Lecture 18 (Oct. 18, 2013)

Learning Goals: (1) Implicit Differentiation

(2) Related Rates: application of chain rule & Implicit differentiation

- Implicit Differentiation:

① Treat y as a function of x. Take the derivative w.r.t. x on both sides of the equation.

② Solve for $\frac{dy}{dx}$

It can be used to find derivatives of some complicated functions

Example 1: Find the derivative of $y=\sqrt{x+\sqrt{x}}$

The function is the same as $y^2 = x+\sqrt{x}$

Take derivatives w.r.t. x on both sides of the equation

$$2y \cdot \frac{dy}{dx} = 1 + \frac{1}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}\left(1+\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right) = \frac{1}{4\sqrt{x^2+x\sqrt{x}}}$$

Example 2: Find the point on the curve $x^3+y^3-9xy=0$ that has horizontal tangent line other than $(0,0)$

(In other words, we want to find (x_0, y_0) on the curve that satisfies $\left.\frac{dy}{dx}\right|_{x=x_0} = 0$)

Use implicit differentiation to find $\frac{dy}{dx}$, we have

$$\underset{\substack{\uparrow \\ \text{chain rule}}}{3x^2 + 3y^2 \cdot \frac{dy}{dx}} - \underset{\substack{\uparrow \\ \text{chain rule \& product rule}}}{\left(9y + 9x \cdot \frac{dy}{dx}\right)} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{9y-3x^2}{3y^2-9x} = 0$$

$$\Rightarrow 9y-3x^2=0 \quad \Rightarrow \begin{cases} y = \frac{1}{3}x^2 \\ x^3+y^3-9xy=0 \leftarrow \text{point on the curve} \end{cases}$$

Solve for (x,y) from two equations: $x^3 + \frac{1}{27}x^6 - 9x \cdot \frac{1}{3}x^2 = 0 \Rightarrow \frac{1}{27}x^3(x^3-54)=0$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{ignore}}}{x=0} \quad \text{or} \quad x=\sqrt[3]{54} = 3\sqrt[3]{2}$$

$\searrow$ give $y = \frac{1}{3}x^2 = \frac{1}{3}(54)^{2/3} = 2^{2/3}$

Example 3: Power rule for fractional powers

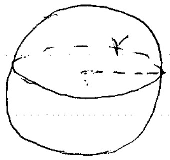
(Sec 7.5 in LKN) Given $y = x^{\frac{m}{n}}$, m, n - nonzero integers

Then $\frac{dy}{dx} = \frac{m}{n} \cdot x^{\frac{m}{n}-1}$

(Hint: The function is the same as $y^n = x^m$)

· Related Rates:

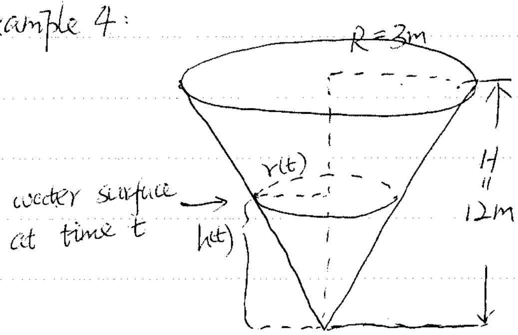
e.g.



A spherical tumor grows over time

Given the growth rate of the radius $\frac{dr}{dt}$ and relation between volume and radius $V = \frac{4}{3}\pi r^3$, we can find $\frac{dV}{dt}$ by $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

Example 4:



Water is flowing into a conical reservoir at a rate of $4\ m^3/min$. How fast is the water surface raising over time?

(In other words, what is $\frac{dh}{dt}$?)

#1: $\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$ with $\frac{dV}{dt} = 4\ (m^3/min)$

To find $\frac{dh}{dV}$, we have $V = \frac{1}{3}\pi r^2 h$ with $r(t)$, $h(t)$ are functions of time

Then by similar triangles, $\frac{R}{H} = \frac{r(t)}{h(t)} \Rightarrow r(t) = \frac{1}{4}h(t) \Rightarrow V = \frac{1}{3}\pi \cdot \frac{1}{16} h^3(t)$

$\Rightarrow h(t) = \sqrt[3]{\frac{48V}{\pi}} \qquad \Rightarrow \frac{dh}{dV} = \sqrt[3]{\frac{48}{\pi}} \cdot \frac{1}{3} \cdot V^{-2/3}$

$\Rightarrow \frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \sqrt[3]{\frac{48}{\pi}} \cdot \frac{1}{3} \cdot (4t)^{-2/3} \cdot 4 = 4\sqrt[3]{\frac{1}{9\pi t^2}}$

#2: From $V = \frac{1}{3}\pi r^2 h$ and $V = 4t$; $r(t) = \frac{1}{4}h(t)$

we have $4t = \frac{1}{48} \cdot \pi h^3(t)$

Use implicit differentiation $4 = \frac{\pi}{48} \cdot 3h^2(t) \cdot \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{64}{\pi h^2(t)} = \frac{64}{\pi\left(\sqrt[3]{\frac{4 \cdot 48t}{\pi}}\right)^2} = 4 \cdot \sqrt[3]{\frac{1}{9\pi t^2}}$