Lecture 7 (Sept. 20, 2013)

Learning Goals: ① Calculate the derivative by definition.
② Second derivative, antiderivative (Position, velocity, acceleration)

- Compute derivative by definition: $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Step 1: write the derivative in the definition form, simplify it as much as possible

Step 2: Compute the limit

Example 1: Find the derivative of $f(x) = \dfrac{4x}{x-1}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{4(x+h)}{x+h-1} - \frac{4x}{x-1}}{h} = \lim_{h \to 0} \frac{4}{h} \cdot \left[\frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \right]$$

$$= \lim_{h \to 0} \frac{4}{h} \cdot \left[\frac{x^2 + hx - x - h - x^2 - hx + x}{(x+h-1)(x-1)} \right] = \lim_{h \to 0} \frac{4}{h} \cdot \frac{-h}{(x+h-1)(x-1)} = \lim_{h \to 0} \frac{-4}{(x+h-1)(x-1)}$$

$$= \cancel{\lim} \frac{-4}{(x-1)^2}$$

Notice: $f(x)$ and $f'(x)$ DNE at $x=1$

- Derivative of Polynomials: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

$\Rightarrow g(x) = p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 2a_2 x + a_1$

we can treat $g(x)$ as a polynomial with domain $(-\infty, +\infty)$

then $g(x)$ is differentiable and we have

$g'(x) = p''(x) = n(n-1) a_n x^{n-2} + (n-1)(n-2) a_{n-1} x^{n-3} + \cdots + 2a_2$

- Second derivative: the second derivative of $f(x)$ can be written as

$$f''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = (f'(x))'$$

Example 2: $p(x) = 2x^2 + 4x + 1$

$\Rightarrow p'(x) = 4x + 4 \Rightarrow p''(x) = \cancel{4x}\ 4 \Rightarrow p^{(3)}(x) = 0 \Rightarrow p^{(4)}(x) = 0 \Rightarrow \cdots$

- Antiderivative / antidifferentiation

$$\text{given } f(x) \underset{\text{antiderivative}}{\overset{\text{take derivative}}{\rightleftarrows}} f'(x) \underset{\text{antiderivative}}{\overset{\text{take derivative}}{\rightleftarrows}} f''(x) \longrightarrow \cdots$$

(?) under $f(x)$; given under $f'(x)$; given under $f''(x)$

Example 3: given $g(x) = f''(x) = ax^2$, which of the following is $f(x)$?

✓ (a) $f(x) = \frac{a}{12}x^4$ (c) $f(x) = \frac{a}{12}x^4 + bx^2$

(b) $f(x) = \frac{a}{12}x^4 + bx^3$ ✓ (d) $f(x) = \frac{a}{12}x^4 + c$ (b, c - nonzero constants)

Notice: Given the function $f(x)$, we can not find a unique antiderivative

Back to example 2, if we start with $p''(x) = 4$

take antiderivative, we can only get $\cancel{p}$ $p'(x) = 4x + c_1$ c_1 - constant

then $p(x) = 2x^2 + c_1 x + c_2$. c_2 - constant

c_1, c_2 can be any value unless extra conditions are given.

- Position, velocity and acceleration.

Suppose $y(t)$ represents the position of a moving object at time t

then $y'(t) = v(t)$ represents the velocity of the object

$y''(t) = v'(t) = a(t)$ represents the acceleration of the object

Given one of those three, we should be able to find the other two

Example 4: Uniformly accelerated motion, $a(t) = c_1$ constant

then take antiderivative, $v(t) = c_1 t + c_2$

$v(0) = c_2 \leftarrow$ initial velocity

take the antiderivative of $v(t)$. $y(t) = \frac{1}{2}c_1 t^2 + c_2 t + c_3$

$y(0) = c_3 \leftarrow$ initial position

Example 5: Free falling object (see Example 4.8 in Leah's notes)

$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$