheet to determine the value of the root you find with the starting guess $x_0 = 4$. (You need not show this on the graph). Do you get to the same value as you did in part (c)? Why or why not?

(e) This function has lots of zeros. From your familiarity with the functions $\sin(x)$ and $e^{-x}$, where do you expect to find the roots for larger $x$ values?

Hand in a graph produced by the spreadsheet, and an additional page showing any calculations you made to get the answers. You can print the spreadsheet contents as part of your hand-in work.

12.16. **Comparing approximate and true solutions:**

(a) Use Euler's method to find an approximate solution to the differential equation

$$\frac{dy}{dx} = y$$

with $y(0) = 1$. Use a step size $h = 0.1$ and find the values of $y$ up to $x = 0.5$. Compare the value you have calculated for $y(0.5)$ using Euler's method with the true solution of this differential equation. What is the **error** i.e. the difference between the true solution and the approximation?

(b) Now use Euler's method on the differential equation

$$\frac{dy}{dx} = -y$$

with $y(0) = 1$. Use a step size $h = 0.1$ again and find the values of $y$ up to $x = 0.5$. Compare the value you have calculated for $y(0.5)$ using Euler's method with the true solution of this differential equation. What is the error this time?

12.17. **Euler's method applied to logistic growth:** Consider the logistic differential equation

$$\frac{dy}{dt} = ry(1 - y)$$

Let $r = 1$. Use Euler's method to find a solution to this differential equation starting with $y(0) = 0.5$, and step size $h = 0.2$. Find the values of $y$ up to time $t = 1.0$.

12.18. Use the spreadsheet and Euler's method to solve the differential equation shown below:

$$dy/dt = 0.5y(2-y)$$

Use a step size of $h = 0.1$ and show (on the same graph) solutions for the following four initial values:

$$y(0) = 0.5,\ y(0) = 1,\ y(0) = 1.5,\ y(0) = 2.25$$

For full credit, you must include a short explanation of what you did (e.g. 1-2 sentences and whatever equations you implemented on the spreadsheet.)

12.19. **Other differential equations:** For each of the following differential equations, find the approximate solution by Euler's method in the specified interval using the given initial condition and step size (display three decimal places for your answer).

(a) $\dfrac{dy}{dx} = y$, $y(0) = \frac{1}{2}$. Find the five successive values of $y(x)$ using a step size of step size $\Delta x = h = 0.1$. (your values will correspond to points along the interval $0 < x < 0.5$).

(b) $\dfrac{dy}{dx} = x + y$, $y(0) = 1$, step size $0.1$ on the interval $[-0.5, 0]$. (Note: starting at $x = 0$, you are going in the negative $x$ direction.)

(c) $\dfrac{dy}{dx} = \sqrt{x+y}$, $y(1) = 2$, find $y$ on $[1, 2]$ in $5$ steps.

12.20. For each of the following differential equations, find value of $y$ at the specific point by Euler's method using the given initial condition and step size (display three decimal places for your answer).

(a) $\dfrac{dy}{dx} = x + y^2$, $y(0) = 1$, step size $h = 0.02$, find $y(0.1)$.

(b) $\dfrac{dy}{dx} = \sqrt{x^2 + y^2}$, $y(0.2) = 0.5$, find $y(0.4)$ in $4$ steps.

## Chapter 13

# More Differential Equations

## 13.1 Introduction

In our discussion of exponential functions, we briefly encountered the idea of a differential equation. We saw that verbal descriptions of the rate of change of a process (for example, the growth of a population) can sometimes be expressed in the format of a differential equation, and that the functions associated with such equations allow us to predict the behaviour of the process over time.

In this chapter, we will develop some of these ideas further, and collect a variety of methods for understanding what differential equations mean, how they can be understood, and how they predict interesting behaviour of a variety of physical and biological systems.

First, a brief review of what we have seen about differential equations so far:

1. A differential equation is a statement linking the rate of change of some state variable with current values of that variable. An example is the simplest population growth model: If $N(t)$ is population size at time $t$:
$$\frac{dN}{dt} = kN.$$
2. A solution to a differential equation is a function that satisfies the equation. For instance, the function $N(t) = Ce^{kt}$ (for any constant $C$) is a solution to the above unlimited growth model. (We checked this by the appropriate differentiation in a previous chapter.) Graphs of such solutions (e.g. N versus t) are called solution curves.
3. To select a specific solution, more information is needed: Namely, some starting value (initial condition) is needed. Given this information, e.g. $N(0) = N_0$, we can fully characterize the desired solution.
4. So far, we have seen simple differential equations with simple functions for their solutions. In general, it may be quite challenging to make the connection between the differential equation (stemming from some application or model) with the solution (which we want in order to understand and predict the behaviour of the system.)

In this chapter we will expand our familiarity with differential equations and assemble a variety of techniques for understanding these. We will encounter both qualitative and quantitative methods. Geometric as well as algebraic techniques will form the core of the concepts here discussed.

## 13.2 Review and simple examples

### 13.2.1 Simple exponential growth and decay

We first review the simplest differential equation representing an exponential growth model.

**Example 13.1 (Exponential growth, revisited)** Characterize the solutions to the exponential growth model

$$\frac{dy}{dt} = y$$

with initial condition

$$y(0) = y_0.$$

■

**Solution:** We know that solutions are

$$y(t) = y_0 e^t.$$

These are functions that grow with time, as shown on the left panel in Figure 13.1.

**Example 13.2 (Exponential decay:)** What are solutions to the differential equation

$$\frac{dy}{dt} = -y$$

with initial condition

$$y(0) = y_0$$

■

**Solution:** This differential equation has solutions of the form

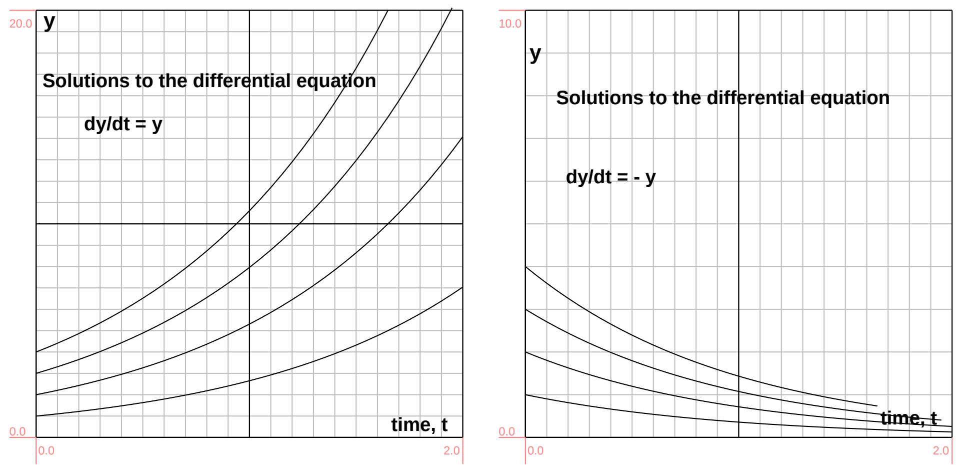
$$y(t) = y_0 e^t,$$

which are functions that decrease with time. We show some of these on the right panel of Figure 13.1. (Both graphs were produced with Euler's method and a spreadsheet.)

**Example 13.3** Suppose we are given a differential equation in a related, but slightly different form

$$\frac{dy}{dt} = 1 - y,$$

with initial condition $y(0) = y_0$. Determine the solutions to this differential equation. ■




**(a)** **(b)**

**Figure 13.1.** *Simple exponential growth and decay*

**Solution:** In this section we display the solutions to this equation, and study its properties. We show how a simple transformation of the variable can lead us to a solution to this equation.

Let $v = 1 - y$. Then

$$\frac{dv}{dt} = -\frac{dy}{dt}$$

But $dy/dt = 1 - y$, so that

$$\frac{dv}{dt} = -(1 - y) = -v.$$

The differential equation has been simplified (when written in terms of the variable $v$): It is just

$$\frac{dv}{dt} = -v.$$

This means that we can write down its solution by inspection, since it has the same form as the exponential decay equation studied previously:

$$v(t) = v_0 e^{-t}$$

Observe, also, that the initial condition for $y$ implies that at time $t = 0$ $v(0) = 1 - y(0) = 1 - y_0$. We now have:

$$v(t) = (1 - y_0)e^{-t}$$
$$1 - y(t) = (1 - y_0)e^{-t}.$$

Finally, we can arrive at an expression for $y$ which is what we were looking for originally:

$$y(t) = 1 - (1 - y_0)e^{-t}.$$

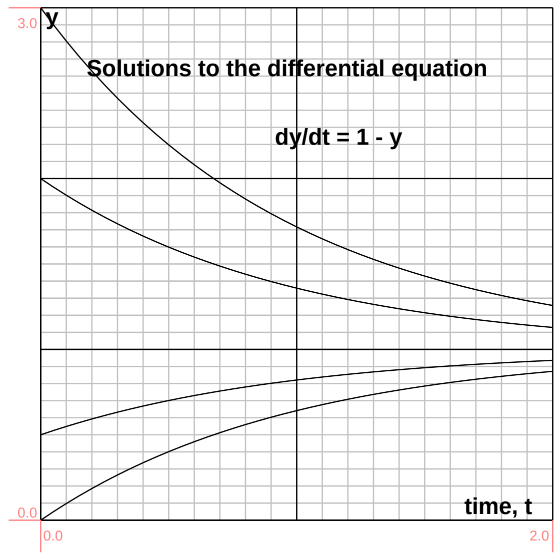



**Figure 13.2.** *Solutions are functions that approach the value* $y = 1$

This is an exact formula that predicts the values of $y$ through time, starting from any initial value.

## 13.3 Newton's law of cooling

Consider an object at temperature $T(t)$ in an environment whose ambient temperature is $E$. Depending on whether the object is cooler or warmer than the environment, the object will heat up or cool down. From common experience we know that after a long time, we should find that the temperature of the object will be essentially equal to that of its environment.

Newton formulated a hypothesis to describe the rate of change of temperature. He assumed that

**The rate of change of temperature $T$ of an object is proportional to the difference between its temperature and the ambient temperature, $E$.**

$$\frac{dT}{dt} \text{ is proportional to } (T(t) - E)$$

so that

$$\frac{dT}{dt} = k(E - T(t)), \quad \text{where} \quad k > 0.$$

Here we have used the proportionality constant $k > 0$ to arrive at the appropriate sign of the Right Hand Side (RHS). (Otherwise, if the expression on the right were $k(T(t) -$

$E$), then the direction of the change would be incorrect (a hotter object would get hotter in a cold room, etc). This is an example of a differential equation linking the current temperature $T(t)$ to its rate of change.

In order to predict what happens, we need to know the starting value of $T$. This is supplied in separate information, called the initial condition. For example, we would be given the value of some constant $T_0$ such that at time $t = 0$ the temperature is $T(0) = T_0$. The differential equation together with an initial condition is called an *initial value problem*. Below, we will show how a solution to this problem can be found in several ways. One technique involves seeking a formula for the function that has the property so described. This is called an analytic solution. Numerical solutions, obtained by an approximation method such as Euler's method, shortcut the need for such functional descriptions. Finally qualitative techniques use the differential equation directly to analyze and understand the overall behaviour.

### An analytic solution to Newton's law of cooling

Considering the temperature $T(t)$ as a function of time, we would like to solve the differential equation

$$\frac{dT}{dt} = k(E - T),$$

together with the initial condition

$$T(0) = T_0.$$

We will assume that the ambient temperature, $E$ is constant, as is the thermal conductivity, $k > 0$. We are hoping to identify a function $T(t)$ that satisfies this equation, i.e. such that when we differentiate this function we find that

$$\frac{dT(t)}{dt} = k(E - T(t)).$$

The same "trick" as before can be used to convert this to an equation that we know how to solve. If we define a new variable, $v(t) = E - T(t)$, we can show that

$$\frac{dv(t)}{dt} = -kv$$

(This is left as an exercise for the reader.) We can also see that $v(0) = E - T(0) = E - T_0$. Just as in the previous example, when the dust clears, we can find the formula for the solution, which turns out to be

$$T(t) = E + (T_0 - E)e^{-kt}.$$

In Figure 13.3 we show a number of the curves that describe this behaviour for five different starting values of the temperature. (We have set $E = 10$ and $k = 0.2$ in this case.) This set of curves is often called the **solution curves** to the differential equation.

It is evident that finding the full analytic solution to a differential equation can involve a bit of trickery. Indeed, many differential equations will pose great challenges, and

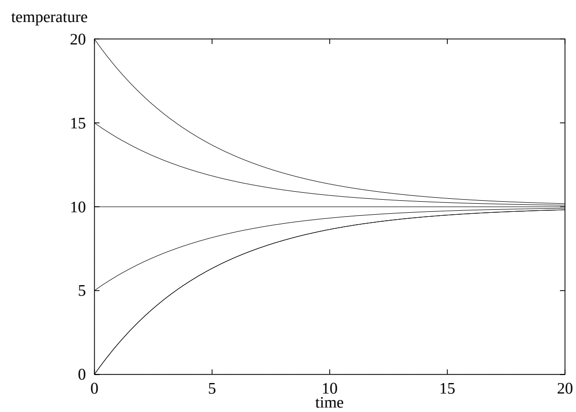



**Figure 13.3.** *Temperature versus time for a cooling object*

some will have no analytic solutions at all. Techniques for solving some of the simpler differential equations forms an important part of mathematics.

Now that we have a detailed solution to the differential equation representing Newton's Law of Cooling, we can apply it to making exact determinations of temperatures over time, or of time at which a certain temperature was attained. An example in which this is done is presented in the following section.

### Application of Newton's law of cooling

**Example 13.4 (Murder mystery:)** It is a dark clear night. The air temperature is $10°$ C. A body is discovered at midnight. Its temperature is then $27°$ C. One hour later, the body has cooled to $24°$ C. Use Newton's law of cooling to determine the time of death. ■

**Solution:** We will assume that the temperature of the person just before death is $37°$ C, i.e. normal body temperature in humans. Letting the time of death be $t = 0$, this would mean that $T(0) = T_0 = 37$. We want to find how much time elapsed until the body was found, i.e. the value of $t$ at which the temperature of the body was $27°$ C. We are told that the ambient temperature is $E = 10$, and we will assume that this was constant over the time span being considered. Newton's law of cooling states that

$$\frac{dT}{dt} = k(10 - T).$$

The solution to this equation is

$$T(t) = 10 + (37 - 10)e^{-kt} = 27,$$

or

$$27 = 10 + 27e^{-kt}, \quad \text{i.e.} \quad 17 = 27e^{-kt}.$$

We do not know the value of the constant $k$, but we have enough information to find it, since we know that at $t+1$ (one hour after discovery) the temperature was $24°$ C, i.e.

$$T(t+1) = 10 + (37-10)e^{-k(t+1)} = 24, \quad \Rightarrow \quad 24 = 10 + 27e^{-k(t+1)}.$$

Thus

$$14 = 27e^{-k(t+1)}.$$

We have two separate equations for the two unknowns $t$ and $k$. We can find both unknowns from these. Taking the ratio of the two equations we obtained we get

$$\frac{14}{17} = \frac{27e^{-k(t+1)}}{27e^{-kt}} = e^{-k}. \quad \Rightarrow \quad -k = \ln\left(\frac{14}{17}\right) = -0.194$$

Thus we have found the constant that describes the rate of cooling of the body. Now to find the time we can use

$$17 = 27e^{-kt} \quad \Rightarrow \quad -kt = \ln\left(\frac{17}{27}\right) = -0.4626$$

so

$$t = \frac{0.4626}{k} = \frac{0.4626}{0.194} = 2.384.$$

Thus the time of discovery of the body was 2.384 hours (i.e. 2 hours and 23 minutes) after death, i.e. at 9:37 pm.

## 13.4 Related examples

The differential equation that we have studied in Newton's Law of Cooling is one representative member of a class of differential equations that share similar behaviour. In this section we describe a more general form, comment on the general aspects of the solutions and list a few other examples. Consider the differential equation

$$\frac{dy}{dt} = a - by$$

where $a$, $b$ are constants together with the initial value

$$y(0) = y_0.$$

While this appears to be an example unrelated to our previous work, by a slight reinterpretation, we will see the connection.

Rewrite the differential equation in the form

$$\frac{dy}{dt} = b\left(\frac{a}{b} - y\right)$$

Now note that our previous differential equation for cooling can be translated into the new equation by the following correspondence:

$$T(t) \to y(t), \qquad E \to \frac{a}{b}, \qquad k \to b.$$

Rewriting Newton's Law of Cooling in this notation produces the equation given above. But we are already familiar with all aspects of the solution to the previous problem of cooling, so, just be reinterpreting it in terms of new quantities, we get the corresponding solution to the more general differential equation:

$$y(t) = \left(y_0 - \frac{a}{b}\right) e^{-bt} + \frac{a}{b}$$

Remarks made in our discussion of cooling should carry over directly to this function and to the behaviour it describes. For example, we find that $dy/dt = 0$ for $y = a/b$. We also note that after a long time the value of $y(t)$ should approach $a/b$.

### Friction and terminal velocity

The velocity of a falling object changes due to the acceleration of gravity, but friction has an effect of slowing down this acceleration. The differential equation satisfied by the velocity $v(t)$ of the falling object is

$$\frac{dv}{dt} = g - kv$$

where $g$ is acceleration due to gravity and $k$ is a constant that represents the effect of friction.

### Production and removal of a substance

An infusion containing a fixed concentration of substance is introduced into a fixed volume. Inside the volume, a chemical reaction results in decay of the substance at a rate proportional to its concentration. Letting $c(t)$ denote the time-dependent concentration of the substance, we would obtain a differential equation of the form

$$\frac{dc}{dt} = K_{\text{in}} - \gamma c$$

where $K_{\text{in}}$ represents the rate of input of substance and $\gamma$ the decay rate.

We can understand the behaviour of these systems by translating our notation from the general to the specific forms given above. For example,

$$c(t) \to y(t), \qquad K_{\text{in}} \to a, \qquad \gamma \to b.$$

Thus the behaviour found in the general case, can be interpreted in each of the specific situations of interest.

## 13.5 Qualitative methods

Finding the formula that described temperature over time involved some convenient recasting of the problem into familiar form. However, not all differential equations are as easily solved analytically. Furthermore, even when we find the analytic solution, it is not always easy to interpret, graph, or understand. This motivates a number of simpler qualitative methods that lead us to an overall understanding of the behaviour directly from information contained in the differential equation, without the challenges of finding a full functional form of the solution. We describe some of these methods below.

### 13.5.1 Rates of change

A differential equation

$$\frac{dy}{dt} = f(y)$$

encodes information about the rate of change $dy/dt$ of the variable $y$, and how this rate of change is linked to the current value of $y$. It can be helpful to sketch the way that the rate of change depends on $y$, because certain interesting conclusions emerge from such diagrams.

**Example 13.5** Consider the differential equation

$$\frac{dy}{dt} = y - y^3. \tag{13.1}$$

Then the rate of change is $f(y) = y - y^3$ . (This is the function on the right hand side of the differential equation.) Use this observation to determine values of $y$ for which $y$ is static (does not change). ■

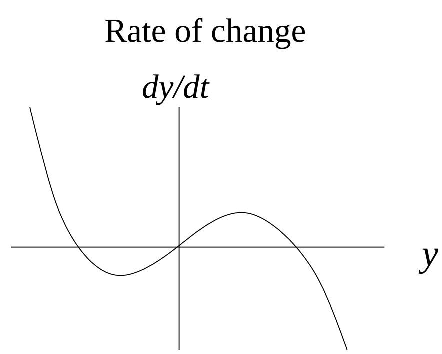



**Figure 13.4.** *The function $y - y^3$ as a rate of change of $y$*

**Solution:** Plotting the rate of change $f(y)$ versus $y$ leads to the sketch shown in Figure 13.4. We see from this sketch that the rate of change is zero for $y = -1, 0, 1$. This means that $y$ does not change at these values, i.e. if we start a system off with $y(0) = 0$, or $y(0) = \pm 1$, the value of $y$ will be static. The three places at which this happens are marked by heavy dots in Figure 13.5(a).

**Example 13.6** Now continue the ideas of Example 13.5 to find the range of values of $y$ for which $y$ decreases, and for which $y$ increases. ■

**Solution:** We also see that $f(y) < 0$ for $-1 < y < 0$ and for $y > 1$. This means that the rate of change of $y$ is negative whenever $-1 < y < 0$ or $y > 1$, which, in turn, implies that if the value of $y(t)$ falls in either of these intervals at any time $t$, then $y(t)$ must be a decreasing function of time. On the other hand, for $0 < y < 1$ or for $y < -1$, we have $f(y) > 0$, i.e., the rate of change of $y$ with respect to $t$ is positive. This says that $y(t)$ must be increasing. We can indicate these observations with arrows marking the direction of change of $y$. Along the $y$ axis (which is now on the horizontal axis of the sketch) increasing $y$ means motion to the right, decreasing $y$ means motion to the left. This has been done in Figure 13.5(b). We see from the directions marked that there is a tendency for $y$ to move away from the value $y = 0$ and to approach either of the values $1$ or $-1$ as time goes by. (What actually happens depends on the initial value of $y$.)

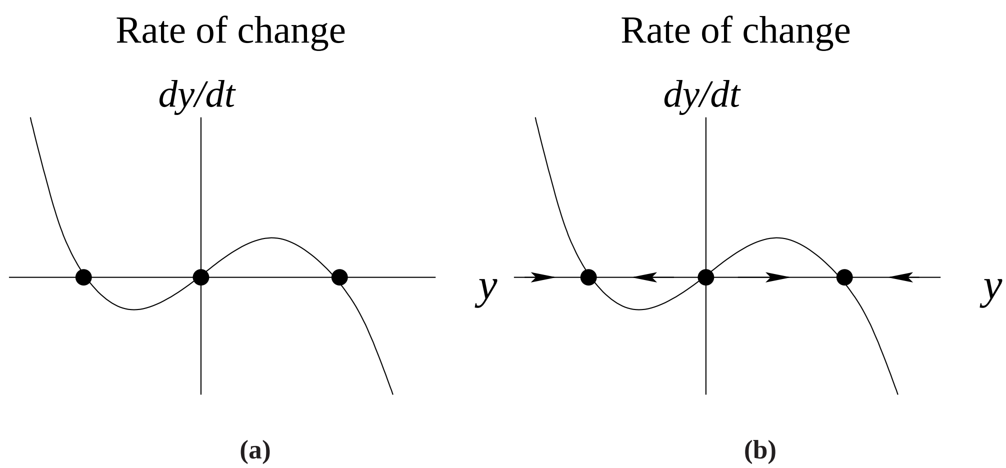



**Figure 13.5.** *Static points and intervals for which $y$ increases or decreases for the differential equation* (13.1)*. See Examples 13.5 and 13.6.*

**Example 13.7 (A cooling object:)** Sketch the same typeof diagrams for the problem of a cooling object and interpret its meaning. $\blacksquare$

**Solution:** Here, the differential equation is

$$\frac{dT}{dt} = 0.2(10 - T).$$

Here, the function $f(T) = 0.2(10 - T)$ is the rate of change associated with a given temperature $T$. A sketch of the rate of change, $F(T)$ versus the temperature $T$ is shown in Figure 13.6(a).

**Example 13.8** Create a similar qualitative sketch for the more general form of linear differential equation

$$\frac{dy}{dt} = a - by. \tag{13.2}$$

$\blacksquare$

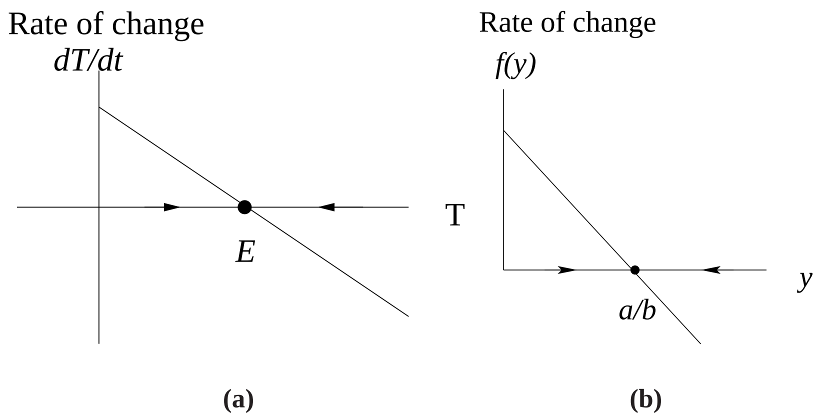



**Figure 13.6.** *(a) Figure for Example 13.7, (b) Qualitative sketch for Eqn.* (13.2) *in Example 13.8.*

**Solution:** The rate of change of $y$ is given by the function $f(y) = a - by$. This is shown in the sketch in Figure 13.6(b). We see that there is one point at which $f(y) = 0$, namely at $y = a/b$. We also see from this figure that the value of $y$ will be approaching this value over time. We can say that, just from the form of the differential equation, even without knowing the formula of the solution, we find that after a long time, the value of $y$ will be approximately $a/b$.

## 13.6 Slope fields

In this section we discuss yet another geometric way of understanding the behaviour predicted by a differential equation. This time, our plots will have a time axis, and we will try to figure out something about the actual solution curves, without resorting to the formulae for their analytic solutions.

We have already seen that solutions to a differential equation of the form

$$\frac{dy}{dt} = f(y)$$

are curves in the $y, t$ plane that describe how $y(t)$ changes over time. (Thus, these curves are graphs of functions of time.) Each initial condition $y(0) = y_0$ is associated with one of these curves, so that together, these curves form a *family* of solutions. What do these curves have in common geometrically?

Simply stated, the slope of the tangent line (which is just $dy/dt$) at any point on any of the curves has to be related to the value of the $y$ coordinate of that point. That is exactly what the differential equation is saying: if the value of $y$ is such and such, then the slope at that point must be $f(y)$. Below we see this related but new way of understanding the differential equations already discussed in this chapter.

| $y$ | $dy/dt$ | slope of tangent line | behaviour of $y$ |
|---|---|---|---|
| -2 | -4 | -ve | decreasing |
| -1 | -2 | -ve | decreasing |
| 0 | 0 | 0 | no change in y |
| 1 | 2 | +ve | increasing |
| 2 | 4 | +ve | increasing |

**Table 13.1.** *Table of derivatives and slopes for the differential equation* (13.3) *of Example 13.9.*

**Example 13.9** Consider the differential equation

$$\frac{dy}{dt} = 2y. \tag{13.3}$$

Compute some of the slopes for various $y$ values and use this to sketch a **slope field**. ■

**Solution:** We create a table of derivative values

Figure 13.7 illustrates the direction field and the corresponding solution curves.

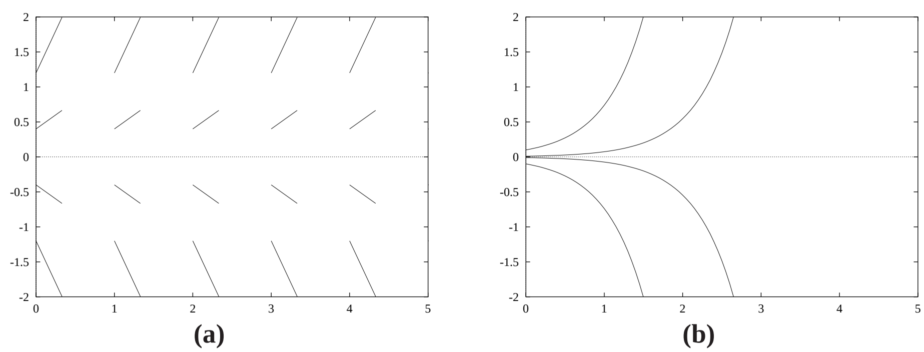



**Figure 13.7.** *Direction field and solution curves for Example 13.9.*

**Example 13.10** For example, consider the differential equation

$$\frac{dy}{dt} = y - y^3.$$

Create a slope field diagram for this differential equation. ■

**Solution:** Any curve that satisfies this equation and that goes through $y = 1$, for example, must have a tangent line of slope $dy/dt = 1 - 1^3 = 0$. This is true regardless of the time

| $y$ | $dy/dt$ |
|---|---|
| 0.0000 | 0.0000 |
| 0.2500 | 0.2344 |
| 0.5000 | 0.3750 |
| 0.7500 | 0.3281 |
| 1.0000 | 0.0000 |
| 1.2500 | -0.7031 |
| 1.5000 | -1.8750 |
| 1.7500 | -3.6094 |
| 2.0000 | -6.0000 |
| 2.2500 | -9.1406 |
| 2.5000 | -13.1250 |

**Table 13.2.** *Table for Example 13.10*

$t$, (since the function $f(y) = y - y^3$ does not depend explicitly on $t$ directly - only on $y$). For example, a curve that goes through the point $y = 2$ would have tangent line of slope $dy/dt = 2 - 2^3 = -6$ at that point. In principle, we could find the correspondence between $y$ values and slopes for many possible $y$ values and use the sketch generated with this information (the *slope field*) to understand what solution curves look like.

In Table 13.2 we show the slopes associated with various points in the $y, t$ plane given by the differential equation in this example.

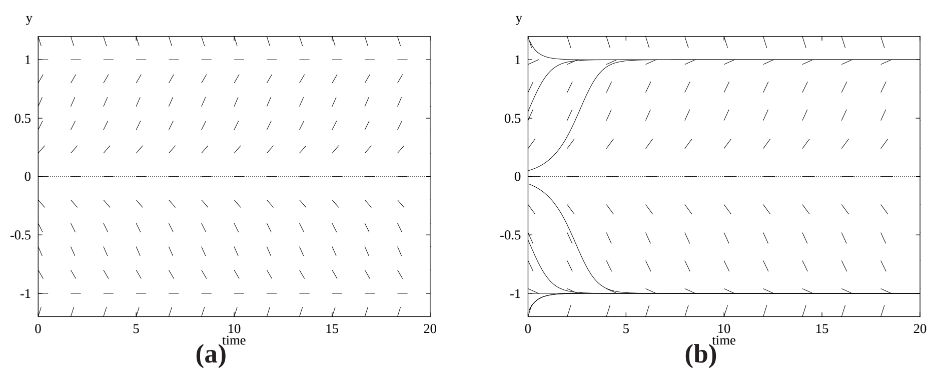



**Figure 13.8.** *Figure for Example 13.10.*

We see above that if $y = 1$ or $y = 0$, the slope of the tangent line is zero (indicating a horizontal tangent line), whereas if $0 < y < 1$, the slopes are positive, indicating that $y$ is increasing. We also see that if $y > 1$, the slopes are negative ($y$ is decreasing), and steeper for larger values of $y$. This agrees with what we have seen earlier with our plot of the rate of change. The picture looks different now, because we are explicitly including the

| temp $T$ | slope $dT/dt$ $= 0.2(10 - T)$ |
|---|---|
| 0.0000 | 2.0000 |
| 2.0000 | 1.6000 |
| 4.0000 | 1.2000 |
| 6.0000 | 0.8000 |
| 8.0000 | 0.4000 |
| 10.0000 | 0.0000 |
| 12.0000 | -0.4000 |
| 14.0000 | -0.8000 |
| 16.0000 | -1.2000 |
| 18.0000 | -1.6000 |
| 20.0000 | -2.0000 |

**Table 13.3.** *Slopes for Example 13.11.*

time axis and showing the curves $y(t)$, whereas in our previous sketch, arrows were used to indicate whether $y$ was increasing or decreasing, without showing a time axis.

**Example 13.11** Sketch a slope field and solution curves for the problem of a cooling object, and specifically for

$$\frac{dT}{dt} = 0.2(10 - T), \tag{13.4}$$

$\blacksquare$

**Solution:** The collection of curves shown in Figure 13.3 are solution curves for the $T(t)$,

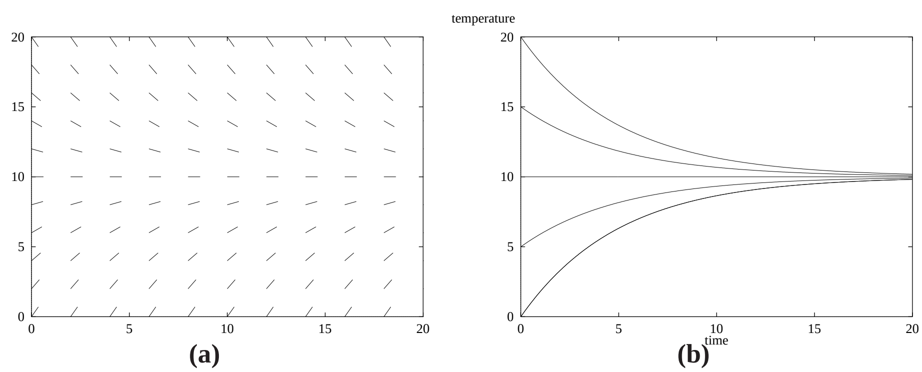



**Figure 13.9.** *Slope field for a cooling object of Example 13.11.*

the function $f(T) = 0.2(10 - T)$ also corresponds to the slope of the tangent lines to the curves in Figure 13.3.

In Table 13.3, we calculate values of the slope $f(T) = 0.2(10 - T)$ for a number of value of $T$, and these are shown plotted as a slope field in Figure 13.9.

## 13.7 Steady states and stability

We notice from Figure 13.9 that for a certain initial temperature, namely $T_0 = 10$ there will be no change with time. Indeed, we find that at this temperature the differential equation specifies that $dT/dt = 0$. Such a value is called a **steady state**.

**Definition 13.12 (Steady state:).** *A Steady state is a state in which a system is not changing.*

**Example 13.13** Find the steady states of the equation

$$\frac{dy}{dt} = y - y^3, \tag{13.5}$$

$\blacksquare$

**Solution:** To find steady states we look for $y$ such that $dy/dt = 0$ so that

$$\frac{dy}{dt} = y - y^3 = 0,$$

i.e. we would have $y = 0$ and $y = \pm 1$ as three steady states.

From Figure 13.5, we see that solutions starting *close to* $y = 1$ tend to get closer and closer to this value. We refer to this behaviour as **stability** of the steady state..

**Definition 13.14 (Stability:).** *We say that a steady state is* **stable** *if states that are initially close enough to that steady state will get closer to it with time. We say that a steady state is* **unstable**, *if states that are initially very close to it eventually move away from that steady state.*

**Example 13.15** Find a stable and an unstable steady state of Eqn. (13.5) in Example 13.13 are stable. $\blacksquare$

**Solution:** From any starting value of $y > 0$ in this example, we see that *after a long time*, the solution curves tend to approach the value $y = 1$. States close to $y = 1$ get closer to it, so this is a stable steady state. For the steady state $y = 0$, we see that initial conditions close to $y = 0$ do not get closer, but rather move away over time. Thus, this steady state is unstable. It turns out that there is also a stable steady state at $y = -1$.

As seen in Example 13.13, even though we do not have any formula that connects $y$ values with specific times, we can say qualitatively what happens to any positive initial values after a long time: they all approach the value $y = 1$.

## 13.8 The logistic equation for population growth

The ideas developed in this chapter, and particularly the qualitative and geometric ideas, can help us to understand a variety of differential equations that stem from biological, physical, or chemical applications. When these equations are **nonlinear**, i.e. when the function $f(y)$ in

$$\frac{dy}{dt} = f(y)$$

is not a simple linear function of $y$, then it can be quite challenging to discover analytic solutions. However, the qualitative methods described above can help to understand what the equations predict.

In Chapter 9, we have seen that the assumption of a constant growth rate leads to a differential equation for population level $N(t)$ of the form

$$\frac{dN}{dt} = rN,$$

which has exponential solutions. This means that only two possible behaviours are obtained: explosive growth if $r > 0$ or extinction if $r < 0$.

Most natural populations are found to attain some level that does not expand continually. This is due to limitations in finding resources or in competing for a fixed territory size. Such populations, under ideal conditions in a fixed environment, would generally stabilize at some typical density, rather than going extinct or continuing an exponential growth. This motivates revising our previous model.

In the modification studied here, we will let $N(t)$ represent the size of a population at time $t$. Consider the differential equation

$$\frac{dN}{dt} = rN\frac{(K-N)}{K}. \tag{13.6}$$

We call this the differential equation the **logistic equation**. Here the parameter $r > 0$ is called the **intrinsic growth rate** and $K > 0$ is the **carrying capacity**. Both these parameters are assumed to be positive constants.

The logistic equation can be justified in one of several ways as a convenient simple model for population growth that has a greater relevance than exponential growth. In the form written above, we could interpret it at

$$\frac{dN}{dt} = R(N)N.$$

where $R(N) = r(K-N)/K$ is a so-called **density dependent growth rate**. (It replaces the previously assumed constant growth rate $rN$, that leads to unlimited growth.) Rewritten in the form

$$\frac{dN}{dt} = rN - bN^2$$

(where $b = r/K$ is a positive quantity), it can be interpreted as the usual linear growth term $rN$, with a superimposed quadratic (nonlinear) rate of death due to overcrowding or competition $bN^2$. We already know that this quadratic term will dominate for larger values of $N$, and this means that when the population is crowded, the loss of individuals is greater than the rate of reproduction.

In this section we will familiarize ourselves with the behaviour predicted by the logistic equation.

**Example 13.16** Find the steady states of the Logistic Equation (13.6). ∎

**Solution:** To determine the steady states of the equation (13.6), i.e. the level of population that would not change over time, we look for values of $N$ such that

$$\frac{dN}{dt} = 0$$

This leads to

$$rN\frac{(K-N)}{K} = 0$$

which has solutions $N = 0$ (no population at all) or $N = K$ (the population is at its carrying capacity).

The logistic equation has a long history in modelling population growth of microorganisms, animals, and human populations. It is justified either by considering it to be a special case of the **density dependent** growth equation

$$\frac{dN}{dt} = R(N)N$$

(In that case the reproductive rate has the form $R(N) = r(K-N)/K$), or, equivalently, it can be considered to fall into a class of equations that have the form

$$\frac{dN}{dt} = rN - bN^2$$

(where the constant is $b = r/K$), which means that a constant rate of reproduction $rN$ is modified by a quadratic mortality rate $bN^2$. The mortality would tend to dominate only for larger values of the population, i.e. if conditions are crowded so that animals have to compete for resources or habitat. (This stems from the fact that the quadratic term is smaller than the linear term near $N = 0$, but dominates for large $N$, as we have already discussed in Chapter 1.)

It is often desirable to formulate the problem in the simplest possible terms. We can do this by a process called rescaling:

**Example 13.17 (Rescaling:)** Define a new variable

$$y(t) = \frac{N(t)}{K}.$$

Interpret what this variable represents and show that the Logistic equation can be written in a simpler form in terms of this variable. ∎

**Solution:** The rescaled variable, $y(t)$, is a population density expressed in units of the carrying capacity. (For example, if the environment can sustain 1000 individuals, and the current population size is $N = 950$ then the value of $y$ is $y = 0.950$.) Since $K$ is assumed constant,

$$\frac{dy}{dt} = \frac{1}{K}\frac{dN}{dt}$$

and we can simplify the equation:

$$\frac{dy}{dt} = ry(1-y). \tag{13.7}$$

We observe that indeed, this equation "looks simpler" and also has only one constant parameter left in it. It is generally the case that rescaling reduces the number of parameters in a differential equation such as seen here.

**Example 13.18** Draw a plot of the rate of change $dy/dt$ versus the value of $y$ for the rescaled logistic equation (13.7). ■

**Solution:** This plot is shown in Figure 13.10. The steady states are located at $y = 0, 1$ (which correspond to $N = 0$ and $N = K$ in the original variable.) We also find that in the interval $0 < y < 1$, the rate of change is positive, so that $y$ increases, whereas for $y > 1$, the rate of change is negative, so $y$ decreases. Since $y$ refers to population size, we need not concern ourselves with behaviour for $y < 0$.

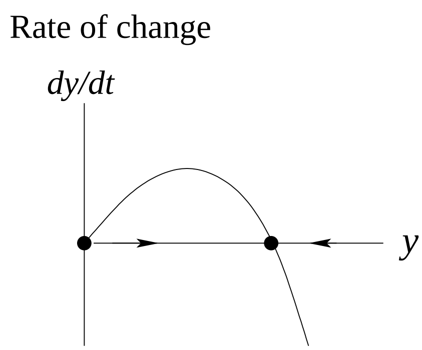



**Figure 13.10.** *Plot of $dy/dt$ versus y for the rescaled logistic equation*(13.7).

From Figure 13.10 we expect to see solutions to the differential equation that approach the value $y = 1$ after a long time. (The only exception to this would be the case where there is no population present at all, i.e. $y = 0$, in which case, there would be no change.) Restated in terms of the original quantities in the model, the population $N(t)$ should approach $K$ after a long time. We now look at the same equation from the perspective of the slope field.

**Example 13.19** Draw a slope field for the rescaled logistic equation with $r = 0.5$, that is for

$$\frac{dy}{dt} = 0.5y(1-y). \tag{13.8}$$

| population $y$ | slope $dy/dt$ $= 0.5y(1-y)$ |
|---|---|
| 0.0000 | 0.0000 |
| 0.1000 | 0.0450 |
| 0.2000 | 0.0800 |
| 0.3000 | 0.1050 |
| 0.4000 | 0.1200 |
| 0.5000 | 0.1250 |
| 0.6000 | 0.1200 |
| 0.7000 | 0.1050 |
| 0.8000 | 0.0800 |
| 0.9000 | 0.0450 |
| 1.0000 | 0.0000 |
| 1.1000 | -0.0550 |
| 1.2000 | -0.1200 |

**Table 13.4.** *Slopes for the logistic equation* (13.8).

$\blacksquare$

**Solution:** We generate slopes in Table 13.4 for different values of $y$ and plot the slope field in Figure 13.11(a).

Finally, we can use the numerical technique of Euler's method to graph out the full solution to this differential equation from some set of initial conditions.

**Example 13.20 (Numerical solutions to the logistic equation:)** Use Euler's method to approximate the solutions to the logistic equation (13.8). $\blacksquare$

**Solution:** In Figure 13.11(b) we show a set of solution curves, obtained by solving the equation numerically using Euler's method and the spreadsheet. To obtain these solutions, a value of $h = \Delta t = 0.1$ was used, the time axis was discretized (subdivided) into steps of size 0.1. A starting value of $y(0) = y_0$ at time $t = 0$ were picked. The successive values of $y$ were calculated as follows:

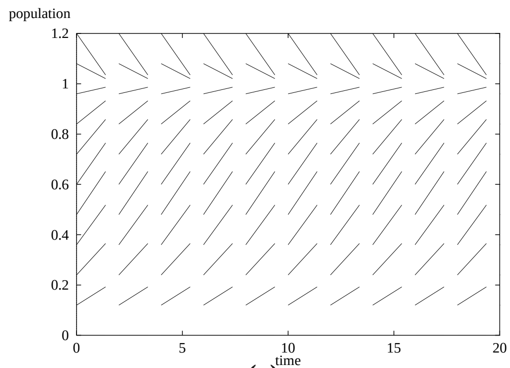
$$y_1 = y_0 + 0.5y_0(1 - y_0)h$$

$$y_2 = y_1 + 0.5y_1(1 - y_1)h$$

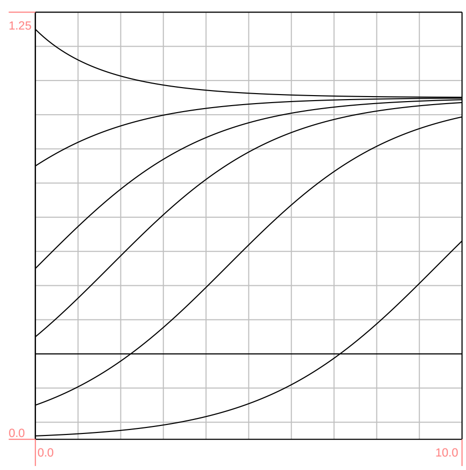
$$\vdots$$

$$y_{k+1} = y_k + 0.5y_k(1 - y_k)h$$

(The attractive feature of using a spreadsheet is that this repetition can be handled automatically by dragging the cell entry containing the results for one iteration down to generate other iterations. Another attractive feature is that once the method is implemented, it is possible to change the initial condition very easily, just by changing a single cell entry.




**(a)**




**(b)**

**Figure 13.11.** *(a) Slope field and (b) solution curves for the logistic equation 13.8.*

From these results, we see that solution curves approach $y = 1$. This means (in terms of the original variable, $N$) that the population will approach the carrying capacity $K$ for all nonzero starting values, i.e. there will be a stable steady state with a fixed level of the population.

**Example 13.21** Some of the curves shown in Figure 13.11(b) have an inflection point, but others do not. Use the differential equation to determine which of the solution curves will have an inflection point. ■

**Solution:** From Figure 13.11(b) we might observe that the curves that emanate from initial values in the range $0 < y_0 < 1$ are all increasing. Indeed, this follows from the fact if $y$ is in this range, the rate of change $ry(1 - y)$ is a positive quantity.

The logistic equation has the form

$$\frac{dy}{dt} = ry(1-y) = ry - ry^2$$

This means that (by differentiating both sides and remembering the chain rule)

$$\frac{d^2y}{dt^2} = r\frac{dy}{dt} - 2ry\frac{dy}{dt} = r\frac{dy}{dt}(1-2y).$$

An inflection point would occur at places where the second derivative changes sign, and in addition

$$\frac{d^2y}{dt^2} = 0.$$

From the above we see that this is possible for $dy/dt = 0$ or for $(1-2y) = 0$. We have already dismissed the first possibility because we have argued that the rate of change in nonzero in the interval of interest. Thus we conclude that an inflection point would occur whenever $y = 1/2$. Any initial condition satisfying $0 < y_0 < 1/2$ would eventually pass through $y = 1/2$ on its way up to the steady state level at $y = 1$, and in so doing, would have an inflection point.

## Exercises

13.1. Consider the differential equation

$$\frac{dy}{dt} = a - by$$

where $a$, $b$ are constants.

(a) Show that the function

$$y(t) = \frac{a}{b} - Ce^{-bt}$$

satisfies the above differential equation for any constant $C$.

(b) Show that by setting

$$C = \frac{a}{b} - y_0$$

we also satisfy the initial condition

$$y(0) = y_0.$$

Remark: You have now shown that the function

$$y(t) = \left(y_0 - \frac{a}{b}\right) e^{-bt} + \frac{a}{b}$$

is a solution to the *initial value problem* (i.e differential equation plus initial condition)

$$\frac{dy}{dt} = a - by, \quad y(0) = y_0.$$

13.2. For each of the following, show the given function $y$ is a solution to the given differential equation.

(a) $t \cdot \dfrac{dy}{dt} = 3y$, $y = 2t^3$.

(b) $\dfrac{d^2y}{dt^2} + y = 0$, $y = -2\sin t + 3\cos t$.

(c) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + y = 6e^t$, $y = 3t^2e^t$.

13.3. Show the function determined by the equation $2x^2 + xy - y^2 = C$, where $C$ is a constant and $2y \neq x$, is a solution to the differential equation $(x-2y)\dfrac{dy}{dx} = -4x-y$.

13.4. Find the constant $C$ that satisfies the given initial conditions.

(a) $2x^2 - 3y^2 = C$, $y|_{x=0} = 2$.

(b) $y = C_1e^{5t} + C_2te^{5t}$, $y|_{t=0} = 1$ and $\frac{dy}{dt}|_{t=0} = 0$.

(c) $y = C_1\cos(t - C_2)$, $y|_{t=\frac{\pi}{2}} = 0$ and $\frac{dy}{dt}|_{t=\frac{\pi}{2}} = 1$.

13.5. **Friction and terminal velocity:** The velocity of a falling object changes due to the acceleration of gravity, but friction has an effect of slowing down this acceleration. The differential equation satisfied by the velocity $v(t)$ of the falling object is

$$\frac{dv}{dt} = g - kv$$

where $g$ is acceleration due to gravity and $k$ is a constant that represents the effect of friction. An object is dropped from rest from a plane.

(a) Find the function $v(t)$ that represents its velocity over time.

(b) What happens to the velocity after the object has been falling for a long time (but before it has hit the ground)?

13.6. **Alcohol level:** Alcohol enters the blood stream at a constant rate $k$ gm per unit time during a drinking session. The liver gradually converts the alcohol to other, non-toxic byproducts. The rate of conversion per unit time is proportional to the current blood alcohol level, so that the differential equation satisfied by the blood alcohol level is

$$\frac{dc}{dt} = k - sc$$

where $k$, $s$ are positive constants. Suppose initially there is no alcohol in the blood. Find the blood alcohol level $c(t)$ as a function of time from $t = 0$, when the drinking started.

13.7. **Newton's Law of Cooling:** Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between the temperature of the object, $T$, and the ambient (environmental) temperature, $E$. This leads to the *differential equation*

$$\frac{dT}{dt} = k(E - T)$$

where $k > 0$ is a constant that represents the material properties and, $E$ is the ambient temperature. (We will assume that $E$ is also constant.)

(a) Show that the function

$$T(t) = E + (T_0 - E)e^{-kt}$$

which represents the temperature at time $t$ satisfies this equation.

(b) The time of death of a murder victim can be estimated from the temperature of the body if it is discovered early enough after the crime has occurred. Suppose that in a room whose ambient temperature is $E = 20$ degrees C, the temperature of the body upon discovery is $T = 30$ degrees, and that a second measurement, one hour later is $T = 25$ degrees. Determine the approximate time of death. (You should use the fact that just prior to death, the temperature of the victim was 37 degrees.)

13.8. **A cup of coffee:** The temperature of a cup of coffee is initially 100 degrees C. Five minutes later, ($t = 5$) it is 50 degrees C. If the ambient temperature is $A = 20$ degrees C, determine how long it takes for the temperature of the coffee to reach 30 degrees C.

13.9. **Glucose solution in a tank:** A tank that holds 1 liter is initially full of plain water. A concentrated solution of glucose, containing 0.25 gm/cm$^3$ is pumped into the tank continuously, at the rate 10 cm$^3$/min and the mixture (which is continuously stirred to keep it uniform) is pumped out at the same rate. How much glucose will there be in the tank after 30 minutes? After a long time? (Hint: write a differential equation for c, the concentration of glucose in the tank by considering the rate at which glucose enters and the rate at which glucose leaves the tank.)

13.10. **Pollutant in a lake:** (From the Dec 1993 Math 100 Exam) A lake of constant volume $V$ gallons contains $Q(t)$ pounds of pollutant at time $t$ evenly distributed throughout the lake. Water containing a concentration of $k$ pounds per gallon of pollutant enters the lake at a rate of $r$ gallons per minute, and the well-mixed solution leaves at the same rate.

(a) Set up a differential equation that describes the way that the amount of pollutant in the lake will change.

(b) Determine what happens to the pollutant level after a long time if this process continues.

(c) If $k = 0$ find the time $T$ for the amount of pollutant to be reduced to one half of its initial value.

13.11. **Slope fields:** Consider the differential equations given below. In each case, draw a slope field, determine the values of $y$ for which no change takes place [such values are called steady states] and use your slope field to predict what would happen starting from an initial value $y(0) = 1$.

(a) $\dfrac{dy}{dt} = -0.5y$

(b) $\dfrac{dy}{dt} = 0.5y(2 - y)$

(c) $\dfrac{dy}{dt} = y(2 - y)(3 - y)$

13.12. Draw a slope field for each of the given differential equations:

(a) $\frac{dy}{dt} = 2 + 3y$

(b) $\frac{dy}{dt} = -y(2 - y)$

(c) $\frac{dy}{dt} = 2 - 3y + y^2$

(d) $\frac{dy}{dt} = -2(3 - y)^2$

(e) $\frac{dy}{dt} = y^2 - y + 1$

(f) $\frac{dy}{dt} = y^3 - y$

(g) $\frac{dy}{dt} = \sqrt{y}(y - 2)(y - 3)^2$, $y \geq 0$.

(h) $\frac{dy}{dt} = 2e^y - 2$

(i) $\frac{dy}{dt} = A - \sin y$ (Hint: consider the cases $A < -1$, $A = -1$, $-1 < A < 1$, $A = 1$ and $A > 1$).

(j) $\frac{dy}{dt} - y = e^t$

13.13. For each of the differential equations (a) to (i) in Problem 12, plot $\frac{dy}{dt}$ as a function of $y$, draw the motion along the $y$ axis, identify the steady state(s) and indicate if the motions are toward or away from the steady state(s).

13.14. **Periodic motion:**

(a) Show that the function $y(t) = A\cos(wt)$ satisfies the differential equation

$$\frac{d^2y}{dt^2} = -w^2y$$

where $w > 0$ is a constant, and $A$ is an arbitrary constant. [Remark: Note that $w$ corresponds to the *frequency* and $A$ to the *amplitude* of an oscillation represented by the cosine function.]

(b) It can be shown using Newton's Laws of motion that the motion of a pendulum is governed by a differential equation of the form

$$\frac{d^2y}{dt^2} = -\frac{g}{L}\sin(y),$$

where $L$ is the length of the string, $g$ is the acceleration due to gravity (both positive constants), and $y(t)$ is displacement of the pendulum from the vertical. What property of the sine function is used when this equation is approximated by the Linear Pendulum Equation:

$$\frac{d^2y}{dt^2} = -\frac{g}{L}y.$$

(c) Based on this Linear Pendulum Equation, what function would represent the oscillations? What would be the frequency of the oscillations?

(d) What happens to the frequency of the oscillations if the length of the string is doubled?

13.15. **A sugar solution:** Sugar dissolves in water at a rate proportional to the amount of sugar not yet in solution. Let $Q(t)$ be the amount of sugar undissolved at time $t$. The initial amount is $100\ kg$ and after $4$ hours the amount undissolved is $70\ kg$.

(a) Find a differential equation for $Q(t)$ and solve it.

(b) How long will it take for $50\ kg$ to dissolve?

13.16. **Infant weight gain:** During the first year of its life, the weight of a baby is given by

$$y(t) = \sqrt{3t + 64}$$

where $t$ is measured in some convenient unit.

(a) Show that $y$ satisfies the differential equation

$$\frac{dy}{dt} = \frac{k}{y}$$

where $k$ is some positive constant.

(b) What is the value for $k$?

(c) Suppose we adopt this differential equation as a model for human growth. State concisely (that is, in one sentence) one feature about this differential equation which makes it a reasonable model. State one feature which makes it unreasonable.

13.17. **Cubical crystal:** A crystal grows inside a medium in a cubical shape with side length $x$ and volume $V$. The rate of change of the volume is given by

$$\frac{dV}{dt} = kx^2(V_0 - V)$$

where $k$ and $V_0$ are positive constants.

(a) Rewrite this as a differential equation for $\frac{dx}{dt}$.

(b) Suppose that the crystal grows from a very small "seed." Show that its growth rate continually decreases.

(c) What happens to the size of the crystal after a very long time?

(d) What is its size (that is, what is either $x$ or $V$) when it is growing at half its initial rate?

13.18. **Leaking water tank:** A cylindrical tank with cross-sectional area $A$ has a small hole through which water drains. The height of the water in the tank $y(t)$ at time $t$ is given by:

$$y(t) = (\sqrt{y_0} - \frac{kt}{2A})^2$$

where $k, y_0$ are constants.

(a) Show that the height of the water, $y(t)$, satisfies the differential equation

$$\frac{dy}{dt} = -\frac{k}{A}\sqrt{y}.$$

(b) What is the initial height of the water in the tank at time $t = 0$ ?

(c) At what time will the tank be empty ?

(d) At what rate is the **volume** of the water in the tank changing when $t = 0$?

13.19. Find those constants $a, b$ so that $y = e^x$ and $y = e^{-x}$ are both solutions of the differential equation

$$y'' + ay' + by = 0.$$

13.20. Let $y = f(t) = e^{-t} \sin t, \ -\infty < t < \infty$.

(a) Show that $y$ satisfies the differential equation $y'' + 2y' + 2y = 0$.

(b) Find all critical points of $f(t)$.

13.21. A biochemical reaction in which a substance $S$ is both produced and consumed is investigated. The concentration $c(t)$ of $S$ changes during the reaction, and is seen to follow the differential equation

$$\frac{dc}{dt} = K_{\text{max}}\frac{c}{k+c} - rc$$

where $K_{\text{max}}, k, r$ are positive constants with certain convenient units. The first term is a concentration-dependent production term and the second term represents consumption of the substance.

(a) What is the maximal rate at which the substance is produced? At what concentration is the production rate 50% of this maximal value?

(b) If the production is turned off, the substance will decay. How long would it take for the concentration to drop by 50%?

(c)

At what concentration does the production rate just balance the consumption rate?

# Appendices

## Appendix A

# A review of Straight Lines

## A.A Geometric ideas: lines, slopes, equations

Straight lines have some important geometric properties, namely:

*The slope of a straight line is the same everywhere along its length.*

### Definition: slope of a straight line:

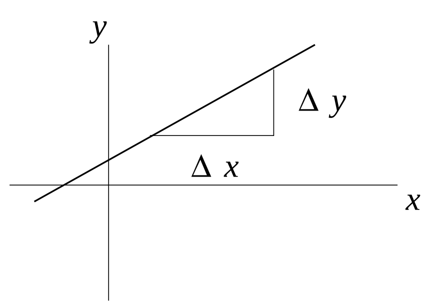



**Figure A.1.** *The slope of a line (usually given the symbol $m$) is the ratio of the change in the y value, $\Delta y$ to the change in the $x$ value, $\Delta x$.*

We define the slope of a straight line as follows:

$$\text{Slope} = \frac{\Delta y}{\Delta x}$$

where $\Delta y$ means "change in the $y$ value" and $\Delta x$ means "change in the $x$ value" between two points. See Figure A.1 for what this notation represents.

### Equation of a straight line

Using this basic geometric property, we can find the equation of a straight line given any of the following information about the line:

- The $y$ intercept, $b$, and the slope, $m$:

$$y = mx + b.$$

- A point $(x_0, y_0)$ on the line, and the slope, $m$, of the line:

$$\frac{y - y_0}{x - x_0} = m$$

- Two points on the line, say $(x_1, y_1)$ and $(x_2, y_2)$:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Remark: any of these can be rearranged or simplified to produce the standard form $y = mx + b$, as discussed in the problem set.

The following examples will refresh your memory on how to find the equation of the line that satisfies each of the given conditions.

**Example A.1** In each case write down the equation of the straight line that satisfies the given statements. (Note: you should also be able to easily sketch the line in each case.)

(a) The line has slope 2 and $y$ intercept 4.

(b) The line goes through the points (1,1) and (3,-2).

(c) The line has $y$ intercept -1 and $x$ intercept 3.

(d) The line has slope -1 and goes through the point (-2,-5).

$\blacksquare$

**Solution:**

(a) We can use the standard form of the equation of a straight line, $y = mx + b$ where $m$ is the slope and $b$ is the $y$ intercept to obtain the equation: $y = 2x + 4$

(b) The line goes through the points (1,1) and (3,-2). We use the fact that the slope is the same all along the line. Thus,

$$\frac{(y - y_0)}{(x - x_0)} = \frac{(y_1 - y_0)}{(x_1 - x_0)} = m.$$

Substituting in the values $(x_0, y_0) = (1, 1)$ and $(x_1, y_1) = (3, -2)$,

$$\frac{(y - 1)}{(x - 1)} = \frac{(1 + 2)}{(1 - 3)} = -\frac{3}{2}.$$

(Note that this tells us that the slope is $m = -3/2$.) We find that

$$y - 1 = -\frac{3}{2}(x-1) = -\frac{3}{2}x + \frac{3}{2},$$
$$y = -\frac{3}{2}x + \frac{5}{2}.$$

(c) The line has $y$ intercept -1 and $x$ intercept 3, i.e. goes through the points (0,-1) and (3,0). We can use the method in (b) to get

$$y = \frac{1}{3}x - 1$$

Alternately, as a shortcut, we could find the slope,

$$m = \frac{\Delta y}{\Delta x} = \frac{1}{3}.$$

(Note that $\Delta$ means "change in the value", i.e. $\Delta y = y_1 - y_0$). Thus $m = 1/3$ and $b = -1$ ($y$ intercept), leading to the same result.

(d) The line has slope -1 and goes through the point (-2,-5). Then,

$$\frac{(y+5)}{(x+2)} = -1,$$

so that

$$y + 5 = -1(x+2) = -x - 2,$$
$$y = -x - 7.$$

## Exercises

1.1. Find the slope and $y$ intercept of the following straight lines:

(a) $y = 4x - 5$

(b) $3x - 4y = 8$

(c) $2x = 3y$

(d) $y = 3$

(e) $5x - 2y = 23$

1.2. Find the equations of the following straight lines

(a) Through the points (2,0) and (1,5).

(b) Through (3,-1) with slope 1/2.

(c) Through (-10,2) with $y$ intercept 10.

(d) The straight line shown in Figure A.2.

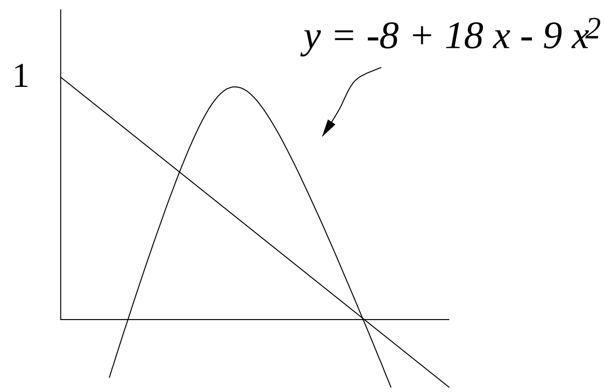



**Figure A.2.** *Figure for problem 2(d)*

1.3. Find the equations of the following straight lines:

(a) Slope $-4$ and $y$ intercept $3$.

(b) Slope $3$ and $x$ intercept $-2/3$.

(c) Through the points $(2, -7)$ and $(-1, 11)$.

(d) Through the point $(1, 3)$ and the origin.

(e) Through the intersection of the lines $3x + 2y = 19$ and $y = -4x + 7$ and through the point $(2, -7)$.

(f) Through the origin and parallel to the line $2x + 8y = 3$.

(g) Through the point $(-2, 5)$ and perpendicular to the line $y = -\dfrac{1}{2}x + 6$.

1.4. **Tangent to a circle:** Shown in Figure A.3 is a circle of radius 1. The $x$ coordinate of the point on the circle at which the line touches the circle is $x = \sqrt{2}/2$. Find the equation of the tangent line. Use the fact that on a circle, the tangent line is perpendicular to the radius vector.

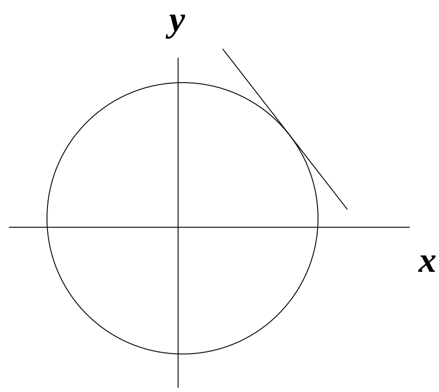



**Figure A.3.** *Figure for problem 4*

# Appendix B

# A Review of Simple Functions

Herer we review a few basic concepts related to functions

## B.A What is a function

A function is just a way of expressing a special relationship between a value we consider as the input ("$x$") value and an associated output ($y$) value. We write this relationship in the form

$$y = f(x)$$

to indicate that $y$ depends on $x$. The only constraint on this relationship is that, for every value of $x$ we can get at most one value of $y$. This is equivalent to the *"vertical line property"*: the graph of a function can intersect a vertical line at most at one point. The set of all allowable $x$ values is called the *domain* of the function, and the set of all resulting values of $y$ are the *range*.

Naturally, we will not always use the symbols $x$ and $y$ to represent independent and dependent variables. For example, the relationship

$$V = \frac{4}{3}\pi r^3$$

expresses a functional connection between the radius, $r$, and the volume, $V$, of a sphere. We say in such a case that "$V$ is a function of $r$".

All the sketches shown in Figure B.1 are valid functions. The first is merely a collection of points, $x$ values and associated $y$ values, the second a histogram. The third sketch is here meant to represent the collection of smooth continuous functions, and these are the variety of interest to us here in the study of calculus. On the other hand, the example shown in Figure B.2 is not the graph of a function. We see that a vertical line intersects this curve at more than one point. This is not permitted, since as we already said, a given value of $x$ should have only one corresponding values of $y$.

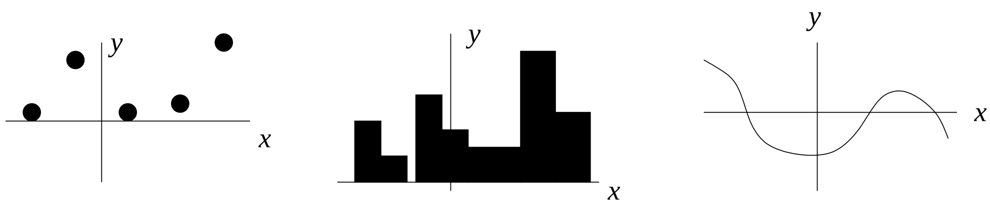



**Figure B.1.** *All the examples above represent functions.*

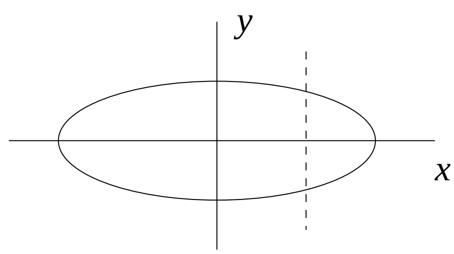
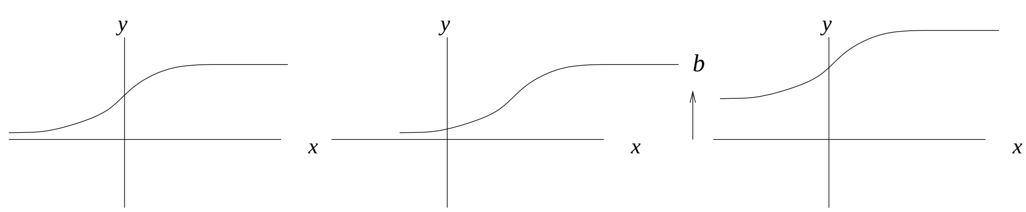



**Figure B.2.** *The above elliptical curve cannot be the graph of a function. The vertical line (shown dashed) intersects the graph at more than one point: This means that a given value of $x$ corresponds to "too many" values of $y$. If we restrict ourselves to the top part of the ellipse only (or the bottom part only), then we can create a function which has the corresponding graph.*

## B.B Geometric transformations

It is important to be able to easily recognize what happens to the graph of a function when we change the relationship between the variables slightly. Often this is called *applying a transformation*. Figures B.3 and B.4 illustrate what happens to a function when shifts, scaling, or reflections occur:




**(a)** $y = f(x)$ **(b)** $y = f(x - a)$ **(c)** $y = f(x) + b$

**Figure B.3.** *(a) The original function $f(x)$, (b) The function $f(x-a)$ shifts $f$ to the right along the positive $x$ axis by a distance $a$, (c) The function $f(x)+b$ shifts $f$ up the $y$ axis by height $b$.*

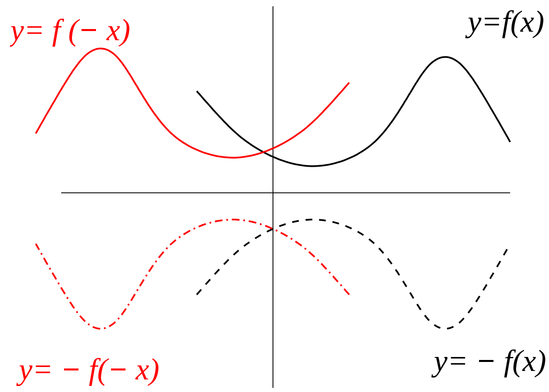



**Figure B.4.** *Here we see a function $y = f(x)$ shown in the black solid line. On the same graph are superimposed the reflections of this graph about the x axis, $y = -f(x)$ (dashed black), about the y axis $y = f(-x)$ (red), and about the y and the x axis, $y = -f(-x)$ (red dashed). The latter is equivalent to a rotation of the original graph about the origin.*

## B.C Classifying

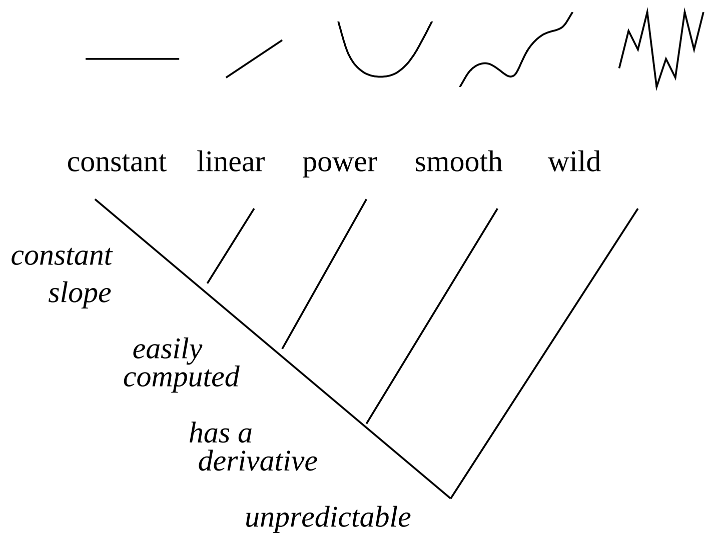



**Figure B.5.** *Classifying functions according to their properties.*

While life offers amazing complexity, one way to study living things is to classify them into related groups. A biologist looking at animals might group them according to certain functional properties - being warm blooded, being mammals, having fur or claws, or having some other interesting characteristic. In the same way, mathematicians often classify the objects that they study, e.g., functions, into related groups. An example of the way that functions might be grouped into very broad classes is also shown in Figure B.5. From left to right, the complexity of behaviour in this chart grows: at left, we see constant and linear functions (describable by one or two simple parameters such as intercepts or slope): these linear functions are "most convenient" or simplest to describe. Further to the right are functions that are smooth and continuous, while at the right, some more irregular, discontinuous function represents those that are outside the group of the "well-behaved". We will study some of the examples along this spectrum, and describe properties that they share, properties they inherit form their "cousins", and new characteristics that appear at distinct branches.

## B.D Power functions and symmetry

We list some of the features of each family of power functions in this section

### Even integer powers

For $n = 2, 4, 6, 8..$ the shape of the graph of $y = x^n$ is as shown in Figure **??**(a).

Here are some things to notice about these graphs:

1. The graphs of all the even power functions intersect at $x = 0$ and at at $x = \pm 1$. The value of $y$ corresponding to both of these is $y = +1$. (Thus, the coordinates of the three intersection points are $(0, 0), (1, 1), (-1, 1)$.)

2. All graphs have a lowest point, also called a *minimum value* at $x = 0$.

3. As $x \to \pm\infty$, $y \to \infty$, We also say that the functions are "unbounded from above".

4. The graphs are all symmetric about the $y$ axis. This special type of symmetry will be of interest in other types of functions, not just power functions. A function with this property is called an **even function**.

### Odd integer powers

For $n = 1, 3, 5, 7, ..$ and other odd powers, the graphs have shapes shown in Figure **??**(b).

1. The graphs of the odd power functions intersect at $x = 0$ and at $x = \pm 1$. The three points of intersection in common to all odd power functions are $(1, 1), (0, 0)$, and $(-1, -1)$.

2. None of the odd power functions have a minimum value.

3. As $x \to +\infty$, $y \to +\infty$. As $x \to -\infty$, $y \to -\infty$. The functions are "unbounded from above and below".

4. The graphs are all symmetric about the origin. This special type of symmetry will be of interest in other types of functions, not just power functions. A function with this type of symmetry is called an **odd function**.

## B.D.1 Further properties of intersections

Here, and in Figure B.6 we want to notice that a horizontal line intersects the graph of a power function only once for the odd powers but possibly twice for the even powers (we have to allow for the case that the line does not intersect at all, or that it intersects precisely at the minimum point). This observation will be important further on, once we want to establish the idea of an inverse function.

A horizontal line has an equation of the form $y = C$ where $C$ is some constant. To find where it intersects the graph of a power function $y = x^n$, we would solve an equation of the form

$$x^n = C \tag{B.1}$$

To do so, we take n'th root of both sides:

$$(x^n)^{1/n} = C^{1/n}.$$

Simplifying, using algebraic operations on powers leads to

$$(x^n)^{1/n} = x^{n/n} = x^1 = x = C^{1/n},$$

However, we have to allow for the fact that there may be more than one solution to equation B.1, as shown for some $C > 0$ in Figure B.6. Here we see the the distinction between odd and even power functions. If $n$ is even then the solutions to equation B.1 are

$$x = \pm C^{1/n},$$

whereas if $n$ is odd, there is but a single solution,

$$x = C^{1/n}.$$

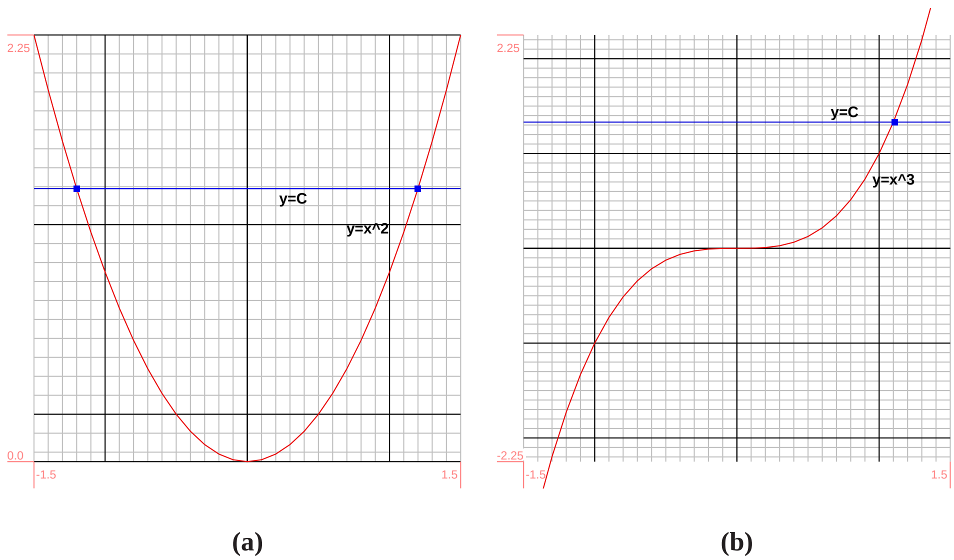



**Figure B.6.** *The even power functions intersect a horizontal line in up to two places, while the odd power functions intersect such a line in only one place.*

**Definition B.1 (Even and odd functions:).** *A function that is symmetric about the $y$ axis is said to be an* even *function. A function that is symmetric about the origin is said to be an* odd *function.*

Even functions satisfy the relationship

$$f(x) = f(-x).$$

Odd functions satisfy the relationship

$$f(x) = -f(-x).$$

Examples of even functions include $y = \cos(x), y = -x^8, y = |x|$. All these are their own mirror images when reflected about the $y$ axis. Examples of odd functions are

$y = \sin(x), y = -x^3, y = x$. Each of these functions is its own double-reflection (about y and then x axes).

In a later calculus course, when we compute integrals, taking these symmetries into account can help to simplify (or even avoid) calculations.

## B.D.2 Optional: Combining even and odd functions

Not every function is either odd or even. However, if we start with symmetric functions, certain manipulations either preserve or reverse the symmetry.

**Example B.2** Show that the product of an even and an odd function is an odd function.

**Solution:** Let $f(x)$ be even. Then

$$f(x) = f(-x).$$

Let $g(x)$ be an odd function. Then $g(x) = -g(-x)$. We define $h(x)$ to be the product of these two functions,

$$h(x) = f(x)g(x).$$

Using the properties of $f$ and $g$,

$$f(x)g(x) = f(-x)[-g(-x)]$$

so, rearranging, we get

$$h(x) = f(x)g(x) = f(-x)[-g(-x)] = -[f(-x)g(-x)].$$

but this is just the same as $-h(-x)$. We have established that

$$h(x) = -h(-x)$$

so that the new function is odd.

A function is not always even or odd. Many functions are neither even nor odd. However, by a little trick, we can show that given any function, $y = f(x)$, we can write it as a sum of an even and an odd function.

**Hint:** Suppose $f(x)$ is not an even nor an odd function. Consider defining the two associated functions:

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)),$$

and

$$f_0(x) = \frac{1}{2}(f(x) - f(-x)).$$

(Can you draw a sketch of what these would look like for the function given in Figure B.3(a)?) Show that $f_e(x)$ is even and that $f_0(x)$ is odd. Now show that

$$f(x) = f_e(x) + f_0(x).$$

## B.E Inverse functions and fractional powers

Suppose we are given a function expressed in the form

$$y = f(x).$$

What this implies, is that $x$ is the independent variable, and $y$ is obtained from it by evaluating a function, i.e. by using the "rule" or operation specified by that function. The above mathematical statement expresses a certain relationship between the two variables, $x$ and $y$, in which the roles are distinct. $x$ is a value we pick, and $y$ is then calculated from it.

However, sometimes we can express a relationship in more than one way: as an example, if the connection between $x$ and $y$ is simple squaring, then provided $x > 0$, we might write either

$$y = x^2$$

or

$$x = y^{1/2} = \sqrt{y}$$

to express the same relationship. In other words

$$y = x^2 \Leftrightarrow x = \sqrt{y}.$$

Observe that we have used two distinct functions in describing the relationship from the two points of view: One function involves squaring and the other takes a square root. We may also notice that for $x > 0$

$$f(g(x)) = (\sqrt{x})^2 = x$$

$$g(f(x)) = \sqrt{(x^2)} = x$$

i.e. that these two functions invert each other's effect.

Functions that satisfy

$$y = f(x) \Leftrightarrow x = g(y)$$

are said to be *inverse functions*. We will often use the notation

$$f^{-1}(x)$$

to denote the function that acts as an inverse function to $f(x)$.

### B.E.1 Graphical property of inverse functions

The graph of an inverse function $y = f^{-1}(x)$ is geometrically related to the graph of the original function: it is a reflection of $y = f(x)$ about the $45°$ line, $y = x$. This relationship is shown in figure B.7 for a pair of functions $f$ and $f^{-1}$.

But why should this be true? The idea is as follows: Suppose that $(a, b)$ is any point on the graph of $y = f(x)$. This means that $b = f(a)$. That, in turn, implies that $a = f^{-1}(b)$, which then tells us that $(b, a)$ must be a point on the graph of $f^{-1}(x)$. But the points $(a, b)$ and $(b, a)$ are related by reflection about the line $y = x$. This is true for any arbitrary point, and so must be true for *all* points on the graphs of the two functions.

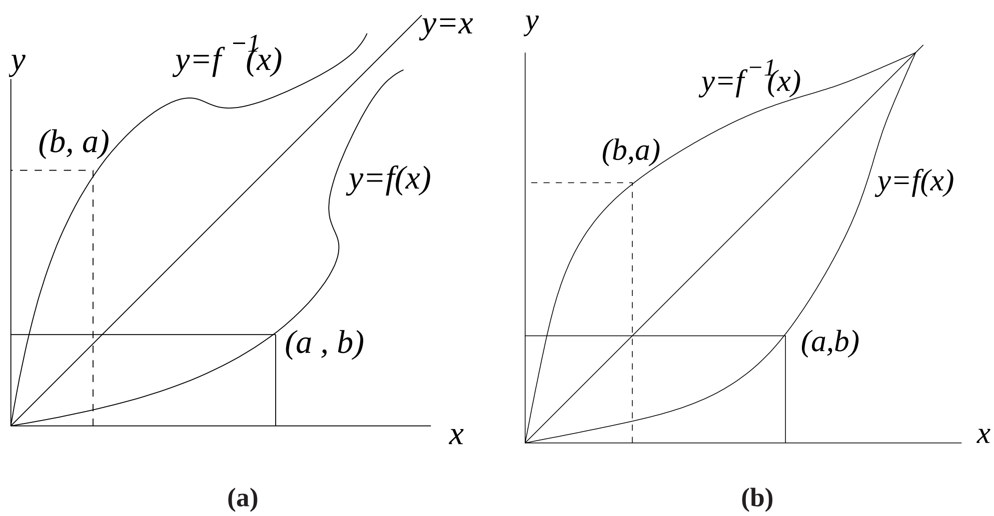



**Figure B.7.** *The point $(a, b)$ is on the graph of $y = f(x)$. If the roles of $x$ and $y$ are interchanged, this point becomes $(b, a)$. Geometrically, this point is the reflection of $(a, b)$ about the line $y = x$. Thus, the graph of the inverse function $y = f^{-1}(x)$ is related to the graph of the original function by reflection about the line $y = x$. In the left panel, the inverse is not a function, as it does not satisfy the vertical line property. In the panel on the right, both $f$ and its reflection satisfy that property, and thus the inverse, $f^{-1}$ is a true function.*

## B.E.2 Restricting the domain

The above argument establishes that, given the graph of a function, its inverse is obtained by reflecting the graph in an imaginary mirror placed along a line $y = x$.

However, a difficulty could arise. In particular, for the function

$$y = f(x) = x^2,$$

a reflection of this type would lead to a curve that cannot be a function, as shown in Figure B.8. (The sideways parabola would not be a function if we included both its branches, since a given value of $x$ would have two associated $y$ values.)

To fix such problems, we simply restrict the domain to $x > 0$, i.e. to the solid parts of the curves shown in Figure B.8. For this subset of the $x$ axis, we have no problem defining the inverse function.

Observe that the problem described above would be encountered for any of the even power functions (by virtue of their symmetry about the $y$ axis) but not by the odd power functions.

$$y = f(x) = x^3 \quad y = f^{-1}(x) = x^{1/3}$$

are inverse functions for all $x$ values: when we reflect the graph of $x^3$ about the line $y = x$ we do not encounter problems of multiple $y$ values.

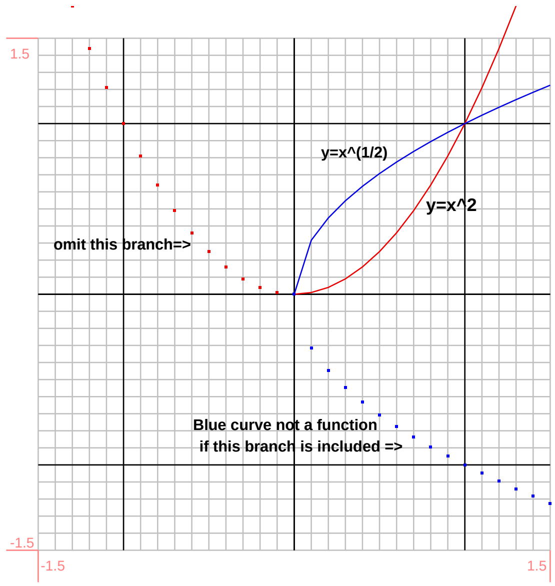



**Figure B.8.** *The graph of $y = f(x) = x^2$ (blue) and of its inverse function. We cannot define the inverse for all $x$, because the red parabola does not satisfy the vertical line property: However, if we restrict to positive $x$ values, this problem is circumvented.*

This follows directly from the horizontal line properties that we discussed earlier, in Figure B.6. When we reflect the graphs shown in Figure B.6 about the line $y = x$, the horizontal lines will be reflected onto vertical lines. Odd power functions will have inverses that intersect a vertical line exactly once, i.e. they satisfy the "vertical line property" discussed earlier.

## B.F Polynomials

A polynomial is a function of the form

$$y = p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

This form is sometimes referred to as *superposition* (i.e. simple addition) of the basic power functions with integer powers. The constants $a_k$ are called coefficients. In practice some of these may be zero. We will restrict attention to the case where all these coefficients are real numbers. The highest power $n$ (whose coefficient is not zero) is called the *degree* of the polynomial.

We will be interested in these functions for several reasons. Primarily, we will find that computations involving polynomials are particularly easy, since operations include only the basic addition and multiplication.

### B.F.1 Features of polynomials

- **Zeros of a polynomial** are values of $x$ such that

  $$y = p(x) = 0.$$

  If $p(x)$ is quadratic (a polynomial of degree 2) then the quadratic formula gives a simple way of finding roots of this equation (also called "zeros" of the polynomial). Generally, for most polynomials of degree higher than 5, there is no analytical recipe for finding zeros. Geometrically, zeros are places where the graph of the function $y = p(x)$ crosses the $x$ axis. We will exploit this fact much later in the course to approximate the values of the zeros using *Newton's Method*.

- **Critical Points:** Places on the graph where the value of the function is locally larger than those nearby (local maxima) or smaller than those nearby (local minima) will be of interest to us. Calculus will be one of the main tools for detecting and identifying such places.

- **Behaviour for very large $x$:** All polynomials are unbounded as $x \to \infty$ and as $x \to -\infty$. In fact, for large enough values of $x$, we have seen that the power function $y = f(x) = x^n$ with the largest power, $n$, dominates over other power functions with smaller powers.For

  $$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  the first (highest power) term will *dominate* for large $x$. Thus for large $x$ (whether positive or negative)

  $$p(x) \approx a_n x^n \quad \text{for large } x.$$

- **Behaviour for small $x$:** Close to the origin, we have seen that power functions with smallest powers dominate. This means that for $x \approx 0$ the polynomial is governed by the behaviour of the smallest (non-zero coefficient) power, i.e,

  $$p(x) \approx a_1 x + a_0 \quad \text{for small } x.$$

## Exercises

2.1. Figure B.9 shows the graph of the function $y = f(x)$. Match the functions (a)-(d) below with their appropriate graph (1)-(4) in Figure B.10.

(a) $y = |f(x)|$,

(b) $y = f(|x|)$,

(c) $y = f(-x)$,

(d) $y = -f(x)$.

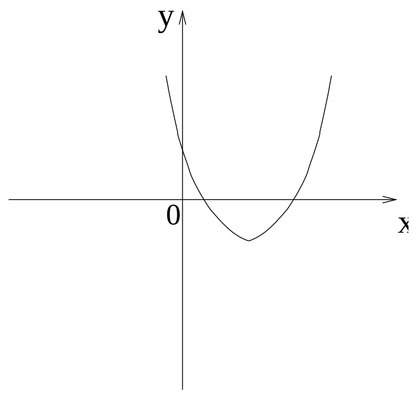



**Figure B.9.** *Plot for problem 1*

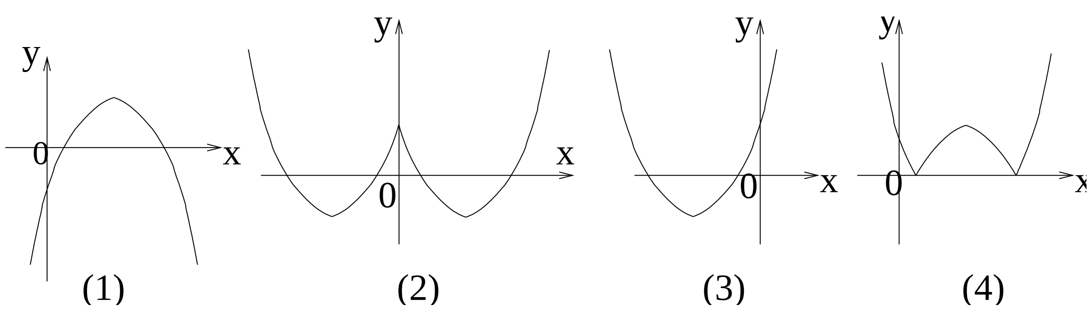



**Figure B.10.** *Plot for problem 1*

2.2. **Even and odd functions:** An even function is a function that satisfies the relationship $f(x) = f(-x)$. An odd function satisfies the relationship $g(x) = -g(-x)$. Determine which of the following is odd, which is even, and which is neither.

(a) $h(x) = 3x$

(b) $p(x) = x^2 - 3x^4$

(c) $q(x) = 2$

(d) $w(x) = \sin(2x)$

(e) $s(x) = x + x^2$

2.3. Figure B.11 shows the graph for the function $y = f(x)$, sketch the graph for $y = f(|x|)$.

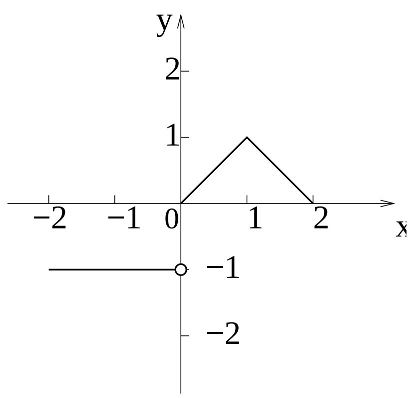



**Figure B.11.** *Plot for problem 3*

2.4. Consider the function $y = Ax^n$ for $n > 0$ an odd integer and $A > 0$ a constant. Find the inverse function. Sketch both functions on the same set of coordinates, and indicate the points of intersection. How would your figure differ if $n$ were an even integer?

## Appendix C

# Limits

We have surreptitiously introduced some notation involving limits without carefully defining what was meant. Here, such technical matters are briefly discussed.

The concept of a **limit** helps us to describe the behaviour of a function close to some point of interest. This proves to be most useful in the case of functions that are either not continuous, or not defined somewhere. We will use the notation

$$\lim_{x \to a} f(x)$$

to denote the value that the function $f$ approaches as $x$ gets closer and closer to the value $a$.

## C.A Limits for continuous functions

If $x = a$ is a point at which the function is defined and continuous (informally, has no "breaks in its graph") the value of the limit and the value of the function at a point are the same, i.e.

**If $f$ is continuous at $x = a$ then**

$$\lim_{x \to a} f(x) = f(a).$$

**Example C.1** Find $\lim_{x \to 0} f(x)$ for the function $y = f(x) = 10$ ■

**Solution:** This function is continuous (and constant) everywhere. In fact, the value of the function is independent of $x$. We conclude immediately that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} 10 = 10.$$

**Example C.2** Find $\lim_{x \to 0} f(x)$ for the function $y = f(x) = \sin(x)$. ■

**Solution:** This function is a continuous trigonometric function, and has the value $\sin(0) = 0$ at the origin. Thus

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \sin(x) = 0$$

Power functions are continuous everywhere. This motivates the next example.

**Example C.3** Compute the limit $\lim_{x\to 0} x^n$ where $n$ is a positive integer. ∎

**Solution:** The function in question, $f(x) = x^n$ is a simple power function that is continuous everywhere. Further, $f(0) = 0$. Hence the limit as $x \to 0$ coincides with the value of the function oat that point, so

$$\lim_{x\to 0} x^n = 0.$$

## C.B Properties of limits

Suppose we are given two functions, $f(x)$ and $g(x)$. We will also assume that both functions have (finite) limits at the point $x = a$. Then the following statements follow.

1. $$\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

2. $$\lim_{x\to a} (cf(x)) = c \lim_{x\to a} f(x)$$

3. $$\lim_{x\to a} (f(x) \cdot g(x)) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$$

4. Provided that $\lim_{x\to a} g(x) \neq 0$, we also have that

$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \left(\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\right).$$

The first two statements are equivalent to linearity of the process of computing a limit.

**Example C.4** Find $\lim_{x\to 2} f(x)$ for the function $y = f(x) = 2x^2 - x^3$.
∎

**Solution:** Since this function is a polynomial, and so continuous everywhere, we can simply plug in the relevant value of $x$, i.e.

$$\lim_{x\to 2} \left(2x^2 - x^3\right) = 2 \cdot 2^2 - 2^3 = 0.$$

Thus when $x$ gets closer to 2, the value of the function gets closer to 0. (In fact, the value of the limit is the same as the value of the function at the given point.)

# C.C Limits of rational functions

## C.C.1 Case 1: Denominator nonzero

We first consider functions that are the quotient of two polynomials, $y = f(x)/g(x)$ at points were $g(x) \neq 0$. This allows us to apply Property 4 of limits together with what we have learned about the properties of power functions and polynomials. Much of this discussion is related to the properties of power functions and dominance of lower (higher) powers at small (large) values of $x$, as discussed in Chapter 1. In the examples below, we consider both limits at the origin (at $x = 0$) and at infinity (for $x \to \infty$). The latter means "very large $x$". See Section 1.5 for examples of the informal version of the same reasoning used to reach the same conclusions.

**Example C.5** Find the limit as $x \to 0$ and as $x \to \infty$ of the quotients

$$\text{(a) } \frac{Kx}{k_n + x}, \quad \text{(b)} \frac{Ax^n}{a^n + x^n}.$$

■

**Solution:** We recognize (a) as an example of the Michaelis Menten kinetics, found in (1.2) and (b) as a Hill function in (1.3) of Chapter 1. We now compute, first for $x \to 0$,

$$\text{(a) } \lim_{x\to 0} \frac{Kx}{k_n + x} = 0, \quad \text{(b)} \lim_{x\to 0} \frac{Ax^n}{a^n + x^n} = 0.$$

This follows from the fact that, provided $a, k_n \neq 0$, both functions are continuous at $x = 0$, so that their limits are the same as the actual values attained by the functions. Now for $x \to \infty$

$$\text{(a) } \lim_{x\to\infty} \frac{Kx}{k_n + x} = \lim_{x\to\infty} \frac{Kx}{x} = K, \quad \text{(b)} \lim_{x\to\infty} \frac{Ax^n}{a^n + x^n} = \lim_{x\to\infty} \frac{Ax^n}{x^n} = A.$$

This follows from the fact that the constants $k_n, a^n$ are always "swamped out" by the value of $x$ as $x \to \infty$, allowing us to obtain the result. Other than the formal limit notation, there is nothing new here that we have not already discussed in Sections 1.6.

Below we apply similar reasoning to other examples of rational functions.

**Example C.6** Find the limit as $x \to 0$ and as $x \to \infty$ of the quotients

$$\text{(a) } \frac{3x^2}{9 + x^2}, \quad \text{(b)} \frac{1 + x}{1 + x^3}.$$

■

**Solution:** For part (a) we note that as $x \to \infty$, the quotient approaches $3x^2/x^2 = 3$. As $x \to 0$, both numerator and denominator are defined and the denominator is nonzero, so we can use the 4th property of limits. We thus find that

$$\text{(a) } \lim_{x\to\infty} \frac{3x^2}{9 + x^2} = 3, \quad \lim_{x\to 0} \frac{3x^2}{9 + x^2} = 0,$$

For part (b), we use the fact that as $x \to \infty$, the limit approaches $x/x^3 = x^{-2} \to 0$. As $x \to 0$ we can apply property 4 yet again to compute the (finite) limit, so that

$$\text{(b)} \lim_{x\to\infty} \frac{1+x}{1+x^3}, \quad \lim_{x\to 0} \frac{1+x}{1+x^3}.$$

**Example C.7** Find the limits of the following function at 0 and $\infty$

$$y = \frac{x^4 - 3x^2 + x - 1}{x^5 + x}.$$

$\blacksquare$

**Solution:** for $x \to \infty$ powers with the largest power dominate, whereas for $x \to 0$, smaller powers dominate. Hence, we find

$$\lim_{x\to\infty} \frac{x^4 - 3x^2 + x - 1}{x^5 + x} = \lim_{x\to\infty} \frac{x^4}{x^5} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

$$\lim_{x\to 0} \frac{x^4 - 3x^2 + x - 1}{x^5 + x} = \lim_{x\to 0} \frac{-1}{x} = -\lim_{x\to 0} \frac{1}{x} = \infty$$

So in the latter case, the limit does not exist.

## C.C.2 Case 2: zero in the denominator and "holes" in a graph

In the previous examples, evaluating the limit, where it existed, was as simple as plugging the appropriate value of $x$ into the function itself. The next example shows that this is not always possible.

**Example C.8** Compute the limit as $x \to 4$ of the function $f(x) = 1/(x-4)$ $\blacksquare$

**Solution:** This function has a vertical asymptote at $x = 4$. Indeed, the value of the function shoots off to $+\infty$ if we approach $x = 4$ from above, and $-\infty$ if we approach the same point from below. We say that the limit **does not exist** in this case.

**Example C.9** Compute the limit as $x \to -1$ of the function $f(x) = x/(x^2-1)$ $\blacksquare$

**Solution:** We compute

$$\lim_{x\to -1} \frac{x}{x^2 - 1} = \lim_{x\to -1} \frac{x}{(x-1)(x+1)}$$

It is evident (even before factoring as we have done) that this function has a vertical asymptote at $x = -1$ where the denominator approaches zero. Hence, the limit does not exist.

Next, we describe an extremely important example where the function has a "hole" in its graph, but where a finite limit exists. This kind of limit plays a huge role in the definition of a derivative.

**Example C.10** Find $\lim_{x \to 2} f(x)$ for the function $y = (x-2)/(x^2-4)$. ■

**Solution:** This function is a quotient of two rational expressions $f(x)/g(x)$ but we note that $\lim_{x\to 2} g(x) = \lim_{x \to 2}(x^2-4) = 0$. Thus we cannot use property 4 directly. However, we can simplify the quotient by observing that for $x \neq 2$ the function $y = (x-2)/(x^2-4) = (x-2)/(x-2)(x+2)$ takes on the same values as the expression $1/(x+2)$. At the point $x = 2$, the function itself is not defined, since we are not allowed division by zero. However, the limit of this function does exist:

$$\lim_{x\to 2} f(x) = \lim_{x \to 2} \frac{(x-2)}{(x^2-4)}.$$

Provided $x \neq 2$ we can factor the denominator and cancel:

$$\lim_{x\to 2} \frac{(x-2)}{(x^2-4)} = \lim_{x\to 2} \frac{(x-2)}{(x-2)(x+2)} = \lim_{x \to 2} \frac{1}{(x+2)}$$

Now we can substitute $x = 2$ to obtain

$$\lim_{x\to 2} f(x) = \frac{1}{(2+2)} = \frac{1}{4}$$

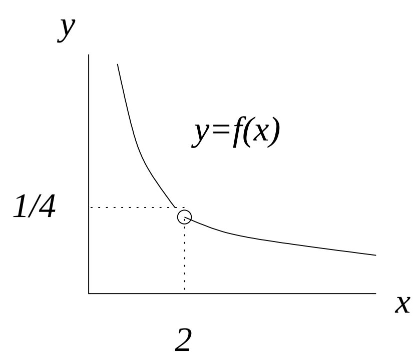



**Figure C.1.** *The function $y = \frac{(x-2)}{(x^2-4)}$ has a "hole" in its graph at $x = 2$. The limit of the function as $x$ approaches 2 does exist, and "supplies the missing point": $\lim_{x\to 2} f(x) = \frac{1}{4}$.*

**Example C.11** Compute the limit

$$\lim_{h\to 0} \frac{K(x+h)^2 - Kx^2}{h}.$$

■

**Solution:** This is a calculation we would perform to compute the derivative of the function $y = Kx^2$ from the definition of the derivative. Details have already been displayed in Example 2.8. The essential idea is that we expand the numerator and simplify algebraically as follows:

$$\lim_{h \to 0} K \frac{(2xh + h^2)}{h} = \lim_{h \to 0} K(2x + h) = 2Kx.$$

Even though the quotient is not defined at the value $h = 0$ (as the denominator is zero there), the limit exists, and hence the derivative can be defined. See also Example 4.1 for a similar calculation for the function $Kx^3$.

## C.D Right and left sided limits

Some functions are discontinuous at a point, but we may still be able to define a limit that the function attains as we approach that point from the right or from the left. (This is equivalent to gradually decreasing or gradually increasing $x$ as we get closer to the point of interest.

Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0; \\ 1 & \text{if } x > 0. \end{cases}$$

This is a step function, whose values is 0 for negative real numbers, and 1 for positive real numbers. The function is not even defined at the point $x = 0$ and has a jump in its graph. However, we can still define a right and a left limit as follows:

$$\lim_{x \to ^+ 0} f(x) = 0, \quad \lim_{x \to ^- 0} f(x) = 1.$$

That is, the limit as we approach from the right is 0 whereas from the left it is 1. We also state the following result:

**If $f(x)$ has a right and a left limit at a point $x = a$ and if those limits are equal, then we say that the limit at $x = a$ exists, and we write**

$$\lim_{x \to ^+ a} f(x) = \lim_{x \to ^- a} f(x) = \lim_{x \to a} f(x)$$

**Example C.12** Find $\lim_{x \to \pi/2} f(x)$ for the function $y = f(x) = \tan(x)$. ■

**Solution:** The function $\tan(x) = \sin(x)/\cos(x)$ cannot be continuous at $x = \pi/2$ because $\cos(x)$ in the denominator takes on the value of zero at the point $x = \pi/2$. Moreover, the value of this function becomes unbounded (grows without a limit) as $x \to \pi/2$. We say in this case that "the limit does not exist". We sometimes use the notation

$$\lim_{x \to \pi/2} \tan(x) = \pm\infty.$$

(We can distinguish the fact that the function approaches $+\infty$ as $x$ approaches $\pi/2$ from below, and $-\infty$ as $x$ approaches $\pi/2$ from higher values.

## C.E Limits at infinity

We can also describe the behaviour "at infinity" i.e. the trend displayed by a function for very large (positive or negative) values of $x$. We consider a few examples of this sort below.

**Example C.13** Find $\lim_{x\to\infty} f(x)$ for the function $y = f(x) = x^3 - x^5 + x$.

**Solution:** All polynomials grow in an unbounded way as $x$ tends to very large values. We can determine whether the function approaches positive or negative unbounded values by looking at the coefficient of the highest power of $x$, since that power dominates at large $x$ values. In this example, we find that the term $-x^5$ is that highest power. Since this has a negative coefficient, the function will approach unbounded negative values as $x$ gets larger in the positive direction, i.e.

$$\lim_{x\to\infty} x^3 - x^5 + x = \lim_{x\to\infty} -x^5 = -\infty.$$

**Example C.14** Determine the following two limits:

$$\text{(a)} \lim_{x\to\infty} e^{-2x}, \quad \text{(b)} \lim_{x\to-\infty} e^{5x},$$

**Solution:** The function $y = e^{-2x}$ becomes arbitrarily small as $x \to \infty$. The function $y = e^{5x}$ becomes arbitrarily small as $x \to -\infty$. Thus we have

$$\text{(a)} \lim_{x\to\infty} e^{-2x} = 0, \quad \text{(b)} \lim_{x\to-\infty} e^{5x} = 0.$$

**Example C.15** Find the limits below:

$$\text{(a)} \lim_{x\to\infty} x^2 e^{-2x}, \quad \text{(b)} \lim_{x\to 0} \frac{1}{x} e^{-x},$$

**Solution:** For part (a) we state here the fact that as $x \to \infty$, the exponential function with negative exponent decays to zero faster than any power function increases. For part (b) we note that for the quotient $e^{-x}/x$ we have that as $x \to 0$ the top satisfies $e^{-x} \to e^0 = 1$, while the denominator has $x \to 0$. Thus the limit at $x \to 0$ cannot exist. We find that

$$\text{(a)} \lim_{x\to\infty} x^2 e^{-2x} = 0, \quad \text{(b)} \lim_{x\to 0} \frac{1}{x} e^{-x} = \infty,$$

## C.F Summary of special limits

As a reference, in the table below, we collect some of the special limits that are useful in a variety of situations.

We can summarize the information in this table informally as follows:

| Function | point | Limit notation | value |
|---|---|---|---|
| $e^{-ax}$, $a > 0$ | $x \to \infty$ | $\lim_{x\to\infty} e^{-ax}$ | 0 |
| $e^{-ax}$, $a > 0$ | $x \to -\infty$ | $\lim_{x\to-\infty} e^{-ax}$ | $\infty$ |
| $e^{ax}$, $a > 0$ | $x \to \infty$ | $\lim_{x\to\infty} e^{ax}$ | $\infty$ |
| $e^{kx}$ | $x \to 0$ | $\lim_{x\to 0} e^{kx}$ | 1 |
| $x^n e^{-ax}$, $a > 0$ | $x \to \infty$ | $\lim_{x\to\infty} x^n e^{-ax}$ | 0 |
| $\ln(ax)$, $a > 0$ | $x \to \infty$ | $\lim_{x\to\infty} \ln(ax)$ | $\infty$ |
| $\ln(ax)$, $a > 0$ | $x \to 1$ | $\lim_{x\to 1} \ln(ax)$ | 0 |
| $\ln(ax)$, $a > 0$ | $x \to 0$ | $\lim_{x\to 0} \ln(ax)$ | $-\infty$ |
| $x\ln(ax)$, $a > 0$ | $x \to 0$ | $\lim_{x\to 0} x\ln(ax)$ | 0 |
| $\dfrac{\ln(ax)}{x}$, $a > 0$ | $x \to \infty$ | $\lim_{x\to\infty} \dfrac{\ln(ax)}{x}$ | 0 |
| $\dfrac{\sin(x)}{x}$ | $x \to 0$ | $\lim_{x\to 0} \dfrac{\sin(x)}{x}$ | 1 |
| $\dfrac{(1-\cos(x))}{x}$ | $x \to 0$ | $\lim_{x\to 0} \dfrac{(1-\cos(x))}{x}$ | 0 |

**Table C.1.** *A collection of useful limits.*

1. The exponential function $e^x$ grows faster than any power function as $x$ increases, and conversely the function $e^{-x} = 1/e^x$ decreases faster than any power of $(1/x)$ as $x$ grows. The same is true for $e^{ax}$ provided $a > 0$.

2. The logarithm $\ln(x)$ is an increasing function that keeps growing without bound as $x$ increases, but it does not grow as rapidly as the function $y = x$. The same is true for $\ln(ax)$ provided $a > 0$. The logarithm is not defined for negative values of its argument and as $x$ approaches zero, this function becomes unbounded and negative. However, it approaches $-\infty$ more slowly than $x$ approaches 0. For this reason, the expression $x\ln(x)$ has a limit of 0 as $x \to 0$.

Appendix D

# Short Answers to Problems

## D..1 Answers to Chapter 1 Problems

- **Problem 1.1:**

  (a) Stretched in $y$ direction by factor $A$; (b) Shifted up by $a$; (c) Shifted in positive $x$ direction by $b$.

- **Problem 1.2:**

  Not Provided

- **Problem 1.3:**

  $y = x^n$; $y = x^{-n}$; $y = x^{1/n}, n = 2,\ 4,\ 6, \ldots$; $y = x^{-n}, n = 1,\ 2,\ 3, \ldots$

- **Problem 1.4:**

  (a) $x = 0, (3/2)^{1/3}$; (b) $x = 0, x = \pm\sqrt{1/4}$.

- **Problem 1.5:**

  if $m - n$ even: $x = \pm\left(\frac{A}{B}\right)^{1/(m-n)}, x = 0$; if $m - n$ odd: $x = \left(\frac{A}{B}\right)^{1/(m-n)}, x = 0$

- **Problem 1.6:**

  (a) $(0, 0)$ and $(1, 1)$; (b) $(0, 0)$; (c) $(\frac{\sqrt{7}}{2}, \frac{3}{4})$, $(-\frac{\sqrt{7}}{2}, \frac{3}{4})$, and $(0, -1)$.

- **Problem 1.7:**

  $m > -1$

- **Problem 1.8:**

  Not Provided

- **Problem 1.9:**

  $$x = \left(\frac{B}{A}\right)^{\frac{1}{b-a}}$$

- **Problem 1.10:**

  (a) $x = 0, -1, 3$; (b) $x = 1$; (c) $x = -2, 1/3$; (d) $x = 1$.

- **Problem 1.11:**

  (b) $a < 0$: $x = 0$; $a \geq 0$: $x = 0, \pm a^{1/4}$; (c) $a > 0$.

- **Problem 1.12:**

  Not Provided

- **Problem 1.13:**

  (a) $V$; (b) $\dfrac{V}{S} = \dfrac{1}{6}a, a > 0$; (c) $a = V^{\frac{1}{3}}$; $a = (\frac{1}{6}S)^{\frac{1}{2}}$; $a = 10$ cm; $a = \frac{\sqrt{15}}{3}$ cm.

- **Problem 1.14:**

  (a) $V$; (b) $\dfrac{r}{3}$; (c) $r = \left(\frac{3}{4\pi}\right)^{1/3} V^{1/3}$; $r = \left(\frac{1}{4\pi}\right)^{1/2} S^{1/2}$; $r \approx 6.2035$ cm; $r \approx 0.8921$ cm.

- **Problem 1.15:**
  $r = 2k_1/k_2 = 12\mu$m.
- **Problem 1.16:**
  (a) $P = C\left(\frac{R}{A}\right)^{d/b}$; (b) $S = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$.
- **Problem 1.17:**
  (a) $a$: $Ms^{-1}$, $b$: $s^{-1}$; (b) $b = 0.2$, $a = 0.002$; (c) $v = 0.001$.
- **Problem 1.18:**
  (a) $v \approx K$, (b) $v = K/2$.
- **Problem 1.19:**
  $K \approx 0.0048$, $k_n \approx 77$ nM
- **Problem 1.20:**
  (a) $x = -1, 0, 1$ (b) 1 (c) $y_1$ (d) $y_2$.
- **Problem 1.21:**
  Line of slope $a^3/A$ and intercept $1/A$
- **Problem 1.22:**
  $K = 0.5$, $a = 2$
- **Problem 1.23:**
  Not Provided
- **Problem 1.24:**
  $m \approx 67$, $b \approx 1.2$, $K \approx 0.8$, $k_n \approx 56$
- **Problem 1.25:**
  $x = \left(\frac{R}{A}\right)^{\frac{1}{r-a}}$

## D..2 Answers to Chapter 1 Problems

- **Problem 1.1:**

  (a) Stretched in $y$ direction by factor $A$; (b) Shifted up by $a$; (c) Shifted in positive $x$ direction by $b$.

- **Problem 1.2:**

  Not Provided

- **Problem 1.3:**

  $y = x^n$; $y = x^{-n}$; $y = x^{1/n}, n = 2,\ 4,\ 6, \ldots$; $y = x^{-n}, n = 1,\ 2,\ 3, \ldots$

- **Problem 1.4:**

  (a) $x = 0, (3/2)^{1/3}$; (b) $x = 0, x = \pm\sqrt{1/4}$.

- **Problem 1.5:**

  if $m - n$ even: $x = \pm\left(\frac{A}{B}\right)^{1/(m-n)}, x = 0$; if $m - n$ odd: $x = \left(\frac{A}{B}\right)^{1/(m-n)}, x = 0$

- **Problem 1.6:**

  (a) $(0, 0)$ and $(1, 1)$; (b) $(0, 0)$; (c) $(\frac{\sqrt{7}}{2}, \frac{3}{4})$, $(-\frac{\sqrt{7}}{2}, \frac{3}{4})$, and $(0, -1)$.

- **Problem 1.7:**

  $m > -1$

- **Problem 1.8:**

  Not Provided

- **Problem 1.9:**

  $$x = \left(\frac{B}{A}\right)^{\frac{1}{b-a}}$$

- **Problem 1.10:**

  (a) $x = 0, -1, 3$; (b) $x = 1$; (c) $x = -2, 1/3$; (d) $x = 1$.

- **Problem 1.11:**

  (b) $a < 0$: $x = 0$; $a \geq 0$: $x = 0, \pm a^{1/4}$; (c) $a > 0$.

- **Problem 1.12:**

  Not Provided

- **Problem 1.13:**

  (a) $V$; (b) $\dfrac{V}{S} = \dfrac{1}{6}a, a > 0$; (c) $a = V^{\frac{1}{3}}$; $a = (\frac{1}{6}S)^{\frac{1}{2}}$; $a = 10$ cm; $a = \frac{\sqrt{15}}{3}$ cm.

- **Problem 1.14:**

  (a) $V$; (b) $\dfrac{r}{3}$; (c) $r = \left(\frac{3}{4\pi}\right)^{1/3} V^{1/3}$; $r = \left(\frac{1}{4\pi}\right)^{1/2} S^{1/2}$; $r \approx 6.2035$ cm; $r \approx 0.8921$ cm.

- **Problem 1.15:**
  $r = 2k_1/k_2 = 12\mu$m.
- **Problem 1.16:**
  (a) $P = C\left(\frac{R}{A}\right)^{d/b}$; (b) $S = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$.
- **Problem 1.17:**
  (a) $a$: $Ms^{-1}$, $b$: $s^{-1}$; (b) $b = 0.2$, $a = 0.002$; (c) $v = 0.001$.
- **Problem 1.18:**
  (a) $v \approx K$, (b) $v = K/2$.
- **Problem 1.19:**
  $K \approx 0.0048$, $k_n \approx 77$ nM
- **Problem 1.20:**
  (a) $x = -1, 0, 1$ (b) 1 (c) $y_1$ (d) $y_2$.
- **Problem 1.21:**
  Line of slope $a^3/A$ and intercept $1/A$
- **Problem 1.22:**
  $K = 0.5$, $a = 2$
- **Problem 1.23:**
  Not Provided
- **Problem 1.24:**
  $m \approx 67$, $b \approx 1.2$, $K \approx 0.8$, $k_n \approx 56$
- **Problem 1.25:**
  $x = \left(\frac{R}{A}\right)^{\frac{1}{r-a}}$

## D..3 Answers to Chapter 2 Problems

- **Problem 2.1:**
  (a) $m = 28°$/min, $b = 50$.
- **Problem 2.2:**
  (a) $-4.91°$F/min. (b) -7, -8, -9 °F/min. (c) -9 °F/min.
- **Problem 2.3:**
  Displacements have same magnitude, opposite signs.
- **Problem 2.4:**
  (b) $9.8$ m/s.
- **Problem 2.5:**
  (a) $-14.7$ m/s; (b) $-gt - \frac{g\epsilon}{2}$; (c) $t = 10$ s.
- **Problem 2.6:**
  $v_0 - g/2$
- **Problem 2.7:**
  $\bar{v} = 13.23\ \mathrm{m/s.}$; secant line is $y = 13.23x - 2.226$
- **Problem 2.8:**
  (a) $2$; (b) $0$; (c) $-2$; (d) $0$.
- **Problem 2.9:**
  (a) $1$; $1$; $1$; (b) $1$; $0$; $1$; (c) $1$; $2$; $4$.
- **Problem 2.10:**
  (a) $3$; (b) $5.55$; (c) $\frac{32}{3}$.
- **Problem 2.11:**
  (a) $\frac{2\sqrt{2}}{\pi}$; (b) $\frac{6(1-\sqrt{2})}{\pi}$; (c) $\pi/4 \le x \le 5\pi/4$ (one solution).
- **Problem 2.12:**
  (a) $2 + h$; (b) 2; (c) $y = 2x$.
- **Problem 2.13:**
  $2h^2 + 25h + 104$; $104$
- **Problem 2.14:**
  (b) $0, -4, -1.9, -2.1, -2 - h$; (c) $-2$.
- **Problem 2.15:**
  (a) $2 + h$; (b) 2; (c) $2.98$.

- **Problem 2.16:**
  (a) $\frac{4}{\pi}$; (b) $\frac{4\sqrt{3}-12}{\pi}$.
- **Problem 2.17:**
  (a) $-1$; (b) $\frac{-2}{1+\epsilon}$; (c) Slope approaches -2; (d) $y = -2x + 4$.
- **Problem 2.18:**
  (a) $v(2) = 12$ m/s; $\bar{v} = 15$ m/s; (b) $v(2) = 0$ m/s; $\bar{v} = 25$ m/s; (c) $v(2) = 13$ m/s; $\bar{v} = 11$ m/s.
- **Problem 2.19:**
  $0$
- **Problem 2.20:**
  $\frac{-1}{(x+1)^2}$

## D..4 Answers to Chapter 3 Problems

- **Problem 3.1:**
  Not Provided
- **Problem 3.2:**
  Not Provided
- **Problem 3.3:**
  $5$; $5$; no change; linear function
- **Problem 3.4:**
  (a) no tangent line; (b) $y = -(x+1)$; (c) $y = (x+1)$.
- **Problem 3.5:**
  (a) increasing: $-\infty < x < 0,\ 1.5 < x < \infty$; decreasing for $0 < x < 1.5$; (b) $0$, local maximum; $1.5$, local minimum; (c) No.
- **Problem 3.6:**
  Not Provided
- **Problem 3.7:**
  Not Provided
- **Problem 3.8:**
  Not Provided
- **Problem 3.9:**
  $y = 2x - 3$
- **Problem 3.10:**
  Not Provided
- **Problem 3.11:**
  (a) $f'(x) = 2x$, $f'(0) = 0$, $f'(1) = 2 > 0$, $f'(-1) = -2 < 0$. Local minimum at $x = 0$; (b) $f'(x) = -3x^2$, $f'(0) = 0$, $f'(1) = -3 < 0$, $f'(-1) = -3 < 0$. No local maxima nor minima; (c) $f'(x) = -4x^3$, $f'(0) = 0$, $f'(1) = -4 < 0$,$f'(-1) = 4 > 0$. Local maximum at $x = 0$.
- **Problem 3.12:**
  Not Provided
- **Problem 3.13:**
  Not Provided
- **Problem 3.14:**
  Not Provided

- **Problem 3.15:**

  $5$

- **Problem 3.16:**

  (a) $y = 3x - 2$; (b) $x = 2/3$; (c) $1.331$; $1.3$.

- **Problem 3.17:**

  (a) $y = -4x + 5$; (b) $x = 5/4$, $y = 5$; (c) $y = 0.6$, smaller.

- **Problem 3.18:**

  (a) $y = f'(x_0)(x - x_0) + f(x_0)$; (b) $x = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

- **Problem 3.19:**

  Not Provided

- **Problem 3.20:**

  Not Provided

- **Problem 3.21:**

  (a) $14.7$ m/s; (b) $-4.9$ m/s.

- **Problem 3.22:**

  (b) $a = 2$.

- **Problem 3.23:**

  $(3, 9)$, $(1, 1)$

## D..5 Answers to Chapter 5 Problems

- **Problem 5.1:**

  (a) zeros: $x = 0$, $x = \pm\sqrt{3}$; loc. max.: $x = -1$; loc. min.: $x = 1$; (b) loc. min.: $x = 2$; loc. max.: $x = 1$; (c) (a): $x = 0$; (b): $x = 3/2$.

- **Problem 5.2:**

  (a) max.: $18$; min.: $0$; (b) max.: $25$; min.: $0$; (c) max.: $0$; min.: $-6$; (d) max.: $-2$; min.: $-17/4$.

- **Problem 5.3:**

  min.: $3/4$

- **Problem 5.4:**

  $x = 0$

- **Problem 5.5:**

  critical points: $x = 0,\ 1,\ 1/2$; inflection points: $x = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{6}$

- **Problem 5.6:**

  Not Provided

- **Problem 5.7:**

  $a = 1$, $b = -6$, $c = 7$

- **Problem 5.8:**

  (a) $v = 3t^2 + 6t$, $a = 6t + 6$; (b) $t = 0$, $\sqrt{3/a}$; (c) $t = 0$, $\sqrt{3/2a}$; (d) $t = 1/\sqrt{2a}$.

- **Problem 5.9:**

  Not Provided

- **Problem 5.10:**

  (a) $t = v_0/g$; (b) $h_0 + \frac{v_0^2}{2g}$; (c) $v = 0$.

- **Problem 5.11:**

  min. at $x = -\sqrt{3}$; max. at $x = \sqrt{3}$; c.u.: $x < -1$, $0 < x < 1$; c.d.: $-1 < x < 0$,$x > 1$; infl.pt.: $x = 0$

- **Problem 5.12:**

  loc. min.:$x = a$ loc. max.: $x = -2a$

- **Problem 5.13:**

  (a) increasing: $x < 0$, $0 < x < 3k$, $x > 5k$; decreasing: $3k < x < 5k$; loc. max.: $x = 3k$; loc. min.:$x = 5k$; (b) c.u.: $0 < x < (3 - \frac{\sqrt{6}}{2})k$, $x > (3 + \frac{\sqrt{6}}{2})k$; c.d.: $x < 0$, $(3 - \frac{\sqrt{6}}{2})k < x < (3 + \frac{\sqrt{6}}{2})k$; infl.pts.: $x = 0,\ (3 \pm \frac{\sqrt{6}}{2})k$.

- **Problem 5.14:**

  (b) $dv/dp = -b\frac{(a+p_0)}{(p+a)^2}$; (c) $p = p_0$.

- **Problem 5.15:**

  Not Provided

- **Problem 5.16:**

  abs. max. of $4.25$ at end points; abs. min. of $2$ at $x = 1$

- **Problem 5.17:**

  (a) $36x^2 - 16x - 15$; (b) $-12x^3 + 3x^2 - 3$; (c) $4x^3 - 18x^2 - 30x - 6$; (d) $3x^2$; (e) $\frac{36x}{(x^2+9)^2}$; (f) $\frac{6x^3-3x^2+6}{(1-3x)^2}$; (g) $\frac{18b^2-7b^{\frac{8}{3}}}{3(2-b^{\frac{2}{3}})^2}$; (h) $\frac{-36m^3+72m^2-36m+5}{(3m-1)^2}$; (i) $\frac{9x^4+8x^3-3x^2-4x+6}{(3x+2)^2}$.

## D..6 Answers to Chapter 6 Problems

- **Problem 6.1:**

  (a) $10, 10$; (b) $10, 10$; (c) $12, 8$.

- **Problem 6.2:**

  (a) $v(t) = 120t^2 - 16t^3$; (b) $t = 5$; (c) $t = 7.5$.

- **Problem 6.3:**

  $9 : 24A.M.$, $15$ km

- **Problem 6.4:**

  (a) $t \approx 1.53$ sec; (b) $v(0.5) = 10.1$ m/sec, $v(1.5) = 0.3$ m/sec, $a(0.5) = -9.8$ m/sec$^2$, $a(1.5) = -9.8$ m/sec$^2$; (c) $t \approx 3.06$ sec.

- **Problem 6.5:**

  $30 \times 10 \times 15$ cm

- **Problem 6.6:**

  (a) $y = (1/\sqrt{3})$; (b) $\sqrt{3}/9$.

- **Problem 6.7:**

  $|a|$ if $a < 4$; $2\sqrt{2a - 4}$ if $a \geq 4$

- **Problem 6.8:**

  $A = 625$ ft$^2$

- **Problem 6.9:**

  All of the fencing used for a circular garden.

- **Problem 6.10:**

  Squares of side $6 - 2\sqrt{3}$ cm.

- **Problem 6.11:**

  Straight lines from $(10, 10)$ to $(\frac{16}{3}, 0)$ then to $(3, 5)$.

- **Problem 6.12:**

  4 °C

- **Problem 6.13:**

  (a) $x = 2B/3$, $R = (4/27)AB^3$; (b) $x = B/3$, $S = AB^2/3$.

- **Problem 6.14:**

  $r = 2k_1/k_2$

- **Problem 6.15:**

  $h = 20$, $r = 5\sqrt{2}$

- **Problem 6.16:**

  (b) $x = \frac{a}{2b}$; (c) $x = 0$; (d) $x = \frac{a-m}{2b}$.

- **Problem 6.17:**

  $x = (A/2B)^{1/3} - 1$

- **Problem 6.18:**

  (b) $N = K/2$.

- **Problem 6.19:**

  $N_{\text{MSY}} = K(1 - qE/r)$

- **Problem 6.20:**

  $E = r/2q$

## D..7 Answers to Chapter 7 Problems

- **Problem 7.1:**
  (a) $\frac{dy}{dx} = \frac{21x^2+2}{6y^5+3}$; (b) $\frac{dy}{dx} = -\frac{2y}{e^y+2x}$; (c) $\frac{dy}{dx} = x^{\cos x}\left(-\sin x \cdot \ln x + \frac{\cos x}{x}\right)$.
- **Problem 7.2:**
  (a) $4\pi r^2 k$; (b) $8\pi rk$; (c) $-\frac{3k}{r^2}$.
- **Problem 7.3:**
  (a) $dA/dt = 2\pi rC$; (b) $dM/dt = \alpha 2\pi rC$.
- **Problem 7.4:**
  $\frac{dM}{dt} = C\pi(3r^2)a$
- **Problem 7.5:**
  $\frac{dV}{dt} = 1 \text{ m}^3/\text{min}$
- **Problem 7.6:**
  (a) $\frac{1}{300\pi}$ cm/s; (b) $\frac{2}{5}$ cm$^2$/s.
- **Problem 7.7:**
  $5$ cm/s
- **Problem 7.8:**
  (a) $\frac{dV}{dt} = \frac{nR}{P}\frac{dT}{dt}$; (b) $\frac{dV}{dt} = -\frac{nRT}{P^2}\frac{dP}{dt}$.
- **Problem 7.9:**
  $\pi^{-1/2}k$
- **Problem 7.10:**
  $-\frac{1}{10\pi}$ cm/min
- **Problem 7.11:**
  1 cm/sec toward lens
- **Problem 7.12:**
  $\frac{dh}{dt} = \frac{-1}{36\pi}$ cm/min
- **Problem 7.13:**
  $k = \frac{1}{10} + \frac{4\pi}{45}$
- **Problem 7.14:**
  $\frac{dh}{dt} = \frac{6}{5\pi}$ ft/min
- **Problem 7.15:**
  $h'(5) = \frac{2}{5\pi}$ m/min

- **Problem 7.16:**
  (a) $\frac{1}{4\pi}$ m/min; (b) $\frac{1}{\pi}$ m/min.
- **Problem 7.17:**
  (a) $-4$ m/s; (b) $-\frac{25}{32}$ per sec.
- **Problem 7.18:**
  $\frac{-1}{4\sqrt{6}}$ m/min
- **Problem 7.19:**
  $\frac{dS}{dA} = \frac{ab}{1+bA}$; no.
- **Problem 7.20:**
  $\frac{dy}{dt} = 3\frac{l_2}{l_1}$ cm/hr
- **Problem 7.21:**
  Not Provided
- **Problem 7.22:**
  Not Provided
- **Problem 7.23:**
  (a) $\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$; (b) $y = \frac{1}{\sqrt{2}}(-x + r\sqrt{3})$; $y = (1/\sqrt{2})(x - r\sqrt{3})$.
- **Problem 7.24:**
  (a) $-3/4$; (b) $y = -(3/4)x + 8$.
- **Problem 7.25:**
  Not Provided
- **Problem 7.26:**
  $(\frac{2}{\sqrt{10}}, \frac{9}{\sqrt{10}})$ and $(-\frac{2}{\sqrt{10}}, -\frac{9}{\sqrt{10}})$
- **Problem 7.27:**
  $m = \frac{4y_p}{x_p}$
- **Problem 7.28:**
  (c) Global minimum occurs at an endpoint, rather than at a critical point.
- **Problem 7.29:**
  Not Provided
- **Problem 7.30:**
  (b) $\frac{dy}{dx} = \frac{(ay-x^2)}{(y^2-ax)}$; (c) $x = 0$, $x = 2^{1/3}a$; (d) No.

- **Problem 7.31:**

  (a) $\frac{dp}{dv} = (2\frac{a}{v^3}) - (p + \frac{a}{v^2})/(v - b)$.

- **Problem 7.32:**

  $(0, 5/4)$

- **Problem 7.33:**

  (a) $y - 1 = -1(x - 1)$; (b) $y'' = \frac{4}{5}$; (c) concave up.

## D..8 Answers to Chapter 8 Problems

- **Problem 8.1:**
  Not Provided
- **Problem 8.2:**
  Not Provided
- **Problem 8.3:**
  (a) $5^{0.75} > 5^{0.65}$; (b) $0.4^{-0.2} > 0.4^{0.2}$; (c) $1.001^2 < 1.001^3$; (d) $0.999^{1.5} > 0.999^{2.3}$.
- **Problem 8.4:**
  Not Provided
- **Problem 8.5:**
  (a) $x = a^2b^3$; (b) $x = \frac{b}{c^{\frac{2}{3}}}$.
- **Problem 8.6:**
  Not Provided
- **Problem 8.7:**
  (a) $x = \frac{3-\ln(5)}{2}$; (b) $x = \frac{e^4+1}{3}$; (c) $x = e^{(e^2)} = e^{e\cdot e}$; (d) $x = \frac{\ln(C)}{a-b}$.
- **Problem 8.8:**
  (a) $\frac{dy}{dx} = \frac{6}{2x+3}$; (b) $\frac{dy}{dx} = \frac{6[\ln(2x+3)]^2}{2x+3}$; (c) $\frac{dy}{dx} = -\frac{1}{2}\tan\frac{1}{2}x$; (d) $\frac{dy}{dx} = \frac{3x^2-2}{(x^3-2x)\ln a}$; (e) $\frac{dy}{dx} = 6xe^{3x^2}$; (f) $\frac{dy}{dx} = -\frac{1}{2}a^{-\frac{1}{2}x}\ln a$; (g) $\frac{dy}{dx} = x^2 2^x(3+x\ln 2)$; (h) $\frac{dy}{dx} = e^{e^x+x}$; (i) $\frac{dy}{dx} = \frac{4}{(e^t+e^{-t})^2}$.
- **Problem 8.9:**
  (a) min.: $x = \frac{2}{\sqrt{3}}$; max.: $x = -\frac{2}{\sqrt{3}}$; infl.pt.: $x = 0$; (b) min.: $x = \frac{1}{\sqrt[3]{3}}$; (c) max.: $x = 1$; inf.pt.: $x = 2$; (d) min.: $x = 0$; (e) min.: $x = 1$; max.: $x = -1$; (f) min.: $x = \ln(2)$; infl. pt.: $x = \ln(4)$.
- **Problem 8.10:**
  $C = 4$, $k = -0.5$
- **Problem 8.11:**
  (a) decreasing; (b) increasing; $y_1(0) = y_2(0) = 10$; $y_1$ half-life $= 10\ln(2)$; $y_2$ doubling-time $= 10\ln(2)$
- **Problem 8.12:**
  Not Provided
- **Problem 8.13:**
  Not Provided

- **Problem 8.14:**
  Not Provided
- **Problem 8.15:**
  crit.pts.: $x = 0$, $x \approx \pm 1.64$; $f(0) = 1$; $f(\pm 1.64) \approx -0.272$
- **Problem 8.16:**
  $x = 1/\beta$
- **Problem 8.17:**
  (a) $x = r$; (c) $x = \frac{ar}{a-r} \ln\left(\frac{R}{A}\right)$; (d) decrease; (e) decrease.
- **Problem 8.18:**
  $x = b\sqrt{\ln((a^2 + b^2)/b^2)}$
- **Problem 8.19:**
  Not Provided

## D..9 Answers to Chapter 9 Problems

- **Problem 9.1:**
  Not Provided
- **Problem 9.2:**
  (a) $C$ any value, $k = -5$; (b) $C$ any value, $k = 3$.
- **Problem 9.3:**
  (a) $y(t) = Ce^{-t}$; (b) $c(x) = 20e^{-0.1x}$; (c) $z(t) = 5e^{3t}$.
- **Problem 9.4:**
  $t = -\frac{\ln 2}{\ln 7 - \ln 10}$
- **Problem 9.5:**
  (a) 57300 years; (b) 22920 years
- **Problem 9.6:**
  (a) 29 years; (b) 58 years; (c) 279.7 years.
- **Problem 9.7:**
  (a) $80.7\%$; (b) 12.3 years.
- **Problem 9.8:**
  $y \approx 707.8 \text{ torr}$
- **Problem 9.9:**
  (a) $P(5) \approx 1419$; (b) $t \approx 9.9years$.
- **Problem 9.10:**
  $\frac{dN}{dt} = 0.05N$; $N(0) = 250$; $N(t) = 250e^{0.05t}$; $2.1 \times 10^{10}$ rodents
- **Problem 9.11:**
  (a) $dy/dt = 2.57y$; (b) $dy/dt = -6.93y$.
- **Problem 9.12:**
  (a) 12990; (b) $30792 \text{ bacteria}$.
- **Problem 9.13:**
  1.39 hours; 9.2 hours
- **Problem 9.14:**
  20 min; 66.44 min
- **Problem 9.15:**
  (a) $y_1$ growing, $y_2$ decreasing; (b) $3.5, 2.3$; (c) $y_1(t) = 100e^{0.2t}$, $y_2(t) = 10000e^{-0.3t}$;
  (d) $t \approx 9.2$ years.

- **Problem 9.16:**

  $12265$ people$/$km$^2$

- **Problem 9.17:**

  (a) $1$ hour; (b) $r = \ln(2)$; (c) 0.25 M; (d) $t = 3.322$ hours.

- **Problem 9.18:**

  $6.93$ years

- **Problem 9.19:**

  $1.7043$ kg

- **Problem 9.20:**

  (a) $\$510$, $\$520.20$, $\$742.97$; $17.5$ years; for 8% interest: $\$520$, $\$540.80$, $\$1095.56$; (b) $\$510.08$, $\$520.37$, $\$745.42$; (c) 5%.

## D..10 Answers to Chapter 10 Problems

- **Problem 10.1:**

  (a) $\frac{dy}{dx} = 2x \ \cos x^2$; (b) $\frac{dy}{dx} = \sin 2x$; (c) $\frac{dy}{dx} = -\frac{2}{3}x^{-\frac{2}{3}}(\cot \sqrt[3]{x})(\csc^2 \sqrt[3]{x})$; (d) $\frac{dy}{dx} = (1-6x)\sec(x-3x^2)\tan(x-3x^2)$; (e) $\frac{dy}{dx} = 6x^2 \tan x + 2x^3 \sec^2 x$; (f) $\frac{dy}{dx} = \frac{\cos x + x \sin x}{\cos^2 x}$; (g) $\frac{dy}{dx} = \cos x - x \sin x$; (h) $\frac{dy}{dx} = \frac{\sin \frac{2}{x}}{x^2 e^{\sin^2 \frac{1}{x}}}$; (i) $\frac{dy}{dx} = 6(2\tan 3x + 3\cos x)(2\sec^2 3x - \sin x)$; (j) $\frac{dy}{dx} = -\sin(\sin x) \cdot \cos x + \cos 2x$.

- **Problem 10.2:**

  (a) $f'(x) = \frac{-(4x^3+10x)\sin(\ln(x^4+5x^2+3))}{(x^4+5x^2+3)}$; (b) $f'(x) = \frac{(3x^2-2\cos(x)\sin(x))\cos(\sqrt{\cos^2(x)+x^3})}{(2\sqrt{\cos^2(x)+x^3})}$;
  (c) $f'(x) = 6x^2 + \frac{1}{x\ln(3)}$; (d) $f'(x) = 4(x^2e^x + \tan(3x))^3(2xe^x + x^2e^x + 3\sec^2(3x))$;
  (e) $f'(x) = 2x\sqrt{\sin^3(x) + \cos^3(x)} + \frac{3x^2(\sin^2(x)\cos(x) - \cos^2(x)\sin(x))}{2\sqrt{\sin^3(x)+\cos^3(x)}}$.

- **Problem 10.3:**

  (a) $180^o$ (b) $300^o$ (c) $164.35^o$ (d) $4320^o$

  (e) $5\pi/9$ (f) $2\pi/45$ (g) $5\pi/2$ (h) $\pi/2$

  (i) $1/2$ (j) $\sqrt{2}/2$ (k) $\sqrt{3}/3$

- **Problem 10.4:**

  Not Provided

- **Problem 10.5:**

  Not Provided

- **Problem 10.6:**

  Not Provided

- **Problem 10.7:**

  $-\sqrt{3}/20$, $1/20$

- **Problem 10.8:**

  (a) $[0, \pi/4], \ [5\pi/4, 2\pi]$; (b) $[3\pi/4, 7\pi/4]$; (c) $x = 3\pi/4, \ 7\pi/4$.

- **Problem 10.9:**

  (a) $T(t) = 37.1 + 0.4\cos[\pi(t-8)/12]$; (b) $W(t) = 0.5 + 0.5\cos[\pi(t-8)/6]$.

- **Problem 10.10:**

  (a) $S = 3\cos\left(\sqrt{\frac{g}{l}}t\right)$; (b) $y = 2\sin\left(\frac{2\pi}{3}t + \frac{\pi}{6}\right) + 10$.

- **Problem 10.11:**

  $\pm(\frac{\pi}{8}, 1)$

- **Problem 10.12:**
  $-0.021$ rad/min
- **Problem 10.13:**
  $0.125$ radians per minute
- **Problem 10.14:**
  Not Provided
- **Problem 10.15:**
  (a) $\pi/8$; (b) $5\pi/8$.
- **Problem 10.16:**
  (a) $\frac{dy}{dt} = -\frac{Cx}{\sqrt{L^2-x^2}}$; (b) $\frac{d\theta}{dt} = \frac{C}{y}$.
- **Problem 10.17:**
  $R = \frac{1}{32}\, v_0^2$
- **Problem 10.18:**
  $8\pi$ m/s; $0$ m/s
- **Problem 10.19:**
  $30\pi$ cm/s; to the right
- **Problem 10.20:**
  (a) $\sqrt{h^2 + 2hR}$; (b) $-v\sqrt{h^2 + 2hR}/R$.
- **Problem 10.21:**
  (a) $\frac{dy}{dx} = \frac{4\sec^2(2x+y)}{1-2\sec^2(2x+y)}$; (b) $\frac{dy}{dx} = \frac{2\sin x}{\cos y}$; (c) $\frac{dy}{dx} = -\frac{y\cos x+\sin y}{x\cos y+\sin x}$.
- **Problem 10.22:**
  $y = -x + 2$

## D..11 Answers to Chapter 11 Problems

- **Problem 11.1:**
  $y' = a/\sqrt{1-a^2x^2}$

- **Problem 11.2:**
  (a) $x$; (b) $x/\sqrt{1-x^2}$; (c) $\sqrt{1-x^2}$.

- **Problem 11.3:**
  (a) $\dfrac{dy}{dx} = \dfrac{1}{3x^{\frac{2}{3}}\sqrt{1-x^{\frac{2}{3}}}}$; (b) $\dfrac{dy}{dx} = \dfrac{1}{3(\arcsin x)^{\frac{2}{3}}\sqrt{1-x^2}}$; (c) $\dfrac{d\theta}{dr} = \dfrac{1}{2r^2+2r+1}$; (d) $\dfrac{dy}{dx} = \text{arcsec}\frac{1}{x} - \frac{x}{\sqrt{1-x^2}}$; (e) $\dfrac{dy}{dx} = \dfrac{-2x^2+a^2-a}{a\sqrt{a^2-x^2}}$; (f) $\dfrac{dy}{dt} = -\left|\dfrac{1+t^2}{1-t^2}\right| \cdot \dfrac{2(1-t^2)}{(1+t^2)^2}$.

- **Problem 11.4:**
  $0.4$ m

- **Problem 11.5:**
  $\frac{5}{26}$ rad/s

## D..12 Answers to Chapter 12 Problems

- **Problem 12.1:**

  (a) $\frac{19}{6}$; (b) $3$; (c) $-0.1$, $-0.1$.

- **Problem 12.2:**

  (a) $0.40208$; (b) $5.99074$.

- **Problem 12.3:**

  $0.99$

- **Problem 12.4:**

  (a) $0.98255$; (b) $0.87475$.

- **Problem 12.5:**

  $-2.998$

- **Problem 12.6:**

  Not Provided

- **Problem 12.7:**

  $1030 \text{ cm}^3$

- **Problem 12.8:**

  $x = 0.357403,\ 2.153292$

- **Problem 12.9:**

  $2.83$

- **Problem 12.10:**

  (a) $(3.41421, 207.237)$, $(0.58580, 0.762)$, $(-0.42858, -0.895)$; (b) $(1.30980, 0.269874)$.

- **Problem 12.11:**

  (a) $x = 0.32219$; (b) $x = 0.81054$; (c) $x = 0.59774$, $x = -0.68045$, $x = -4.91729$; (d) $x = 2.34575$.

- **Problem 12.12:**

  $x = 0,\ \pm 1.895$

- **Problem 12.13:**

  (a) loc.max.: $x = 1.1397,\ -1.9100$; loc.min.: $x = -0.2297$; (b) $f''(x)$ positive: $(-\infty, -2.1902]$, $[-0.7634, 0.3801]$, $[1.5735, \infty)$.

- **Problem 12.14:**

  $-0.9012$

- **Problem 12.15:**

  Not Provided

- **Problem 12.16:**

  (a) $y_5 = 1.61051$; $y(0.5) = 1.6487213$; error $= 0.03821$; (b) $y_5 = 0.59049$; $y(0.5) = 0.60653$; error $= 0.01604$.

- **Problem 12.17:**

  $0.55$, $0.5995$, $0.6475$, $0.6932$, $0.7357$

- **Problem 12.18:**

  Not Provided

- **Problem 12.19:**

  (a) $0.806$; (b) $0.681$; (c) $4.027$.

- **Problem 12.20:**

  (a) $1.112$; (b) $0.622$.

## D..13 Answers to Chapter 13 Problems

- **Problem 13.1:**
  Not Provided
- **Problem 13.2:**
  Not Provided
- **Problem 13.3:**
  Not Provided
- **Problem 13.4:**
  (a) $C = -12$; (b) $C_1 = 1$, $C_2 = -5$; (c) $C_1 = -1$, $C_2 = 0$.
- **Problem 13.5:**
  (a) $v(t) = -\frac{g}{k}e^{-kt} + \frac{g}{k}$; (b) $v = \frac{g}{k}$.
- **Problem 13.6:**
  $c(t) = -\frac{k}{s}e^{-st} + \frac{k}{s}$
- **Problem 13.7:**
  (b) $46$ minutes before discovery.
- **Problem 13.8:**
  $10.6$ min
- **Problem 13.9:**
  $64.795$ gm, $250$ gm
- **Problem 13.10:**
  (a) $Q'(t) = kr - \frac{Q}{V}r = -\frac{r}{V}[Q - kV]$; (b) $Q = kV$; (c) $T = V\ln 2/r$.
- **Problem 13.11:**
  Not Provided
- **Problem 13.12:**
  Not Provided
- **Problem 13.13:**
  Not Provided
- **Problem 13.14:**
  (c) $y(t) = A\cos(\sqrt{g/L}t)$.
- **Problem 13.15:**
  (a) $\frac{dQ}{dt} = kQ$; $Q(t) = 100e^{(-8.9\times10^{-2})t}$; (b) $7.77$ hr.

- **Problem 13.16:**

  (b) $k = 3/2$.

- **Problem 13.17:**

  (a) $\frac{dx}{dt} = \frac{k}{3}(V_0 - x^3)$; (d) $V = \frac{1}{2}V_0$.

- **Problem 13.18:**

  (b) $y_0$; (c) $t = \frac{2A\sqrt{y_0}}{k}$; (d) $-k\sqrt{y_0}$.

- **Problem 13.19:**

  $a = 0, b = -1$

- **Problem 13.20:**

  (b) $t = \pi/4 + n\pi$.

- **Problem 13.21:**

  (a) $K_{\text{max}}$, $c = k$; (b) $\ln(2)/r$; (c) $c = 0$, $c = \frac{K_{\text{max}}}{r} - k$.

## D..14 Answers to Appendix A Problems

- **Problem 1:**

  (a) slope $4$, $y$ intercept $-5$; (b) slope $\frac{3}{4}$, $y$ intercept $-2$; (c) slope $\frac{2}{3}$, $y$ intercept $0$; (d) slope $0$, $y$ intercept $3$; (e) slope $\frac{5}{2}$, $y$ intercept $-\frac{23}{2}$.

- **Problem 2:**

  (a) $y = -5(x-2) = -5x+10$; (b) $y = \frac{1}{2}x - \frac{5}{2}$; (c) $y = \frac{4}{5}x + 10$; (d) $y = -(3/4)x + 1$.

- **Problem 3:**

  (a) $y = -4x+3$; (b) $y = 3x+2$; (c) $y = -6x+5$; (d) $y = 3x$; (e) $y = -6x+5$; (f) $y = -x/4$; (g) $y = 2x+9$.

- **Problem 4:**

  $y = \sqrt{2} - x$

## D..15 Answers to Appendix B Problems

- **Problem 1:**
  Not Provided
- **Problem 2:**
  (a) Odd; (b) Even; (c) Even; (d) Odd; (e) Neither.
- **Problem 3:**
  Not Provided
- **Problem 4:**
  $y = [(1/A)x]^{1/n}; x = 0, \pm(1/A)^{1/(n-1)}$

# Index