

Lecture 7 (Sept. 20, 2013)

Learning Goals: ① Calculate the derivative by definition.

② Second derivative antiderivative (Position, velocity, acceleration)

• Compute derivative by definition: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

Step 1: write the derivative in the definition form, simplify it as much as possible

Step 2: Compute the limit

Example 1: Find the derivative of $f(x) = \frac{4x}{x-1}$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{4(x+h)}{x+h-1} - \frac{4x}{x-1}}{h} = \lim_{h \rightarrow 0} \frac{4}{h} \cdot \left[\frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \right] \\ &= \lim_{h \rightarrow 0} \frac{4}{h} \cdot \left[\frac{x^2 + hx - x - h - x^2 - hx + x}{(x+h-1)(x-1)} \right] = \lim_{h \rightarrow 0} \frac{4}{h} \cdot \frac{-h}{(x+h-1)(x-1)} = \lim_{h \rightarrow 0} \frac{-4}{(x+h-1)(x-1)} \\ &= \lim_{h \rightarrow 0} \frac{-4}{(x-1)^2} \end{aligned}$$

Notice: $f(x)$ and $f'(x)$ DNE at $x=1$

• Derivative of Polynomials: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$

$$\Rightarrow g(x) = p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots + 2 a_2 x + a_1$$

we can treat $g(x)$ as a polynomial with domain $(-\infty, +\infty)$

then $g(x)$ is differentiable and we have

$$g'(x) = p''(x) = n(n-1) a_n x^{n-2} + (n-1)(n-2) a_{n-1} x^{n-3} + \dots + 2 a_2$$

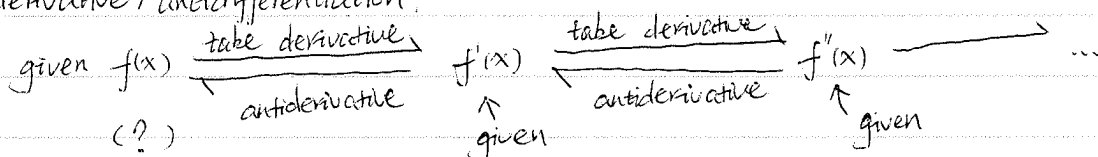
• Second derivative: the second derivative of $f(x)$ can be written as

$$f''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx} \left(\frac{df}{dx} \right) = (f'(x))'$$

Example 2: $p(x) = 2x^2 + 4x + 1$

$$\Rightarrow p'(x) = 4x + 4 \Rightarrow p''(x) = 4 \Rightarrow p^{(3)}(x) = 0 \Rightarrow p^{(4)}(x) = 0 \Rightarrow \dots$$

• Antiderivative / antidifferentiation



Example 3: given $g(x) = f''(x) = ax^2$, which of the following is $f(x)$?

✓ (a) $f(x) = \frac{a}{12}x^4$

(c) $f(x) = \frac{a}{12}x^4 + bx^2$

(b) $f(x) = \frac{a}{12}x^4 + bx^3$

✓ (d) $f(x) = \frac{a}{12}x^4 + C$

(b, c - nonzero constants)

Notice: Given the function $f(x)$, we can not find a unique antiderivative

Back to example 2, if we start with $f''(x) = 4$

take antiderivative, we can only get $f'(x) = 4x + C_1$ C_1 - constant

then $f(x) = 2x^2 + C_1x + C_2$, C_2 - constant

C_1, C_2 can be any value unless extra conditions are given.

• Position, velocity and acceleration.

Suppose $y(t)$ represents the position of a moving object at time t

then $y'(t) = v(t)$ represents the velocity of the object

$y''(t) = v'(t) = a(t)$ represents the acceleration of the object

Given one of those three, we should be able to find the other two.

Example 4: Uniformly accelerated motion, $a(t) = C_1$ constant.

then take antiderivative, $v(t) = C_1t + C_2$

$v(0) = C_2 \leftarrow$ initial velocity

take the antiderivative of $v(t)$, $y(t) = \frac{1}{2}C_1t^2 + C_2t + C_3$

$y(0) = C_3 \leftarrow$ initial position

Example 5: Free falling object (see Example 4.8 in Leach's notes)

$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$

