Today

- Derivatives of exponential functions.

Exponential functions

Which of the following is an exponential function?

(A) x^n

(B) 2^x

(C) e^2

(D) $\ln(x)$

Exponential functions

Which of the following is an exponential function?

(A) x^n <--power function

(B) 2^x <--variable in the exponent!

(C) e^2 <--an exp. calculation but not an exp. function

(D) ln(x) <--the INVERSE of an exp. function

Exponential functions a^x where $a>1$...

(A) All go through the point (1,1).

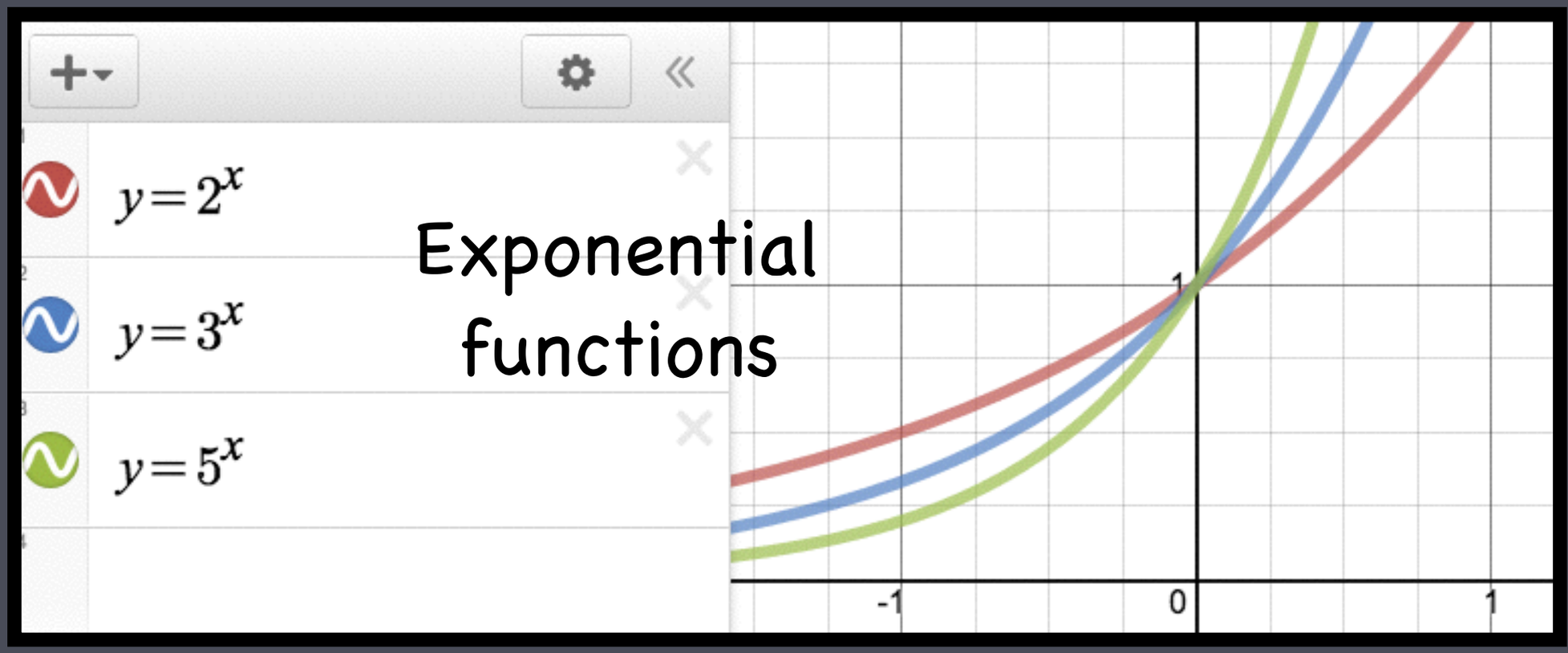
(B) All go through the point (0,0).

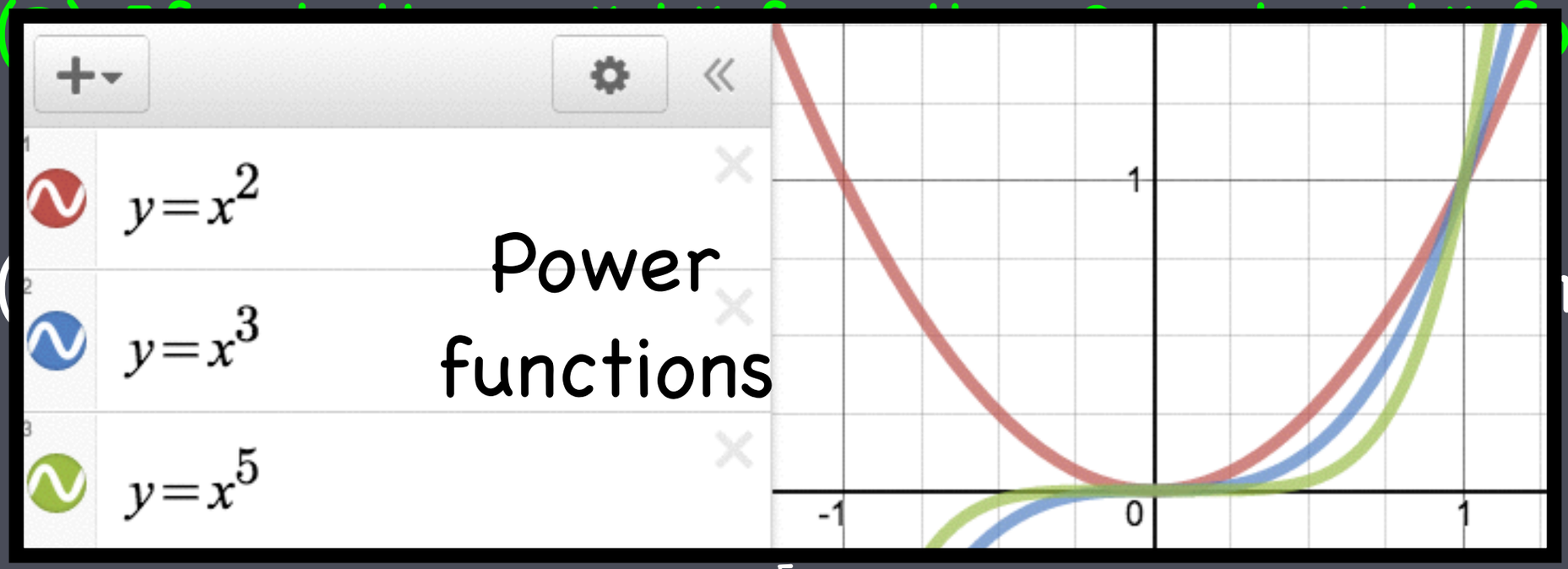
(C) All go through the point (1,0).

(D) If $a<b$ then $a^x<b^x$ for all $x>0$ and $a^x>b^x$ for all $x<0$.

(E) If $a<b$ then $a^x<b^x$ for all $x>1$ and $a^x>b^x$ for all $x<1$.

Exponential functions a^x





The derivative of $f(x)=a^x$ is...

(A) $f'(x) = xa^{x-1}$.

(B) $f'(x) = ax^{a-1}$.

(C) $f'(x) = (a^{x+h}-a^x)/h$

(D) $f'(x) = a^x$.

(E) $f'(x) = Ca^x$.

Checking your high school memory here.

Derivative of $f(x)=2^x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} = \lim_{h \to 0} \frac{2^x(2^h - 1)}{h}$$

$$= 2^x \lim_{h \to 0} \frac{2^h - 1}{h} = C_2\, 2^x$$

$$\lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.69338746258 = C_2$$

Derivative of $f(x)=a^x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= a^x \lim_{h \to 0} \frac{a^h - 1}{h} \quad = C_a\, a^x$$

$$\lim_{h \to 0} \frac{a^h - 1}{h} \approx ?? \qquad = C_a$$

Find a special value of a.

- When is $C_a = \lim_{h \to 0} \frac{a^h - 1}{h} = 1$?

Want $\frac{a^h - 1}{h} \approx 1$ (for h small)

$$a^h - 1 \approx h$$

$$a^h \approx h + 1$$

$$a \approx (h+1)^{\frac{1}{h}}$$

With h=0.1, a≈2.5937424601.

With h=0.01, a≈2.70481382942.

With h=0.001, a≈2.71692393224.

With h=0.0001, a≈2.71814592682.

With h=0.00001, a≈2.71826823719.

What is this special a value? a=e!

We just found a function that is its own derivative! $f(x)=e^x$.

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h} \quad \to 1$$

$$= e^x$$

This is precisely how e is defined - the number whose exponential function is its own derivative.

Differential equations

- What real number is the same as its own square?
  - Equivalent to asking "what x satisfies the equation $x=x^2$?"
    - Call this an algebraic equation.
- What function is equal to its own derivative?
  - Equivalent to asking "what f(x) satisfies $f'(x)=f(x)$?"
  - Call this a "differential equation". (DE)

DE example: Which of the following satisfies $f'(x)=f(x)$?

(A) $f(x) = 2^x$

(B) $f(x) = e^x$

(C) $f(x) = x^{-1}$

(D) $f(x) = -x^{-1}$

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A note about units

- e is a "pure" number without units, called dimensionless.
- This means e^a for any a is also dimensionless.
- Furthermore, the exponent, a, must also be dimensionless.
- If $y(t) = y_0e^{-kt}$, and t is time in seconds, what must be the units of k?

What is the definition of the inverse function of f(x)?

(A) The function g(x) for which g(f(x))=x.

(B) The mirror image of graph of f(x) in the line y=x.

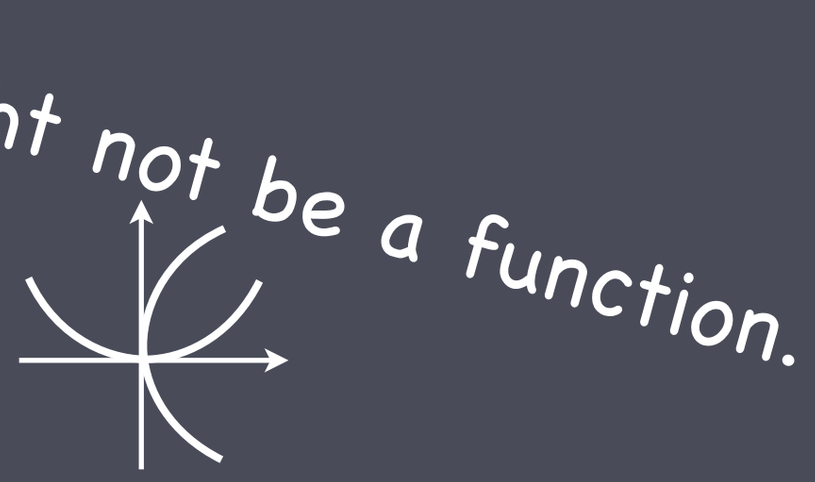
(C) 1/f(x)

(D) -f(x)

What is the definition of the inverse function of f(x)?

(A) The function g(x) for which g(f(x))=x. $g(x) = f^{-1}(x)$

(B) The mirror image of graph of f(x) in the line y=x.



(C) 1/f(x)

(D) -f(x)

$f^{-1}(x)$ is the function that goes backwards through f(x). If you plug the output of f(x) into $f^{-1}(x)$, you will get back to x.

Let $f(x)=e^x$. Define $\ln(x)$ to be $f^{-1}(x)$.

Which of the following is false?

(A) If $a=e^b$ and $c=e^d$ then $\ln(a/c) = b-d$.

(B) If $a=e^b$ and $c=e^d$ then $\ln(a-c) = b/d$.

(C) If $c=a^d$ then $\ln(c) = d\ \ln(a)$.

(D) If $a=e^b$ and $c=a^d$ then $\ln(c) = bd$.

(E) If $a=e^b$ and $c=e^d$ then $\ln(ac) = b+d$.

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$\ln(x)$ = natural logarithm of x = $\log_e(x)$)