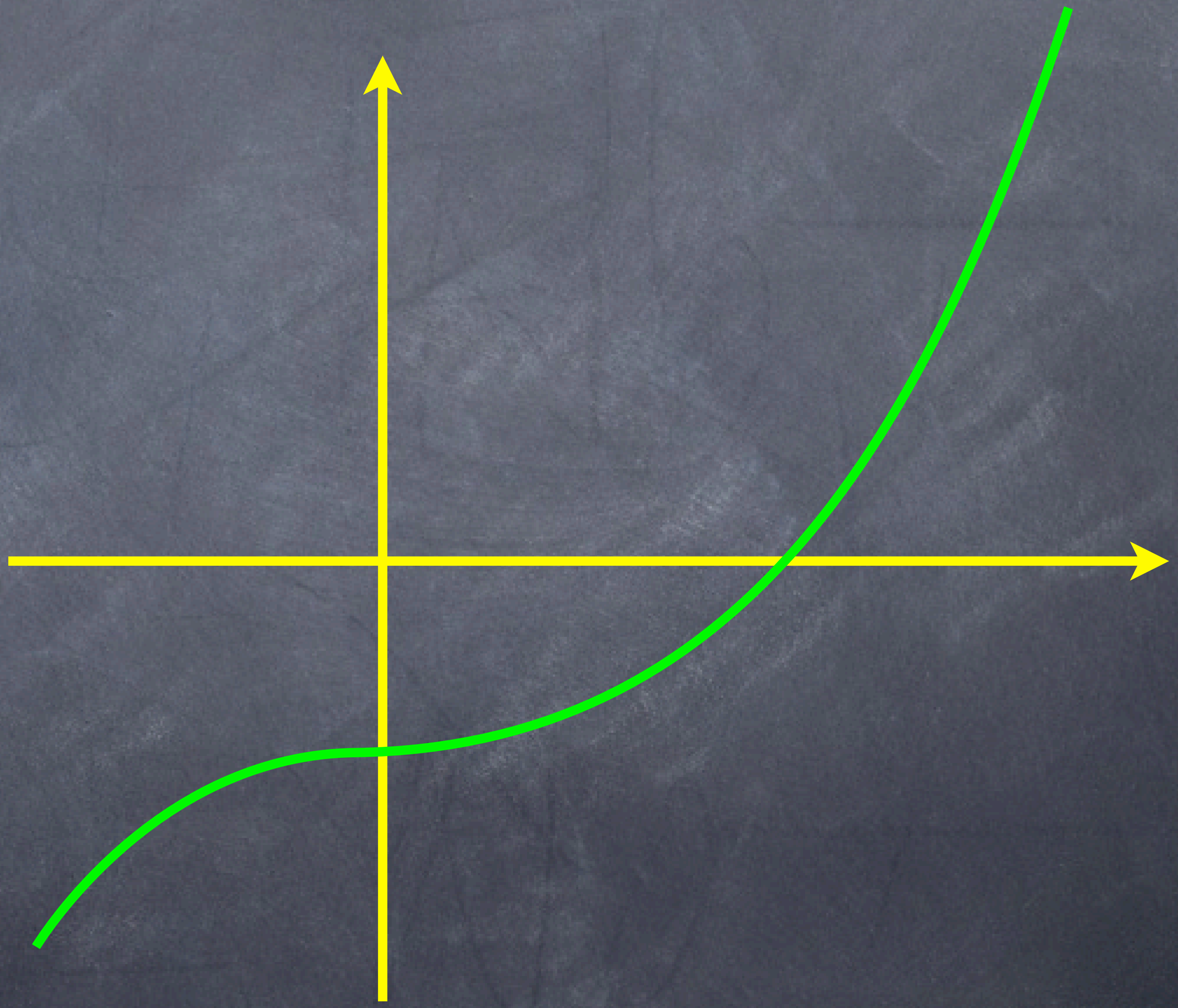
# Today

- Newton's method (cont)
- Qualitative analysis of differential equations
  - Steady states
  - Slope fields
  - Stability of steady states
  - Velocity (y') versus position (y)

# Newton's method

- NM is used to find zeros of a function: $f(x)=0$.
- It can be applied to finding approximates of
  - critical points of a function $g(x)$:
    - define $f(x)=g'(x)$,
  - intersections of functions, $g(x)=h(x)$:
    - define $f(x) = g(x)-h(x)$,
  - irrational numbers: e.g. sqrt(2):
    - define $f(x)=x^2-2$.

# Find the zero of $f(x)=x^3-2$.



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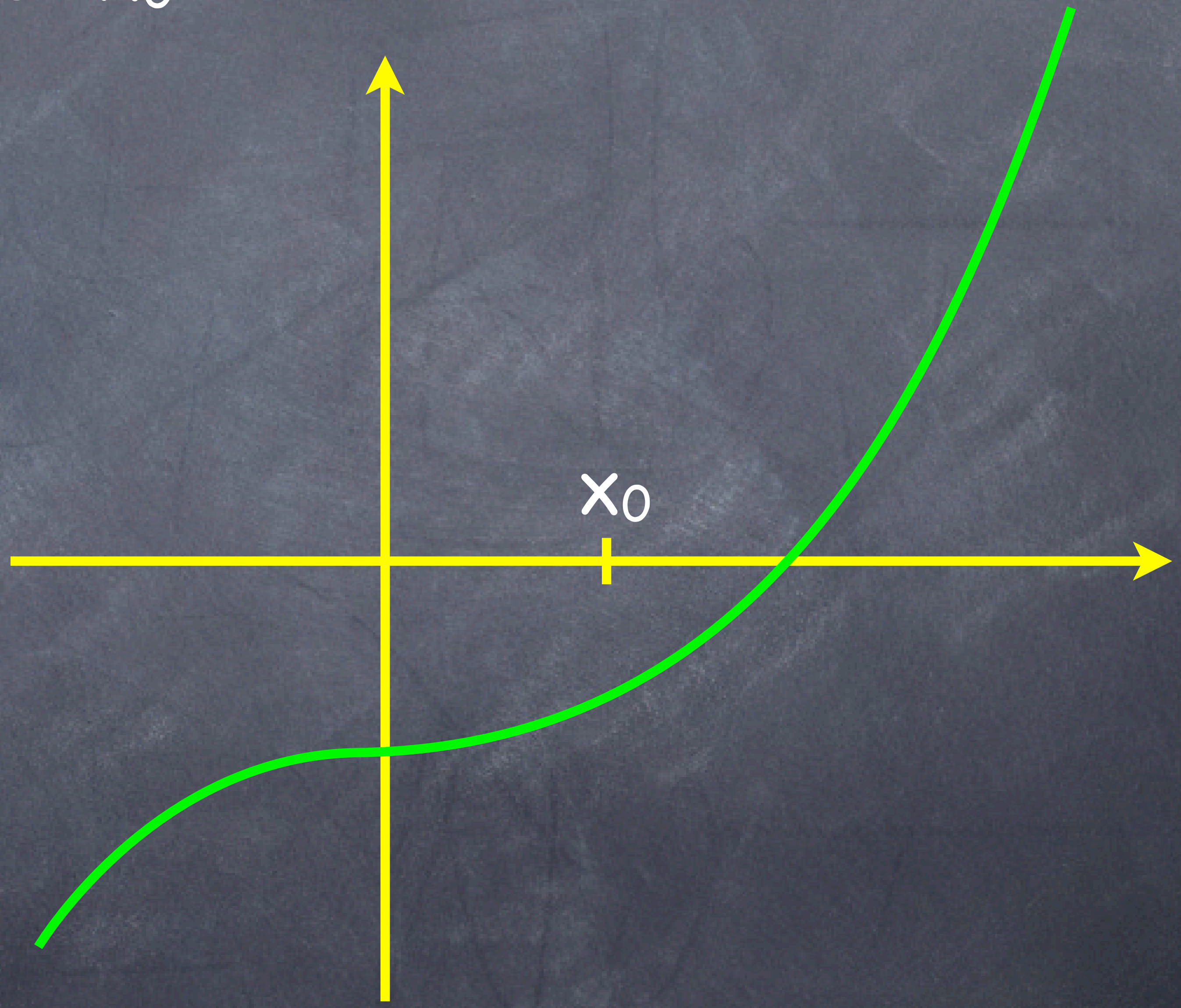

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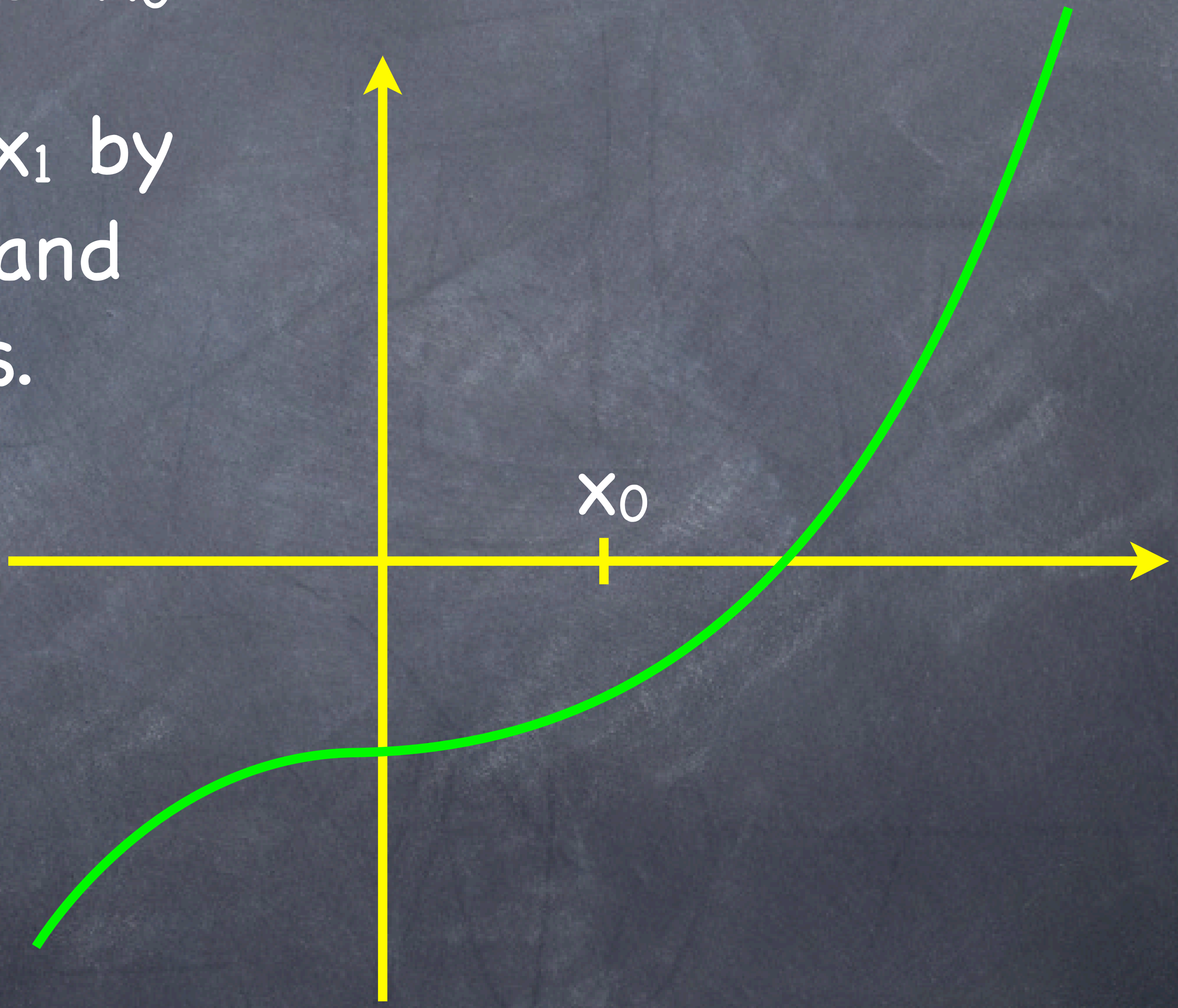
- Get a "better" estimate $x_1$ by finding the tangent line and following it to the x-axis.



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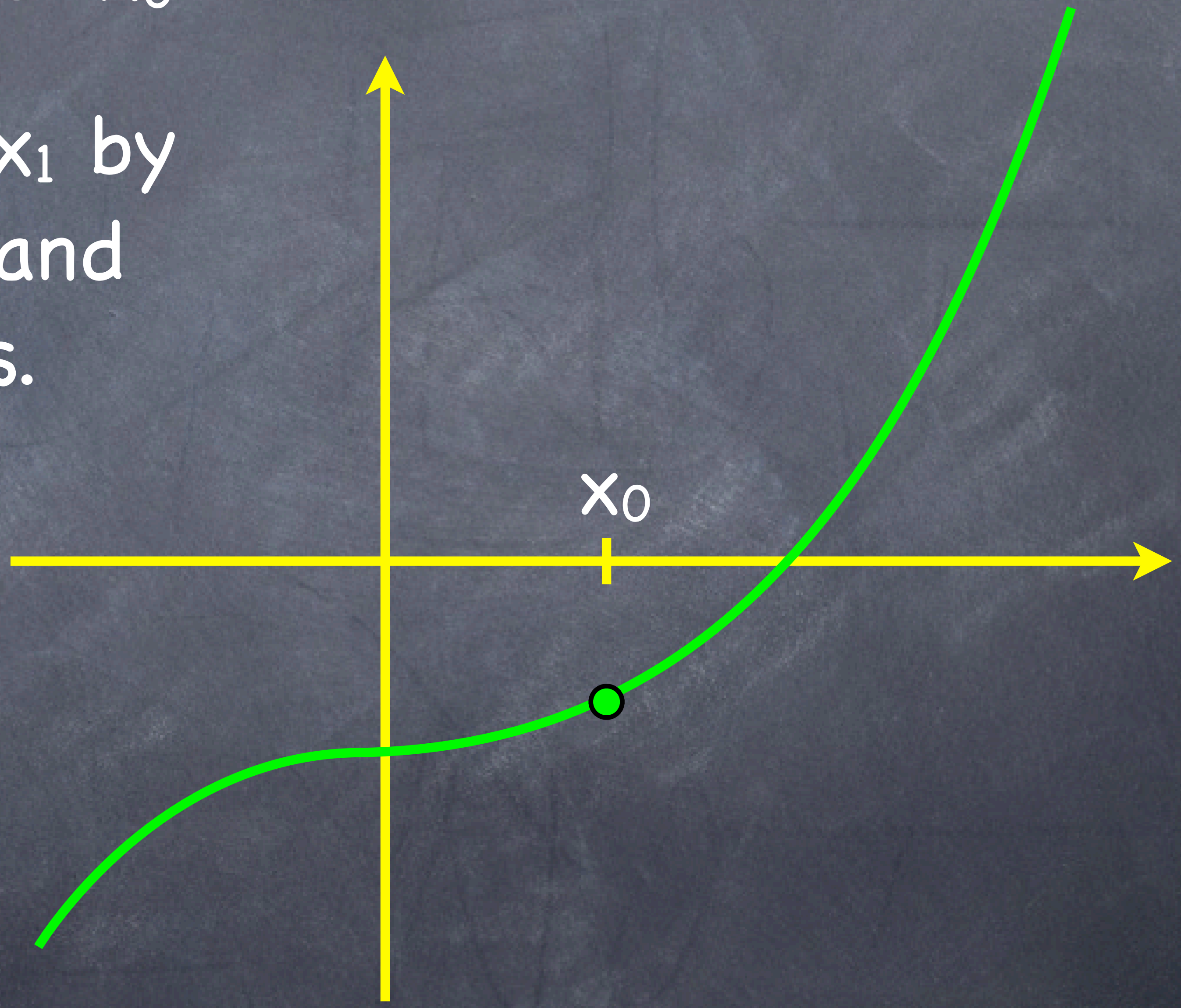

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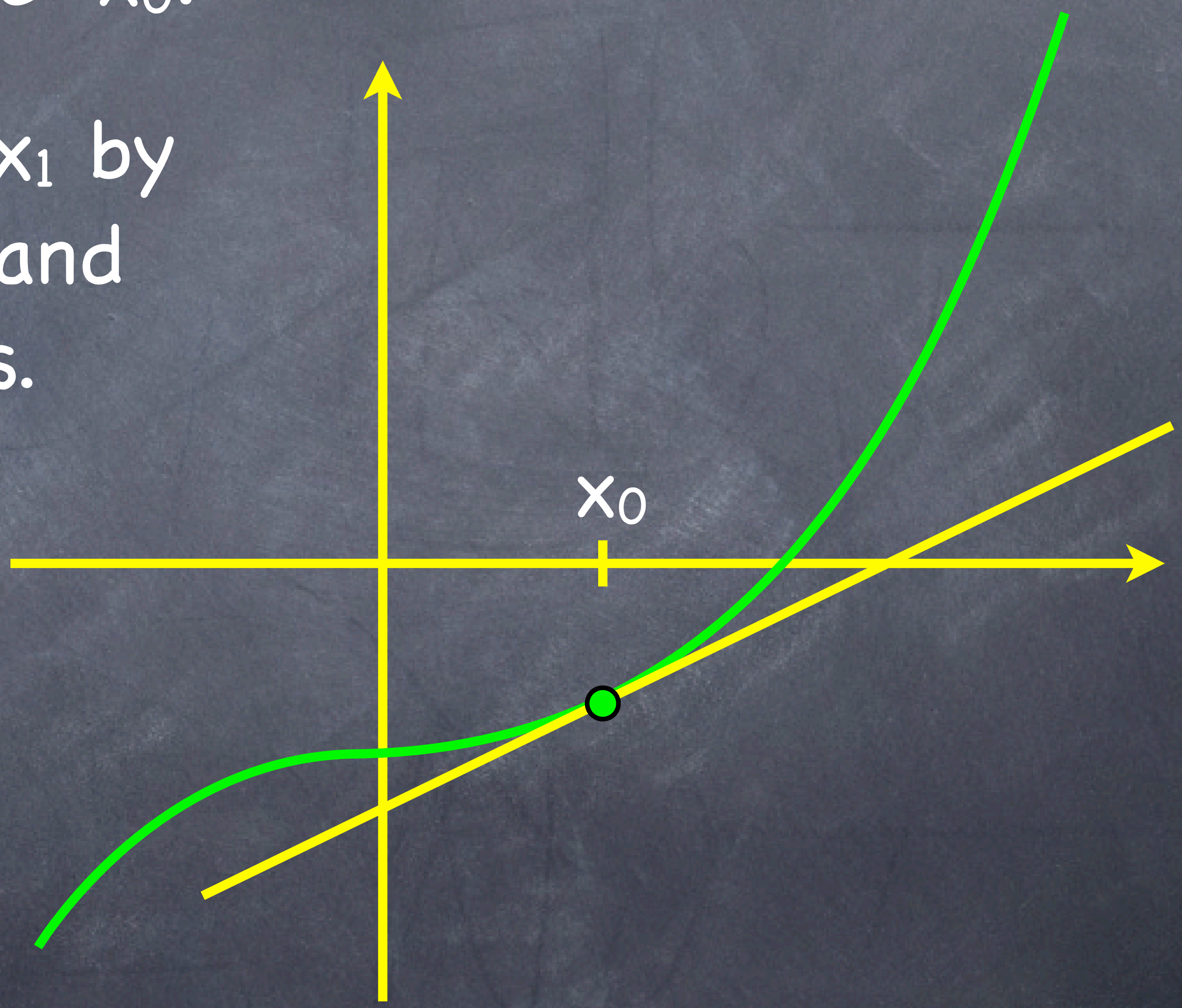

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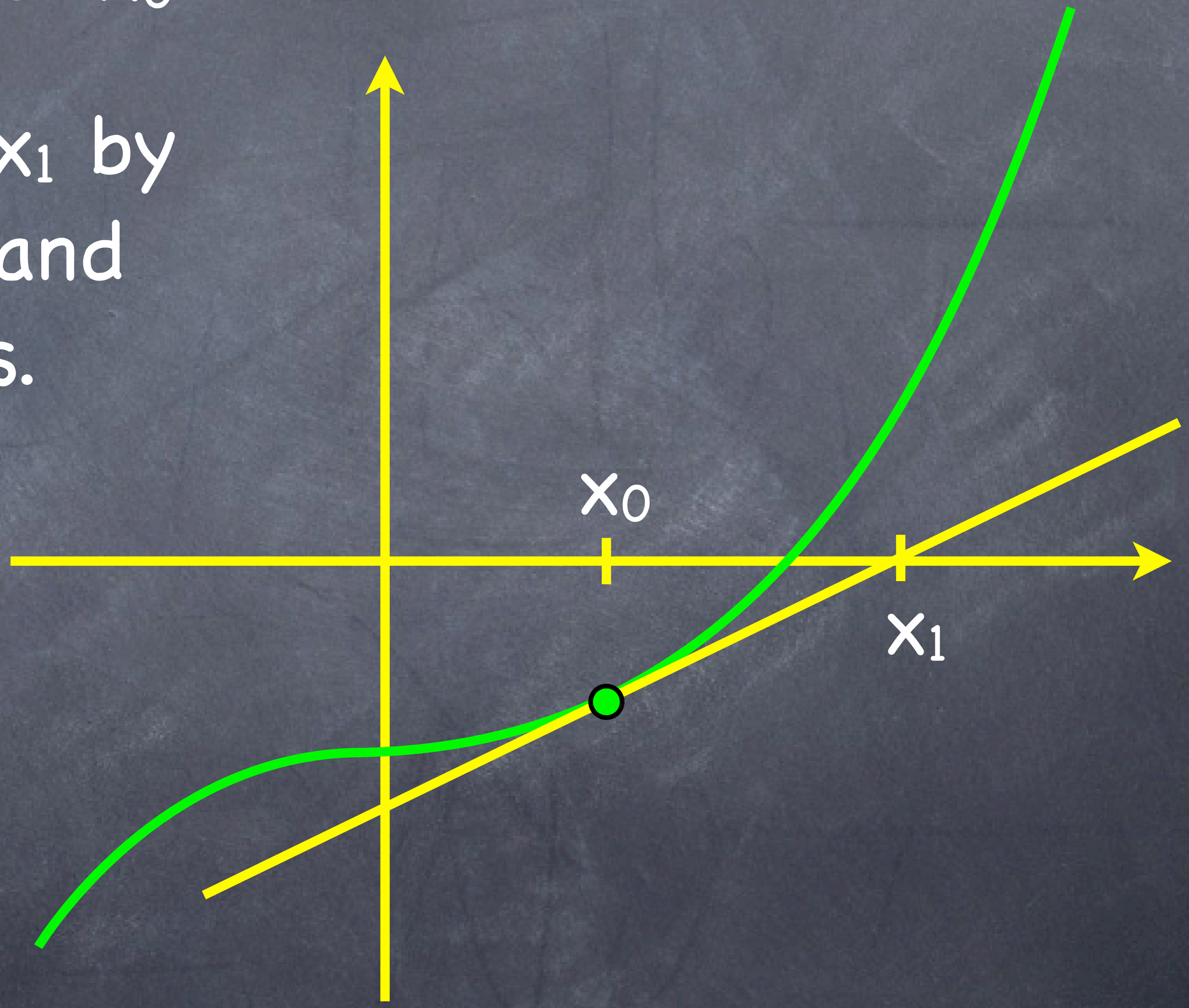

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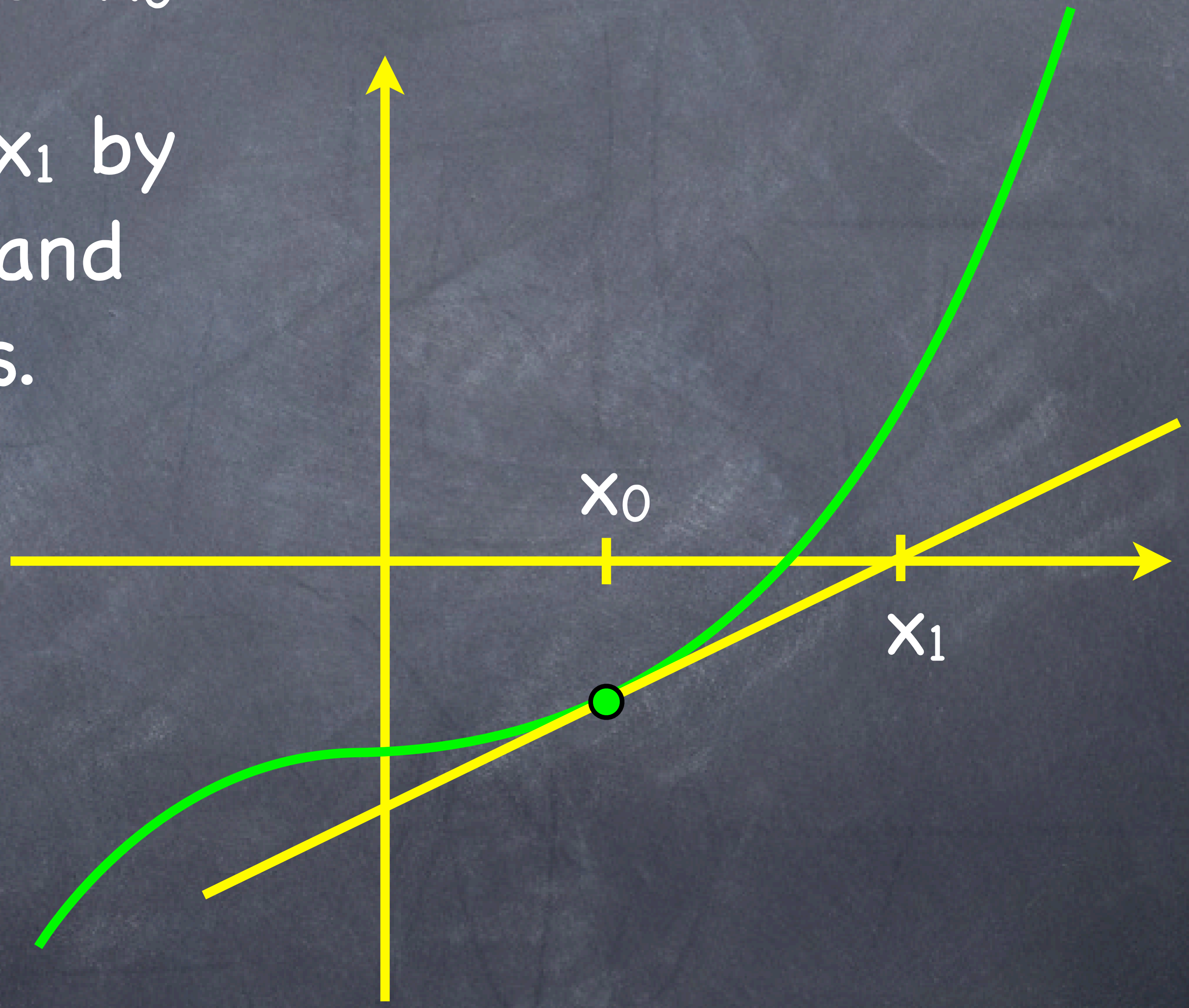

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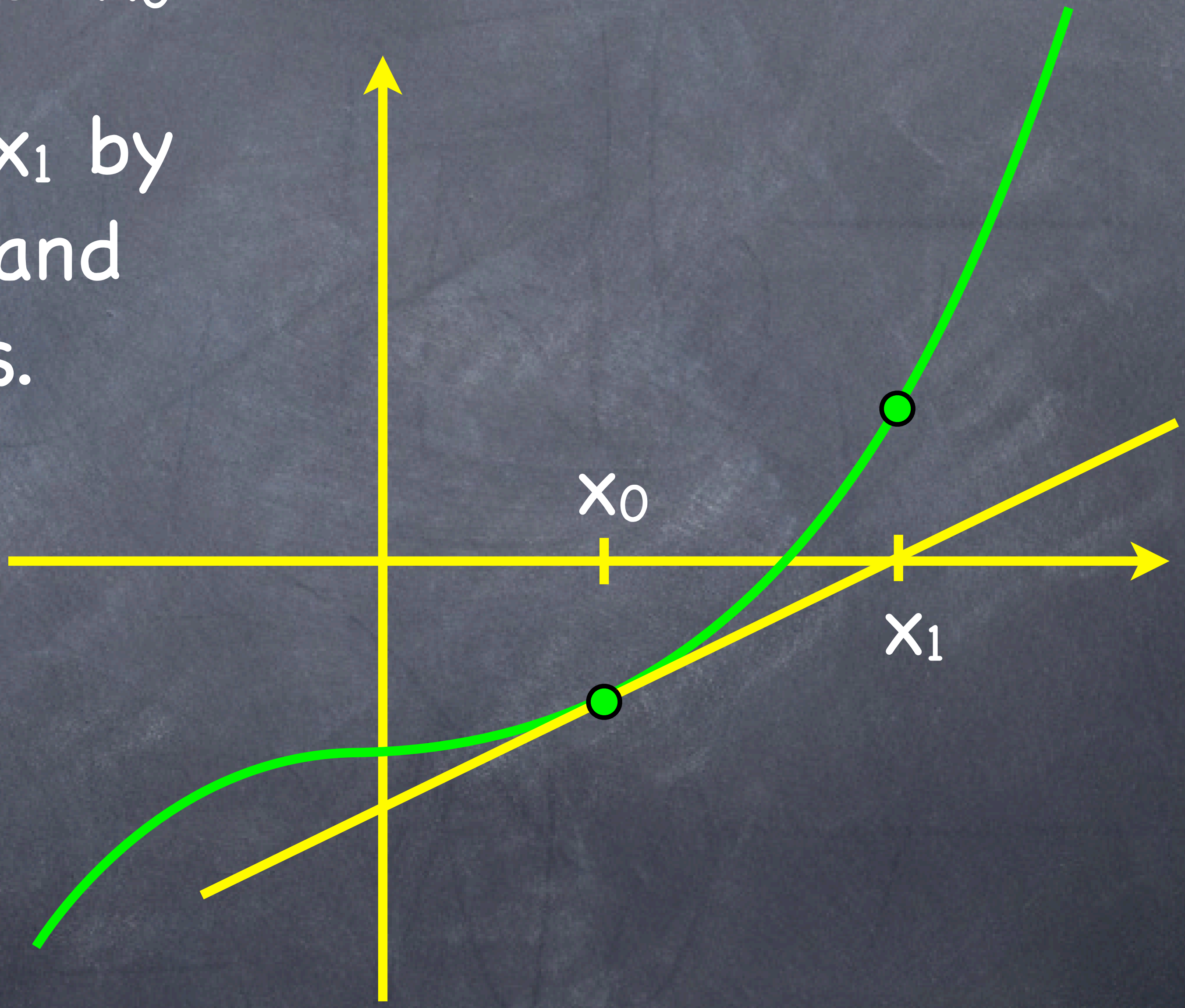

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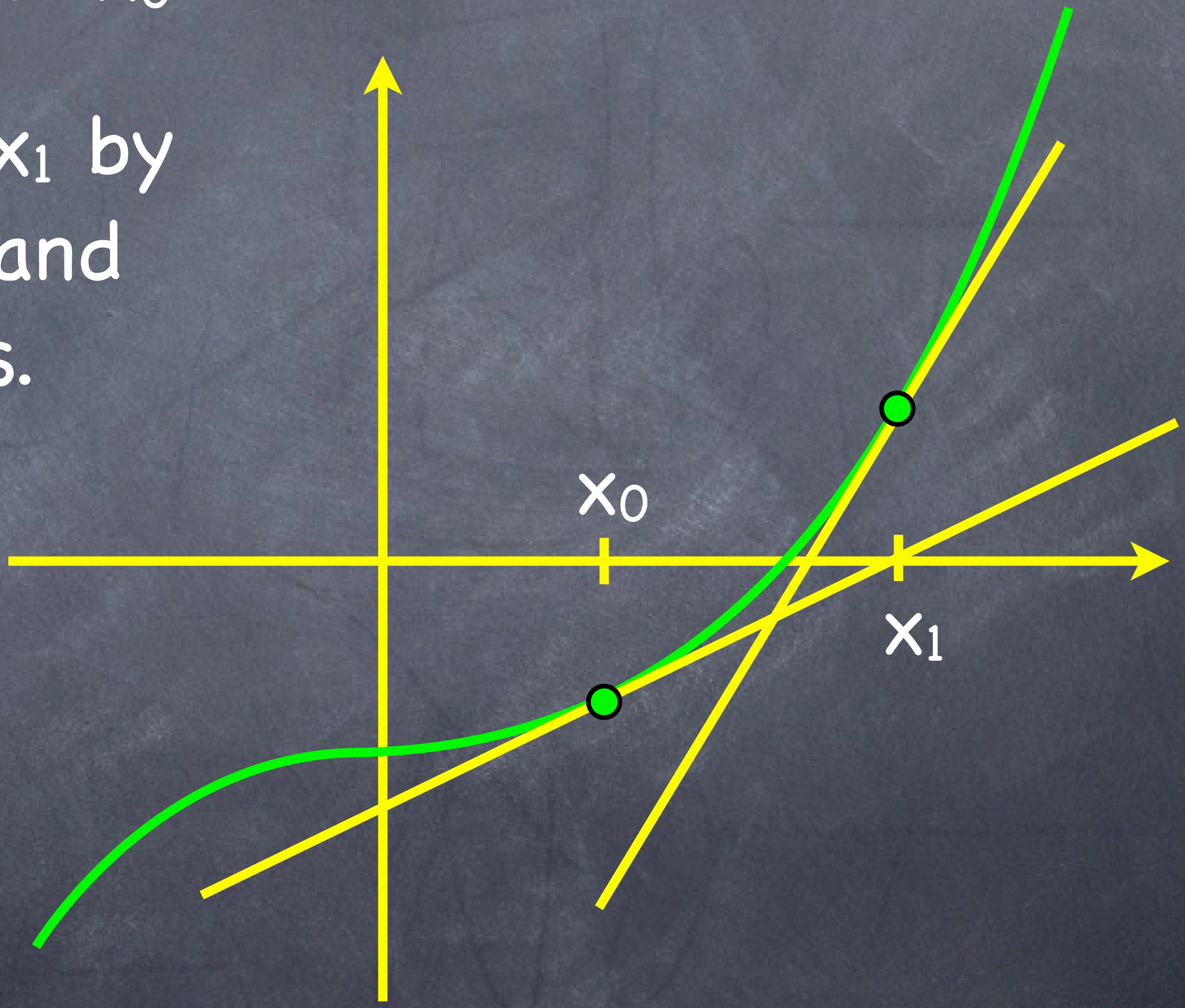

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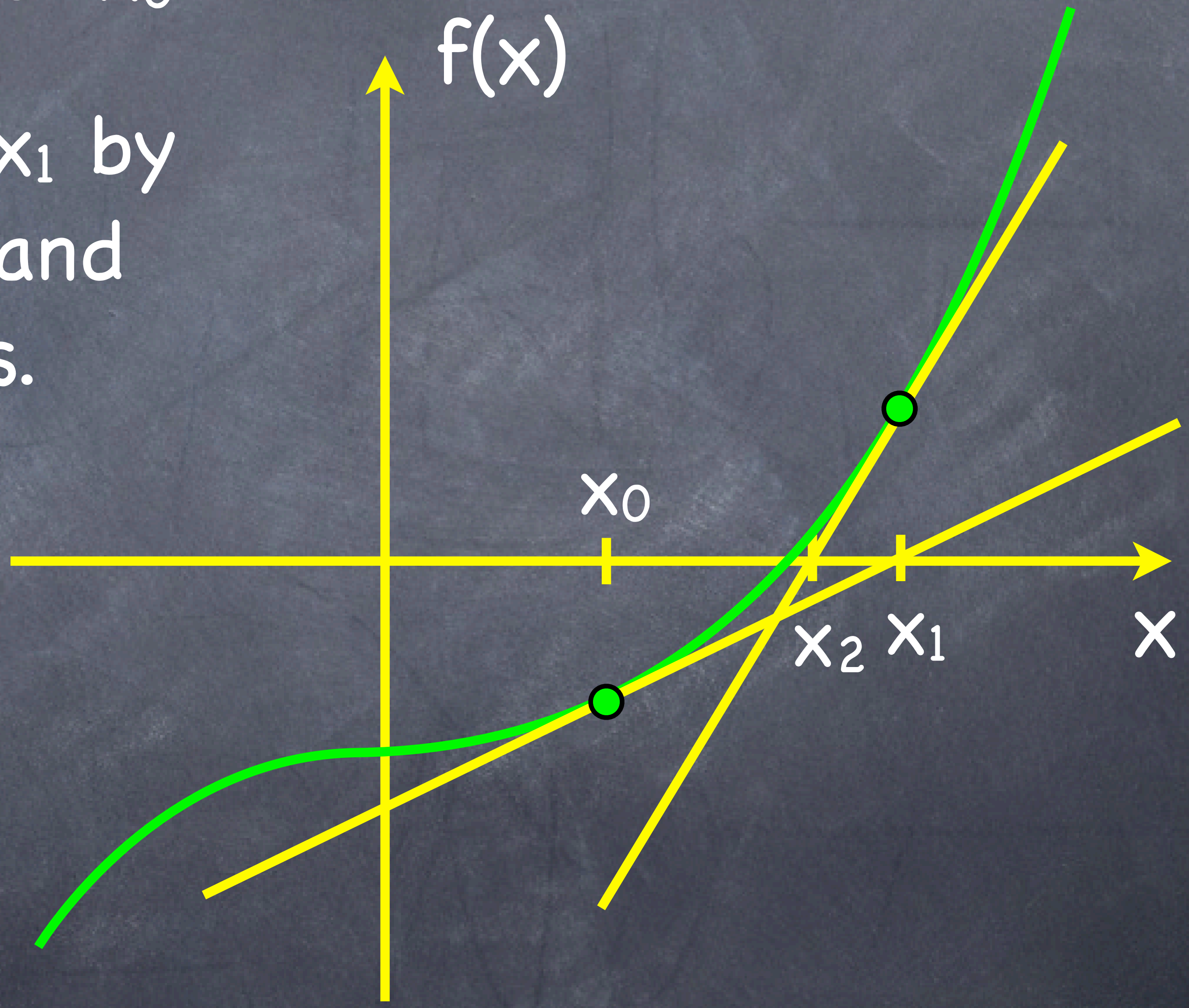
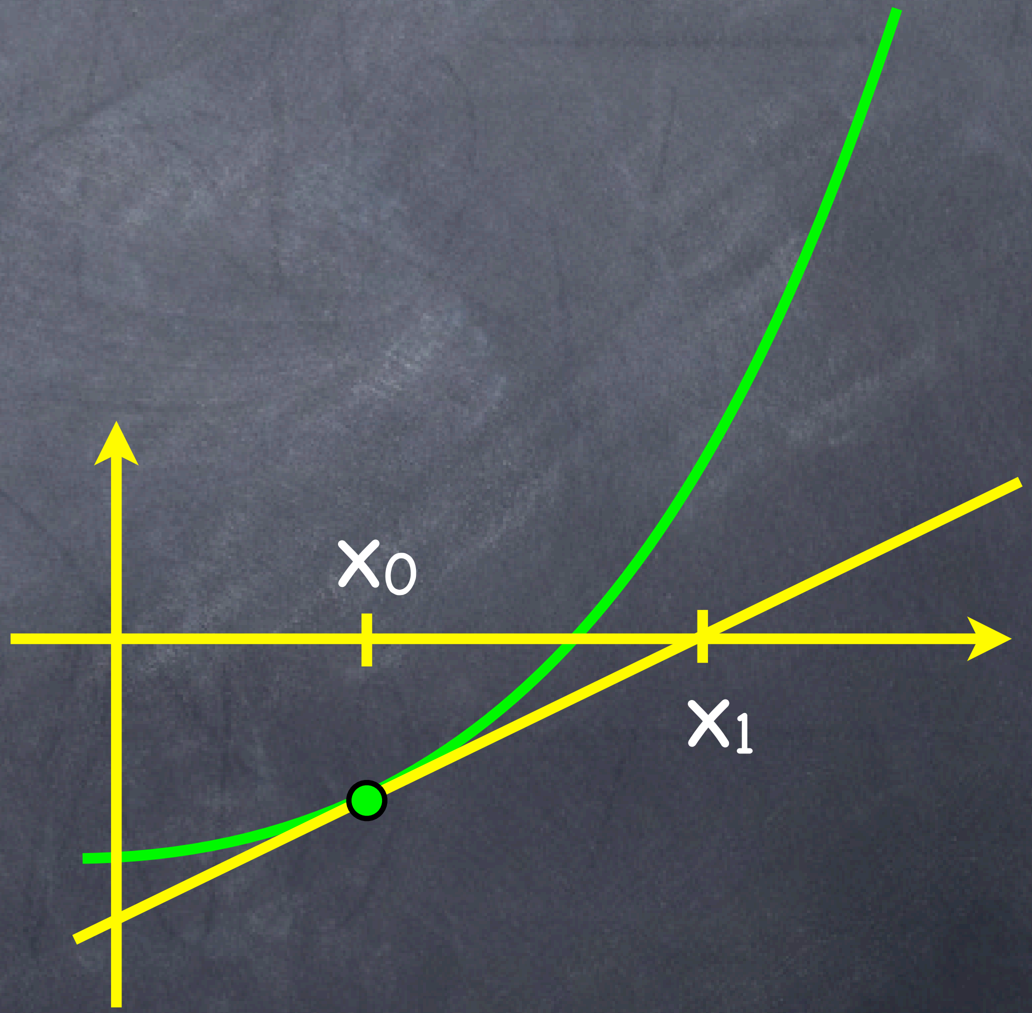


# Calculating successive estimates



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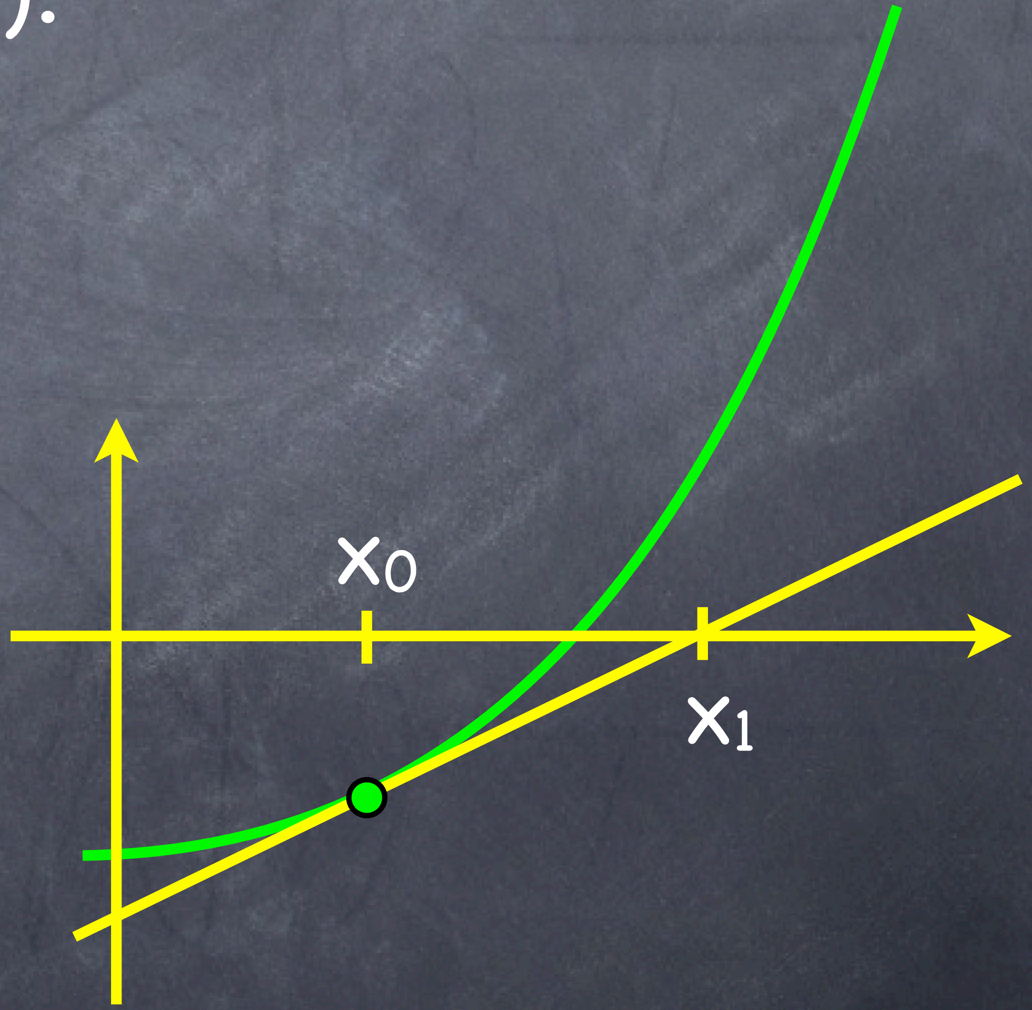

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(A) $x_{n+1} = x_n + f(x_n) / f'(x_n)$.

(B) $x_{n+1} = x_n - f(x_n) / f'(x_n)$.

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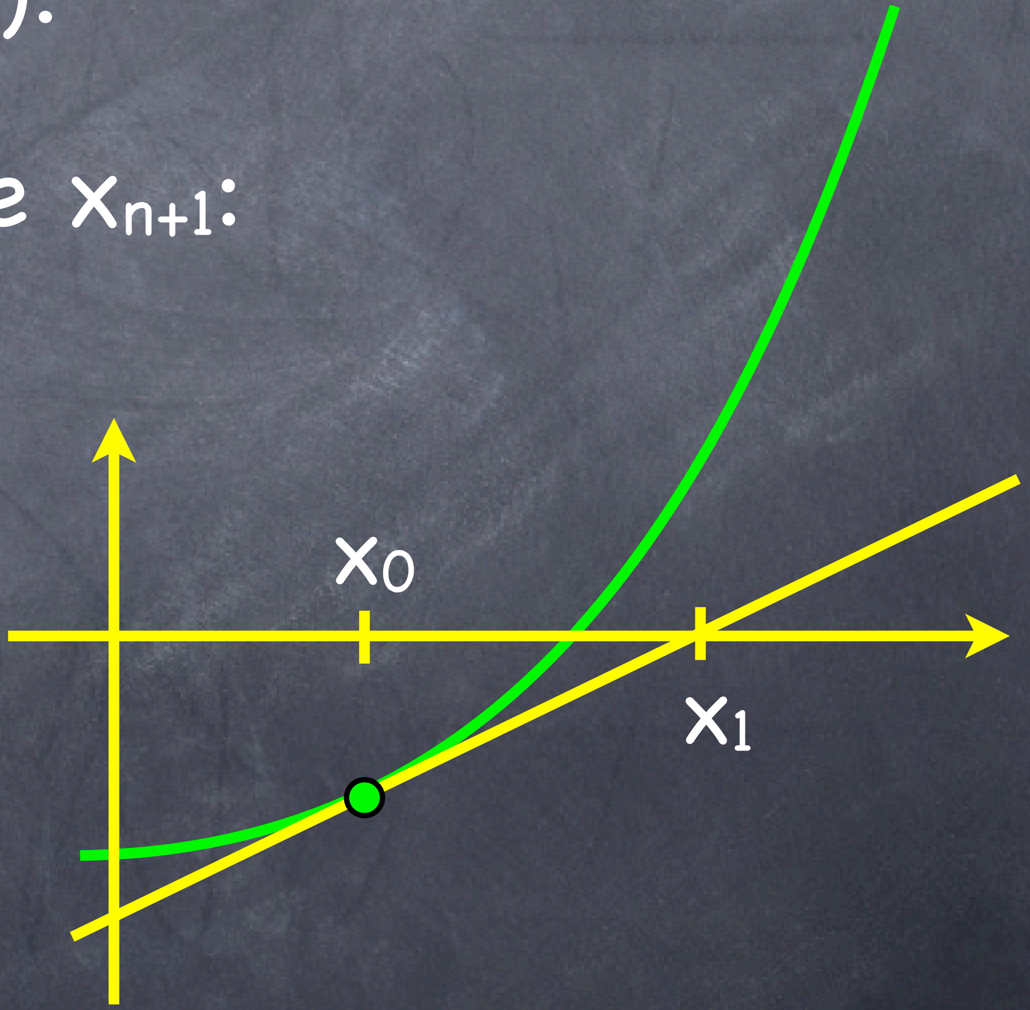

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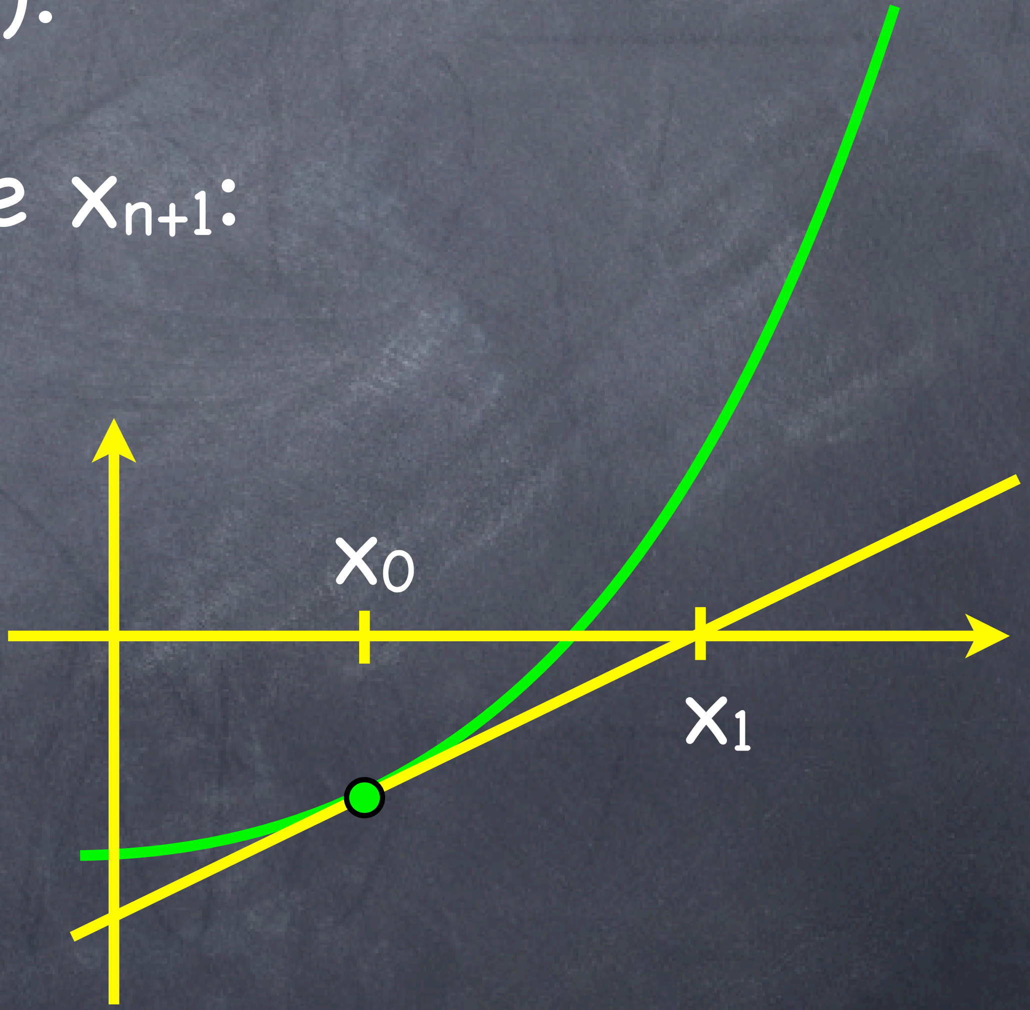

# To estimate $\sqrt{3}$, which function would you apply Newtons' method to?

(A) $f(x) = x^{1/2}$

(B) $f(x) = x^{1/2} - 3$

(C) $f(x) = x^2$

(D) $f(x) = x^2 - 3$

(E) $f(x) = (x-3)^{1/2}$

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(D) $f(x) = x^2 - 3$ <--- This one has a root at $\sqrt{3}$.

(E) $f(x) = (x-3)^{1/2}$

# Estimate $\sqrt{3}$ using Newton's method with initial guess $x_0=2$.

(A) 7/4

(B) 97/56

(C) 1.7

(D) 1.73205080757

$$x_{n+1} = x_n - f(x_n) / f'(x_n).$$

Finished already? Now use linear approximation. Which approach is better?

# Estimate $\sqrt{3}$ using Newton's method with initial guess $x_0=2$.

(A) 7/4 = 1.75 <--- $x_1$

(B) 97/56 = 1.73214 <--- $x_2$

(C) 1.7

(D) 1.73205080757 <--- first 11 digits of $\sqrt{3}$ .

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- Qualitative analysis - extract information about the general solution without solving.
    - Steady states
    - Slope fields
    - Stability of steady states
    - Plotting y' versus y (phase line)

$$\underset{\underset{\text{velocity}}{\uparrow}}{x'} = x(1-\underset{\underset{\text{position}}{\uparrow}}{x})$$

- **Steady state.** Where can you stand so that the DE tells you not to move?

(A) x=-1

(B) x=0

(C) x=1/2

(D) x=1

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$$x' = x(1-x)$$

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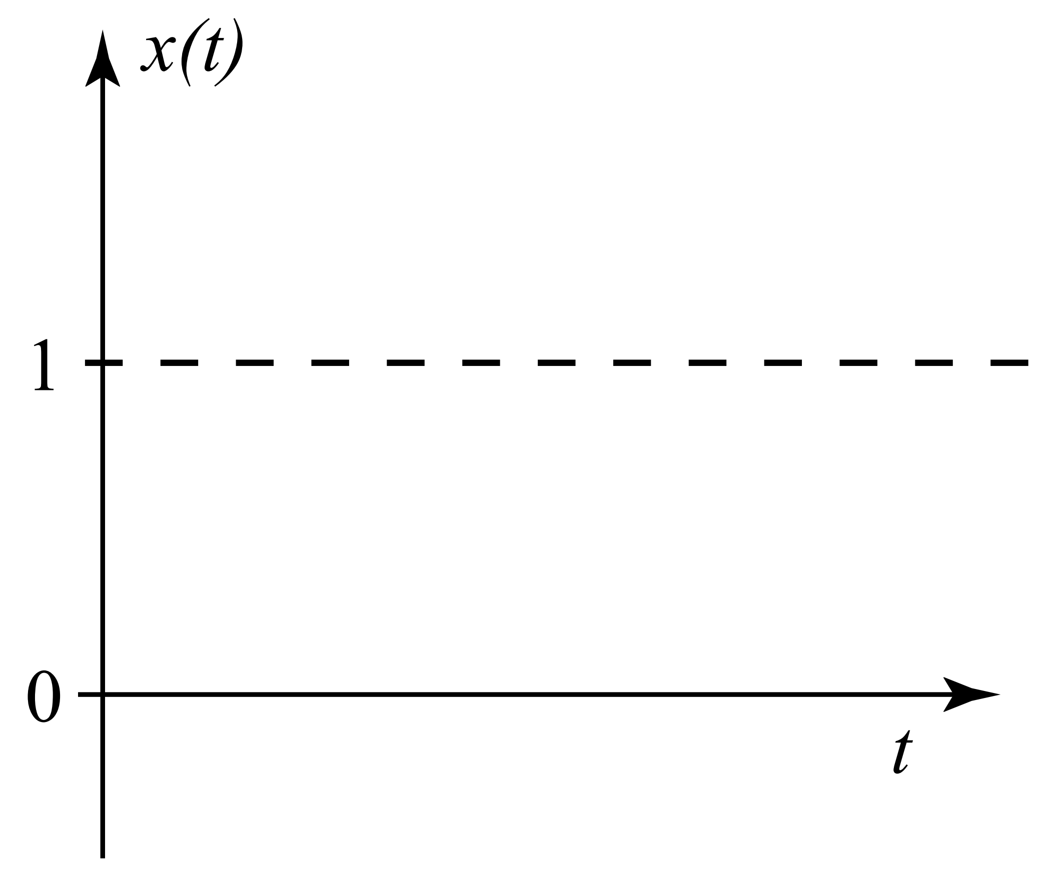

A steady state is a constant solution.

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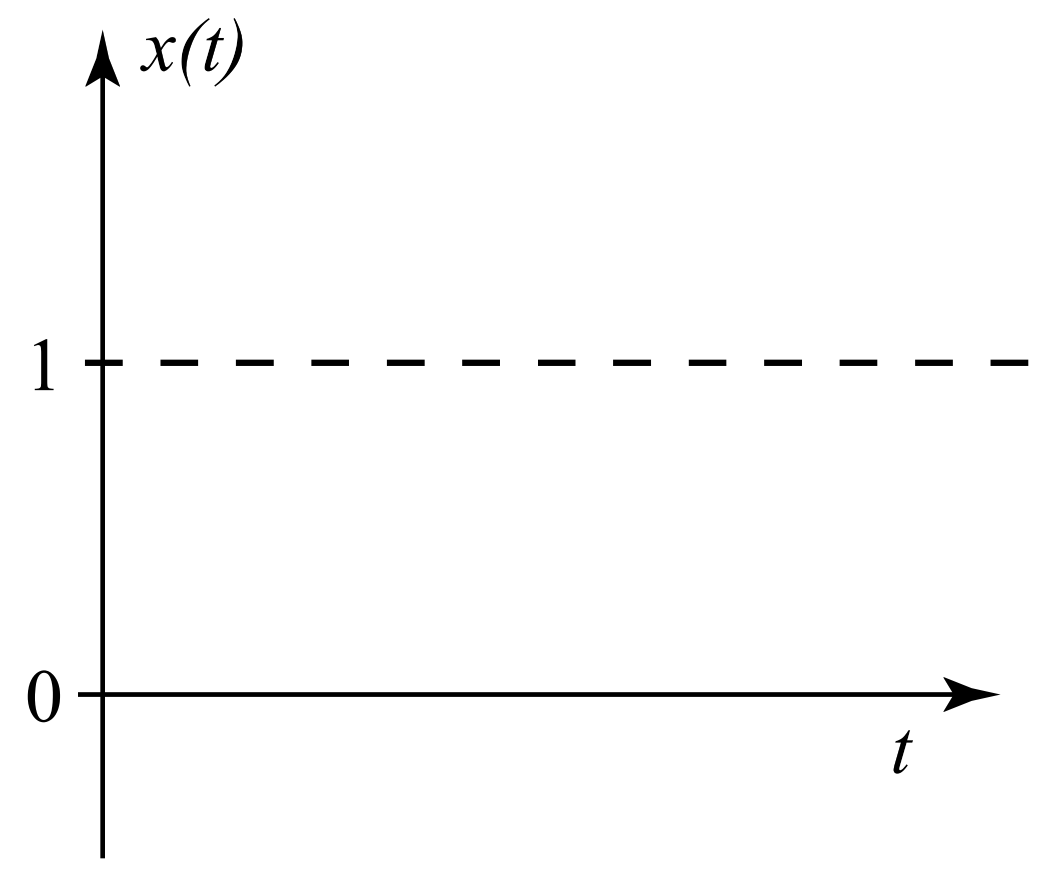

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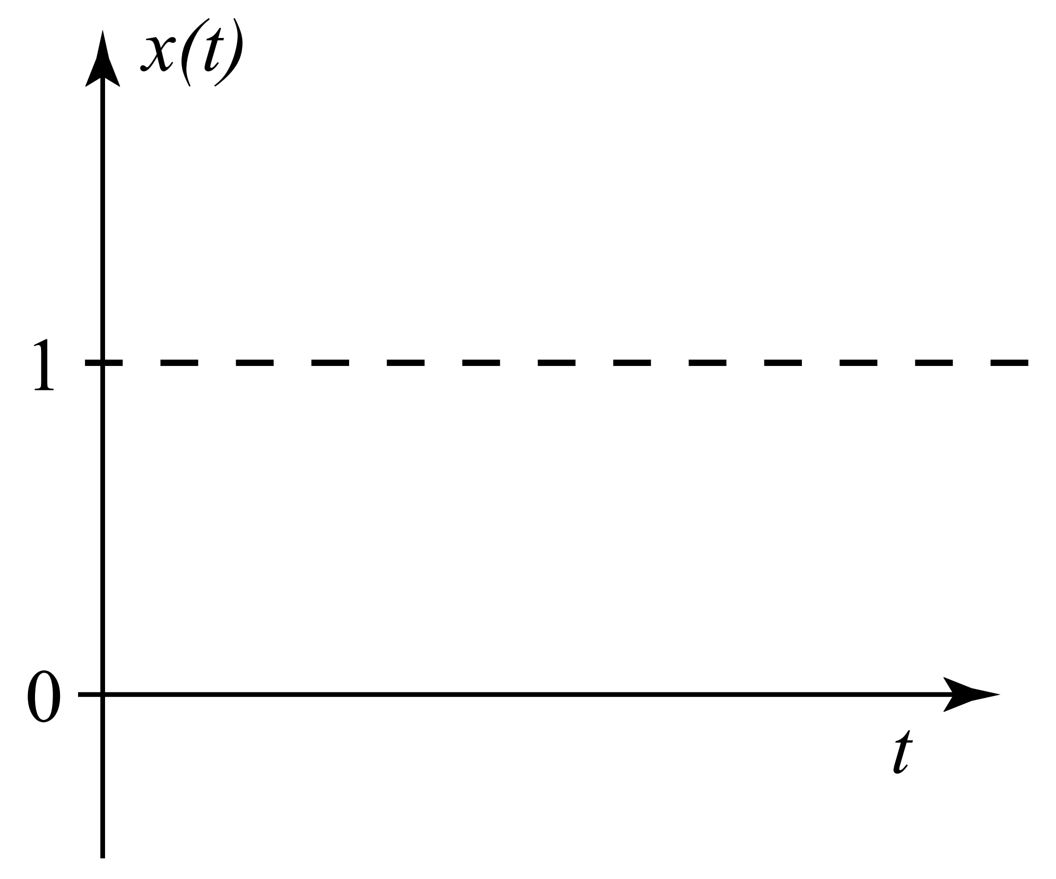

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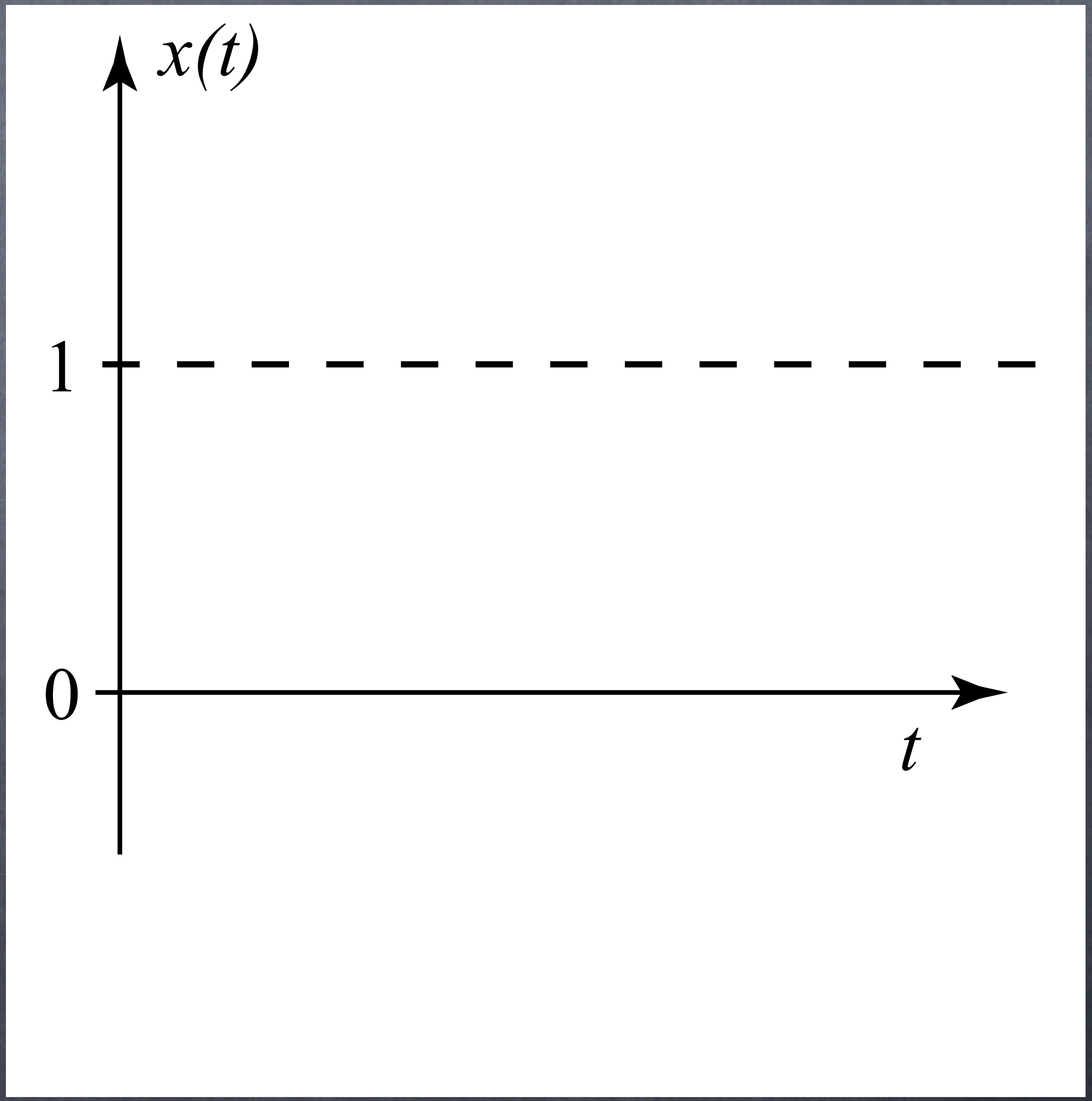

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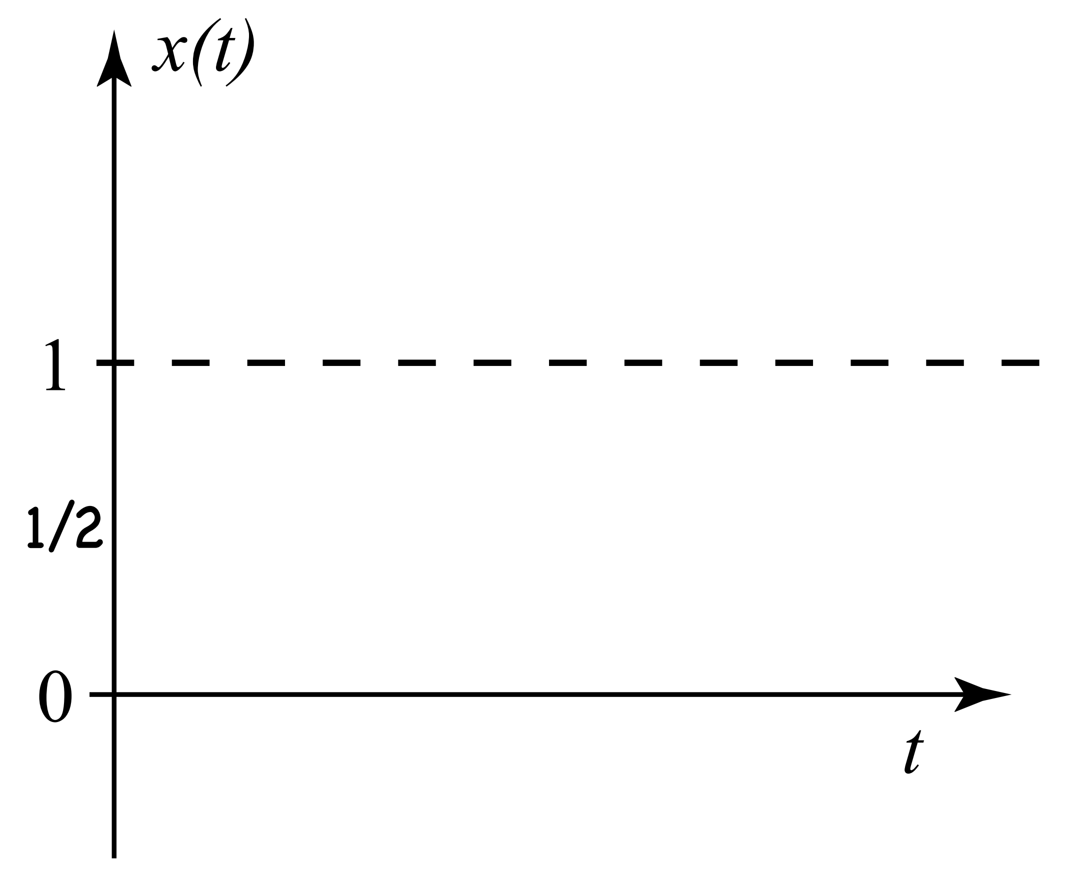

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  - When x(t)=1/2 what is x'?

    (A) 0

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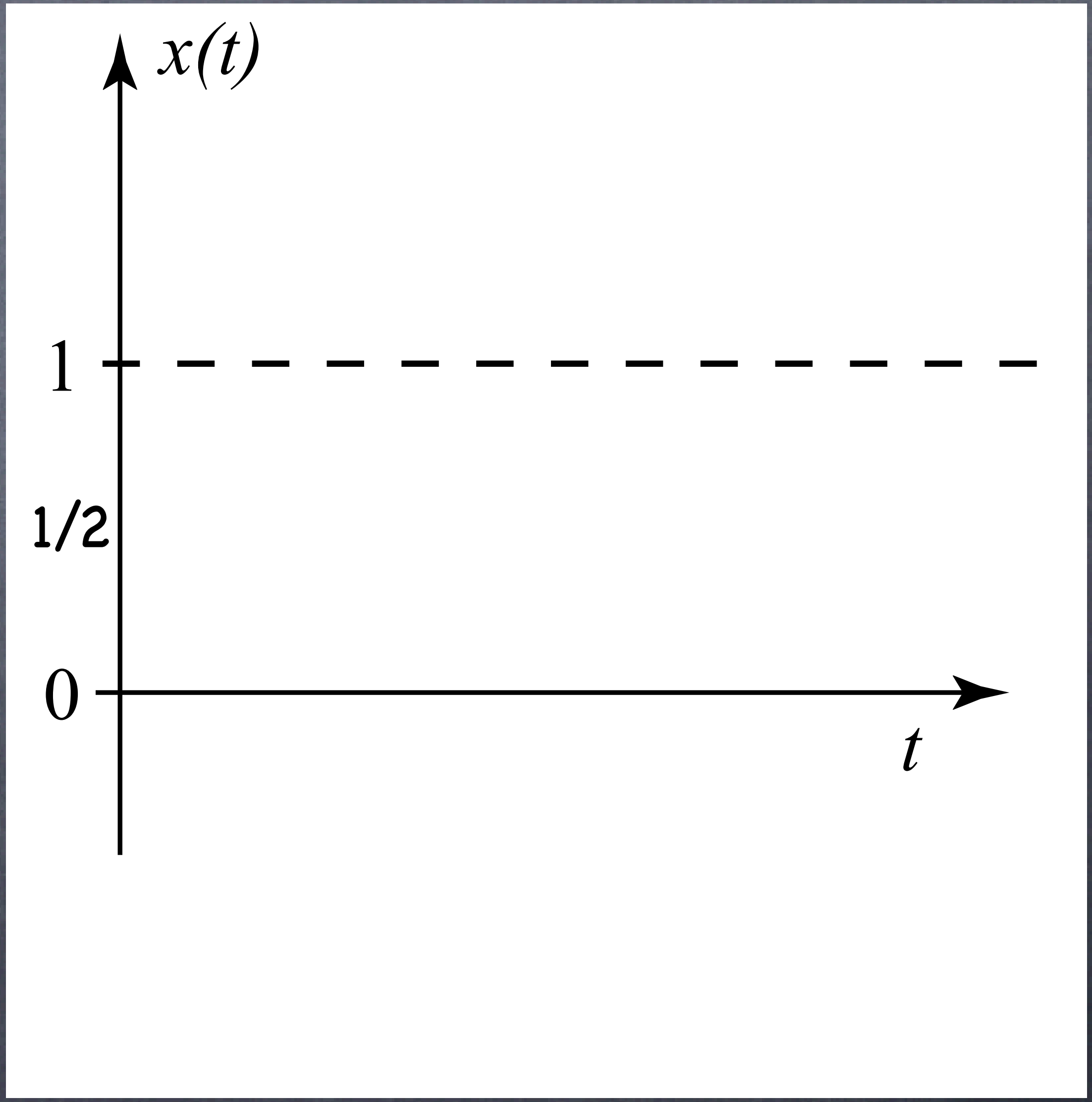

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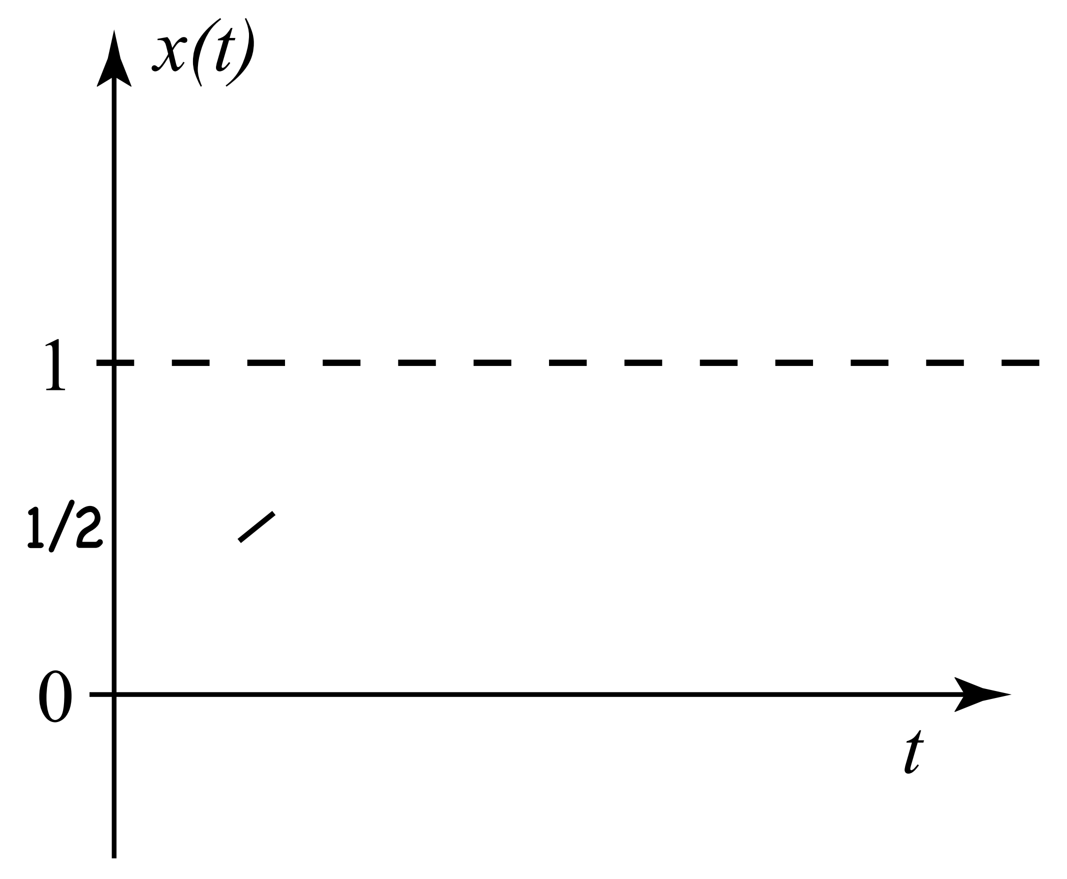

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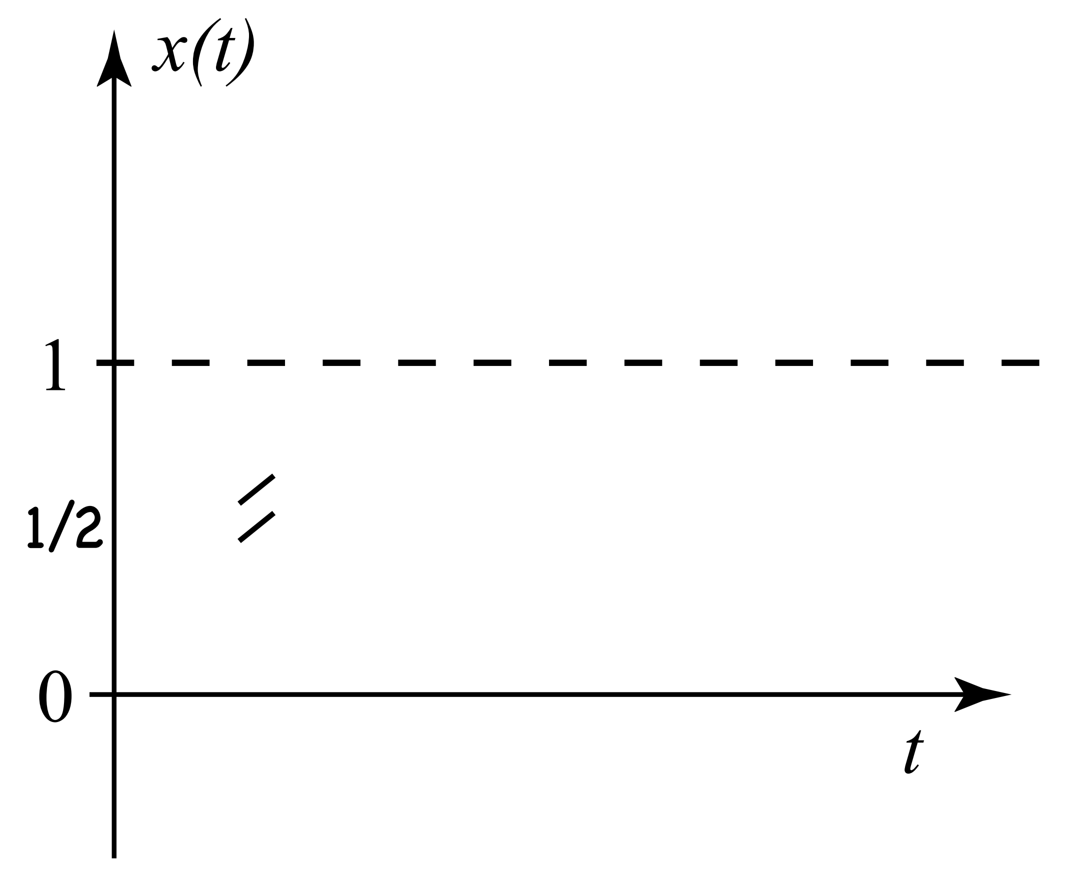

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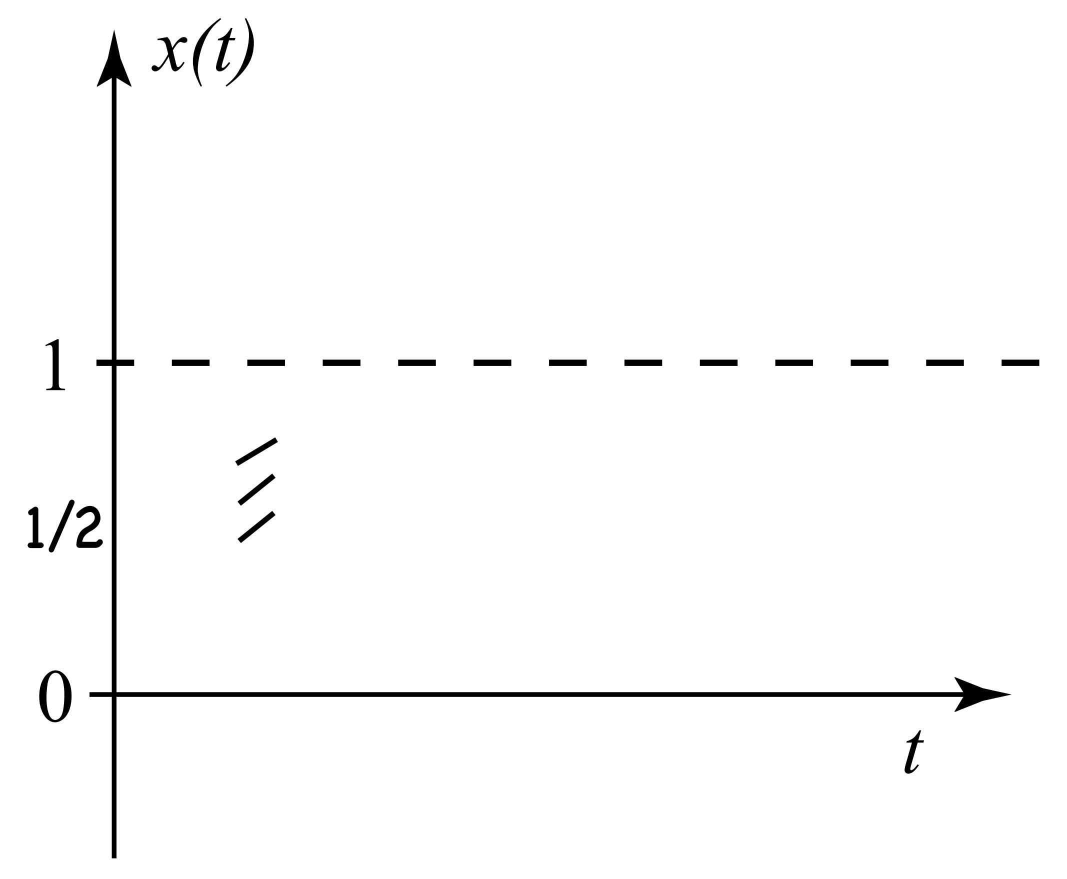

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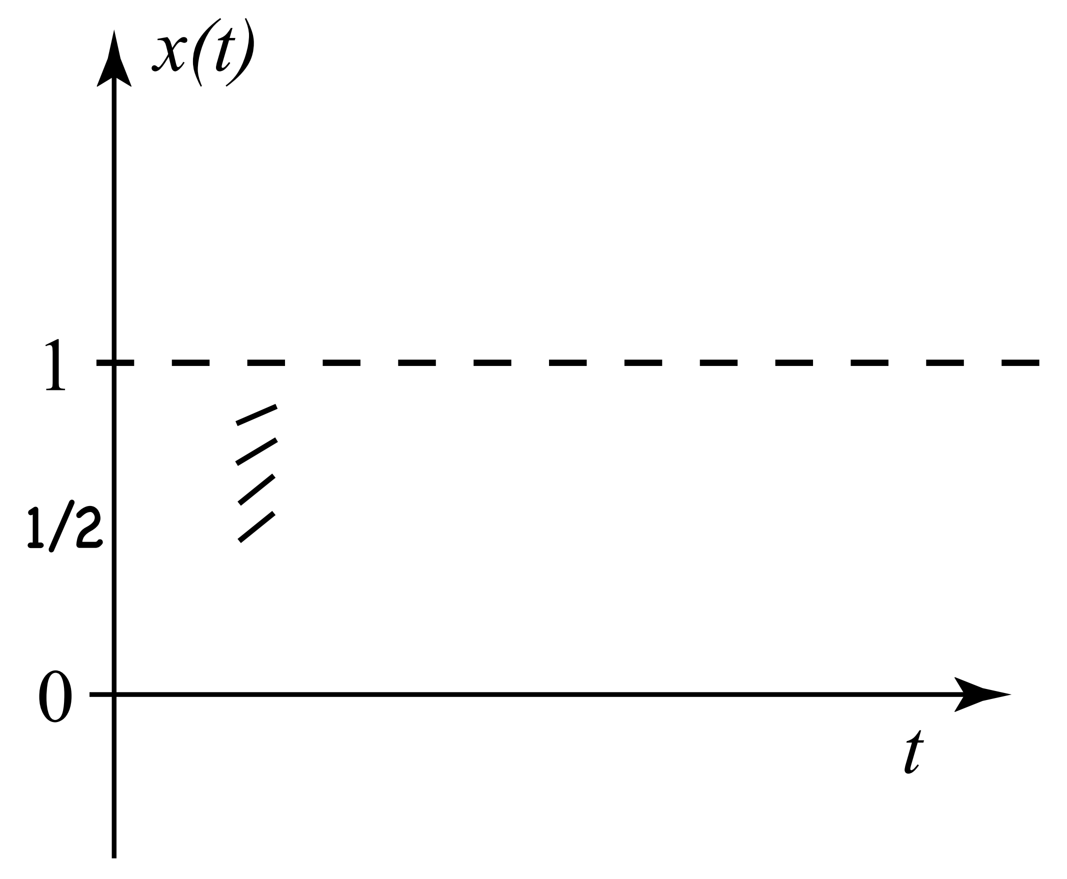

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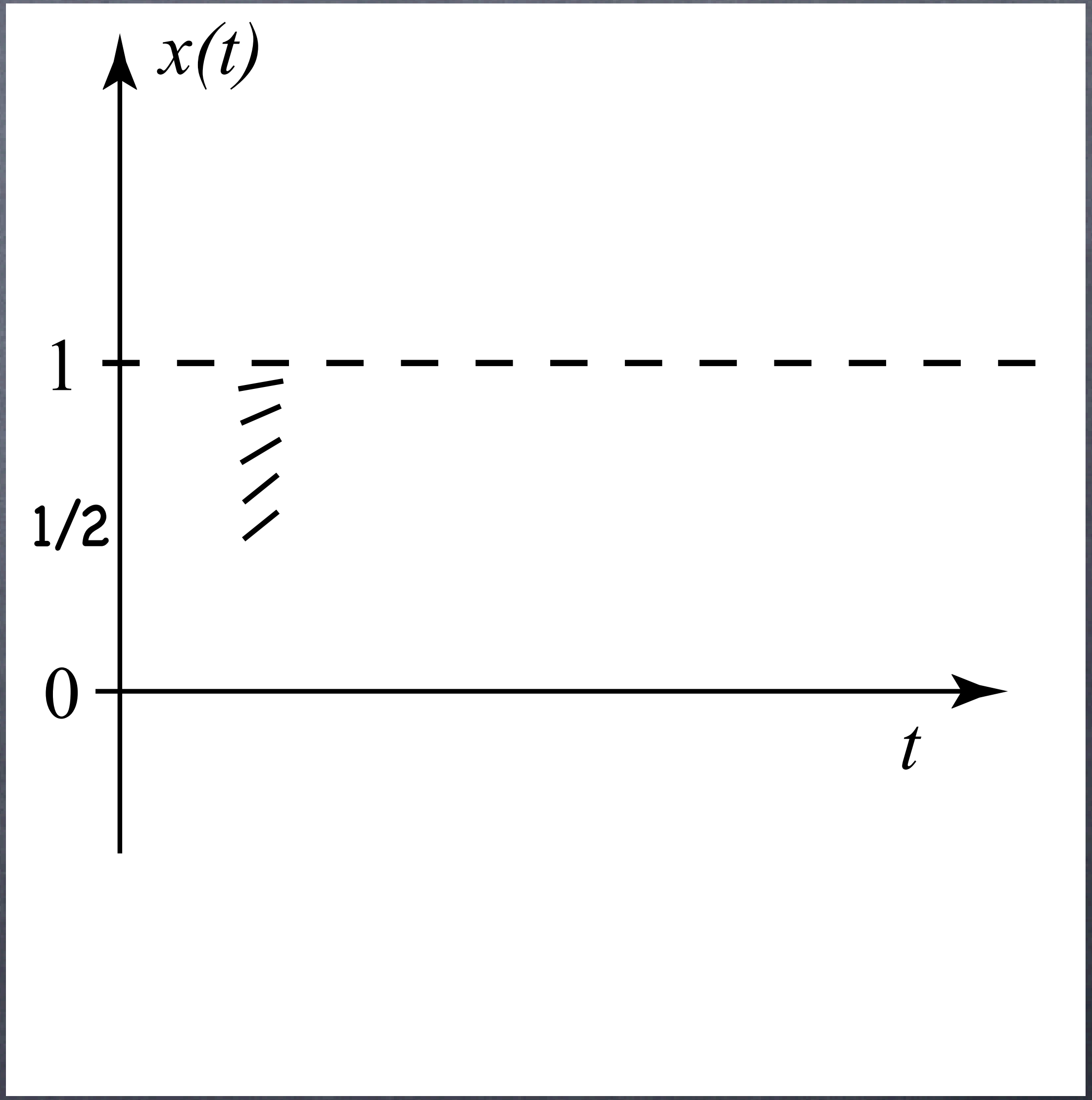

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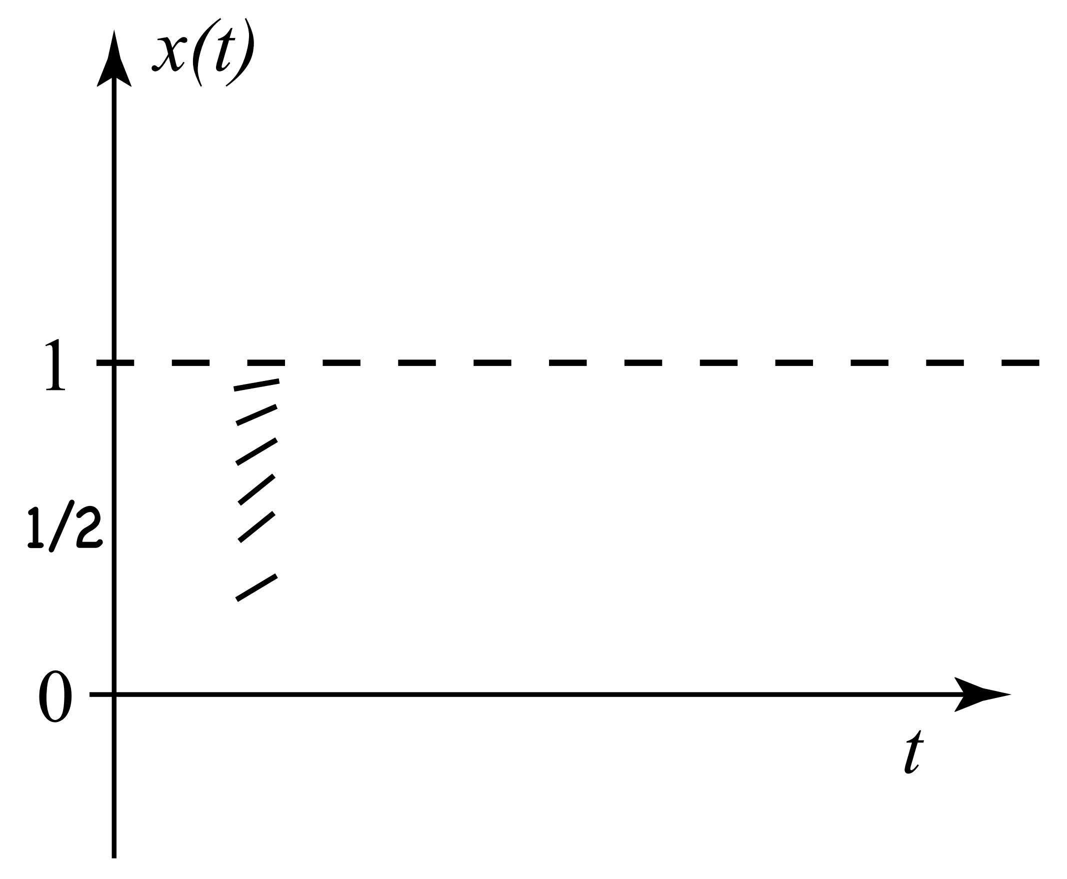

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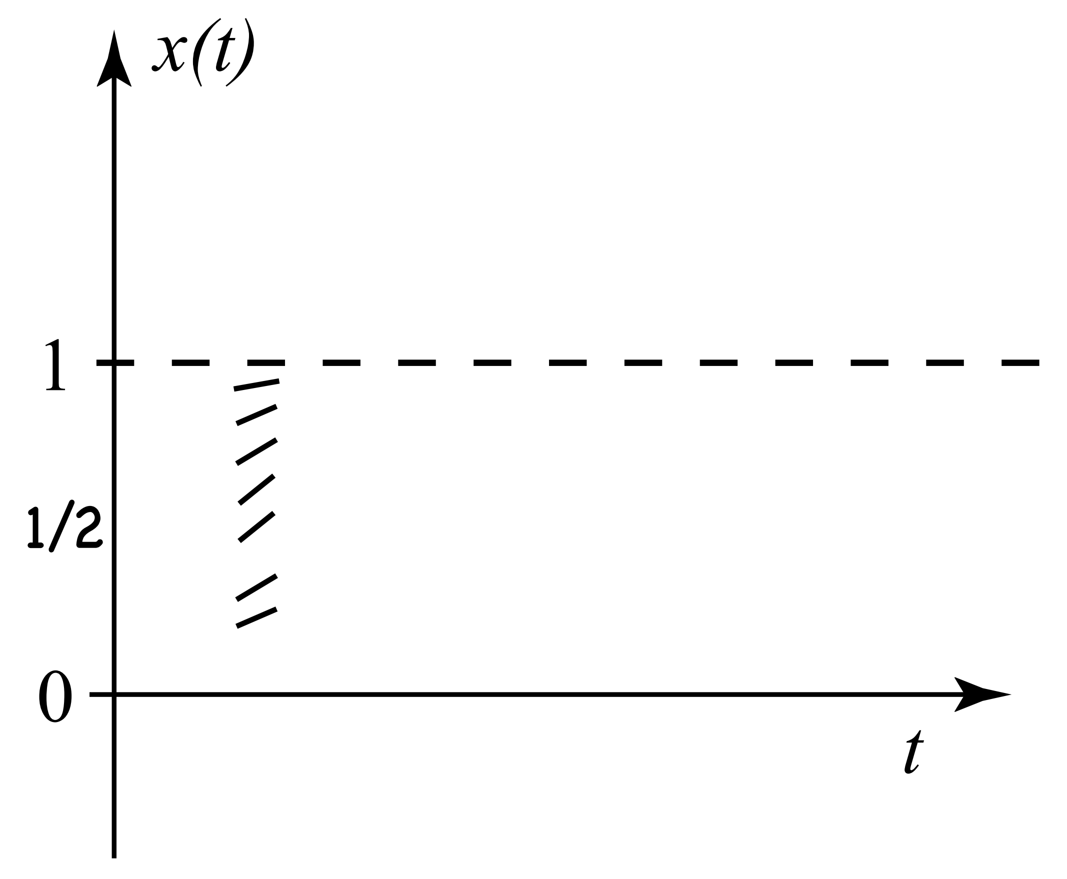

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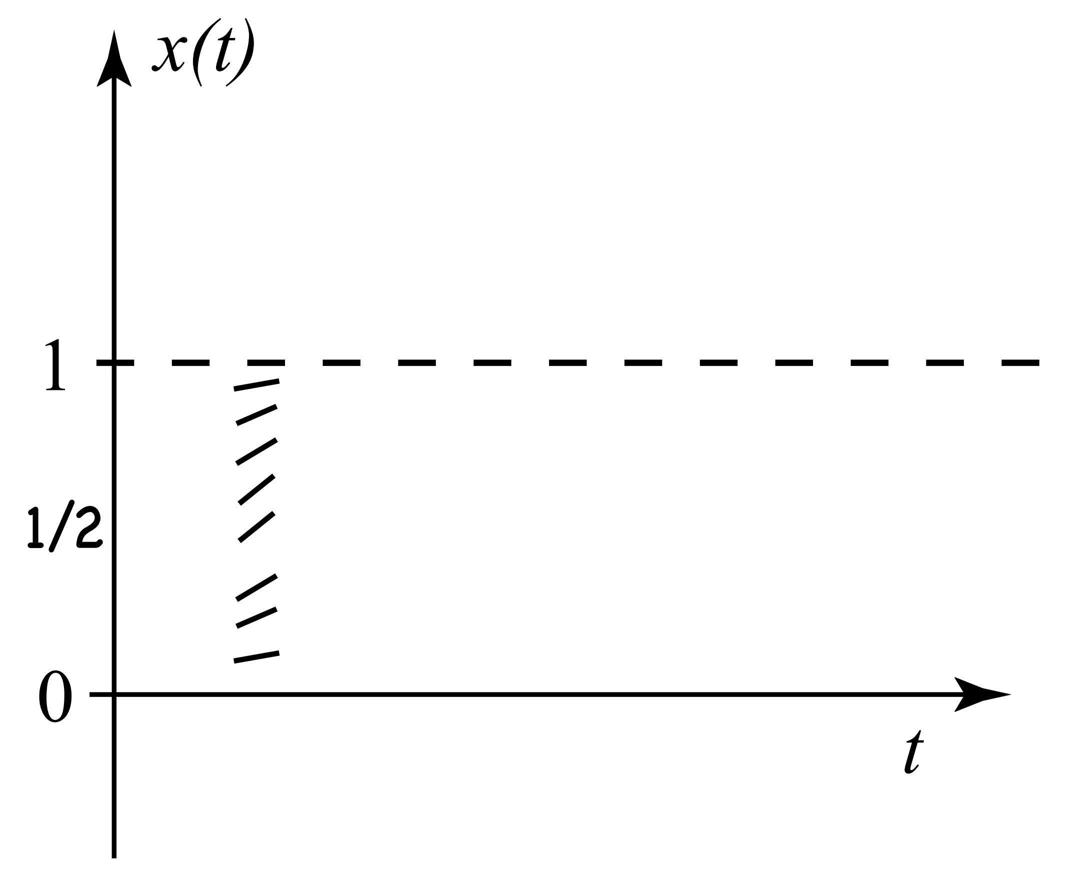

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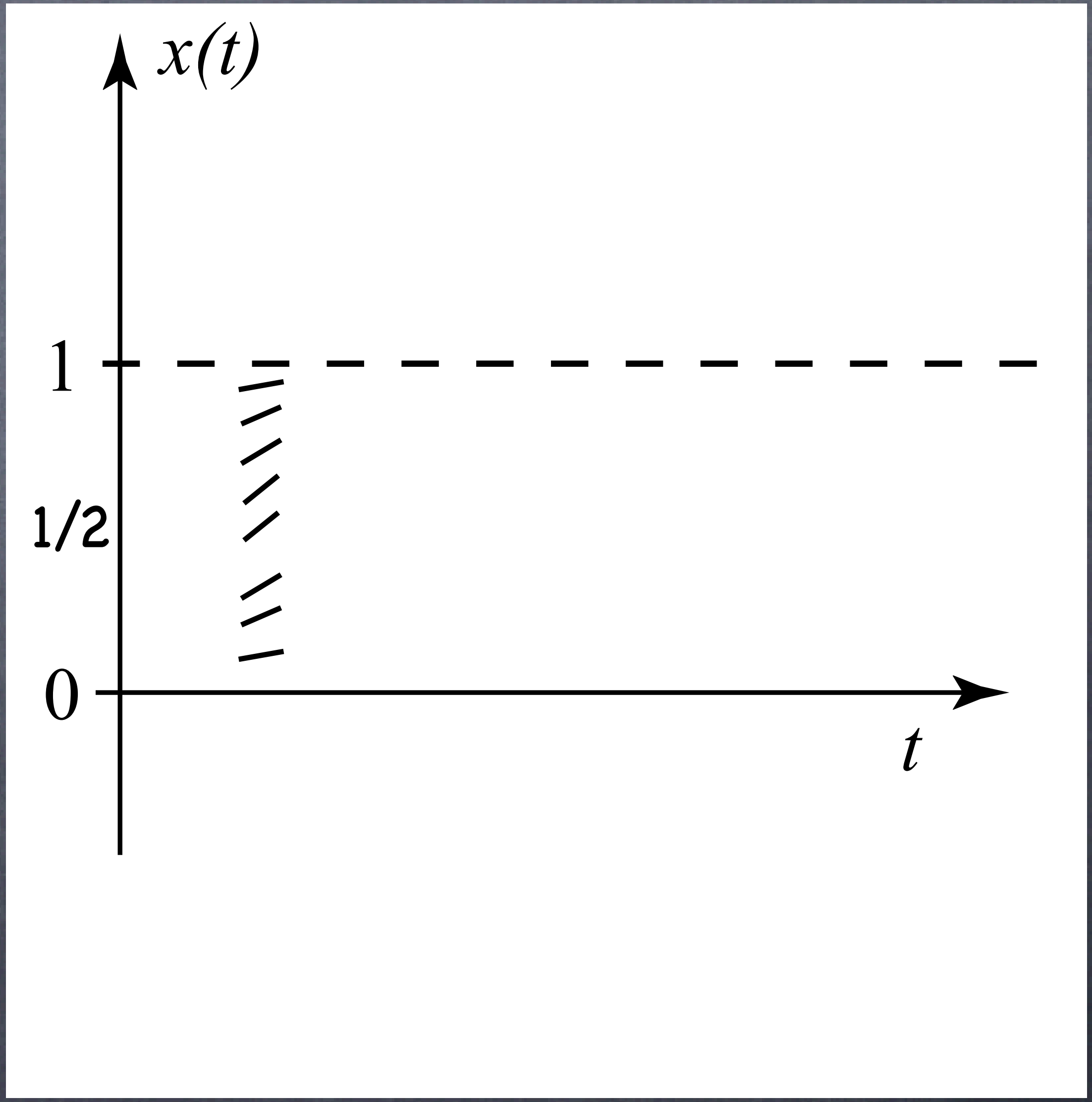

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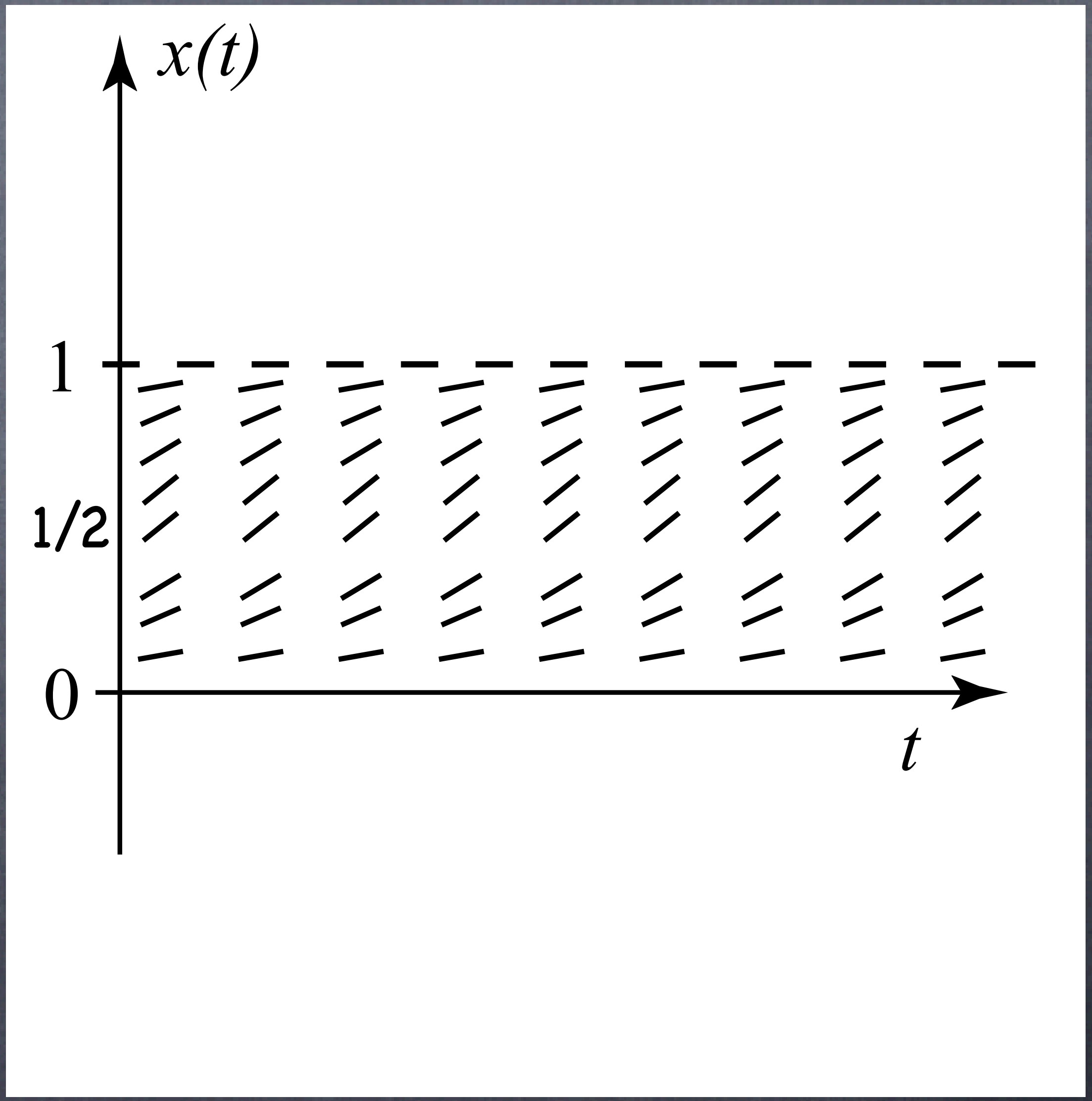

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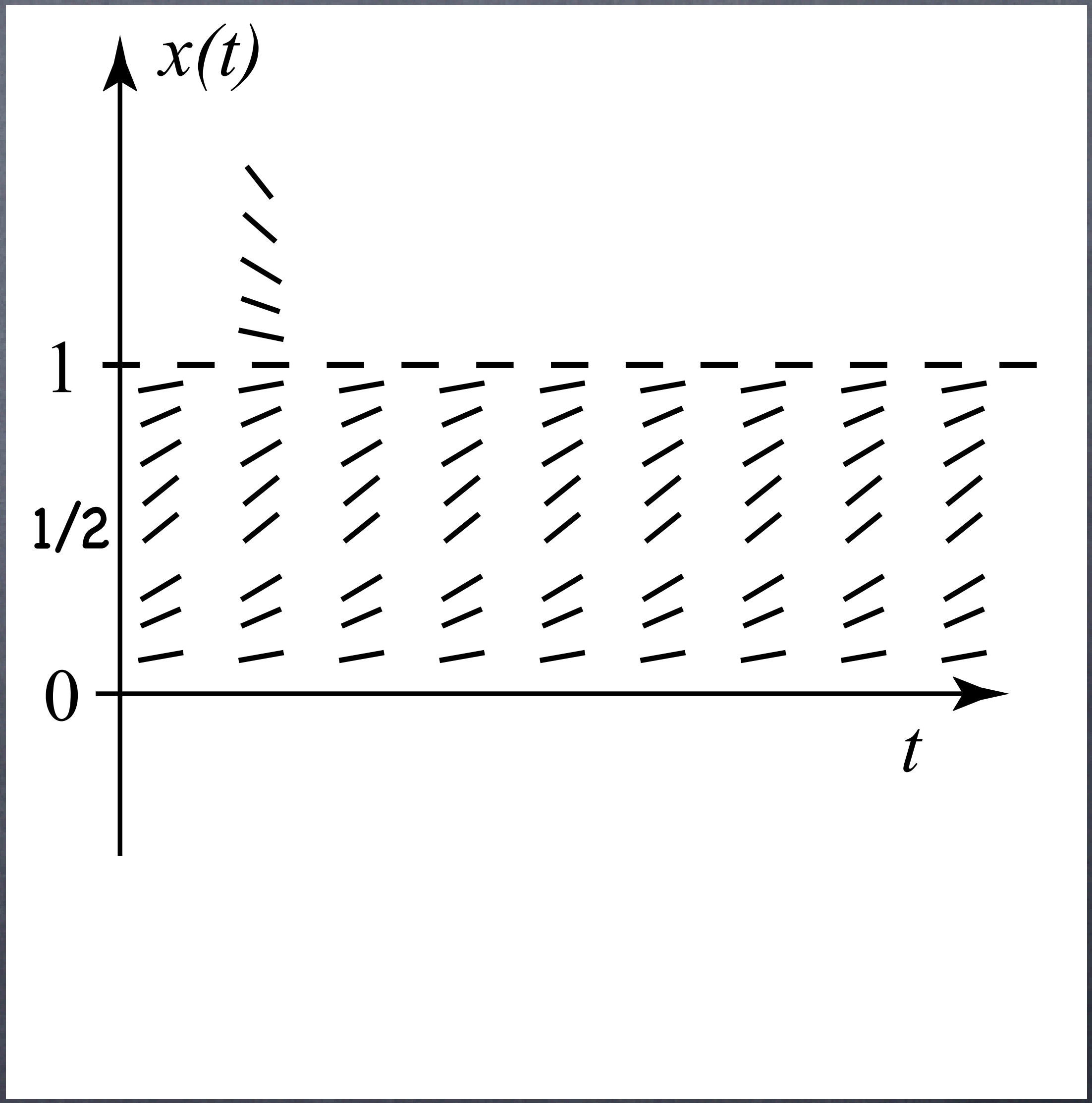

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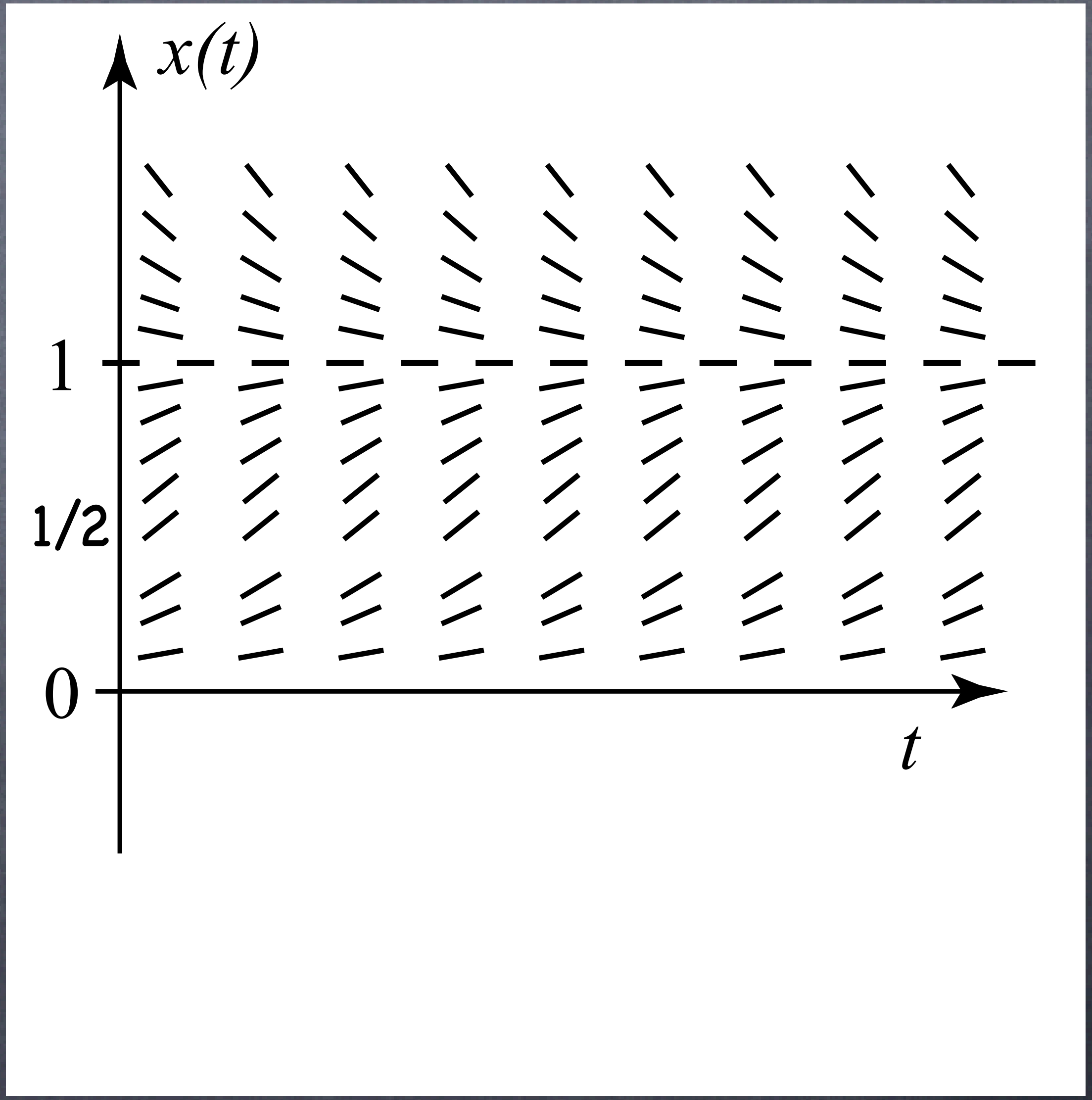

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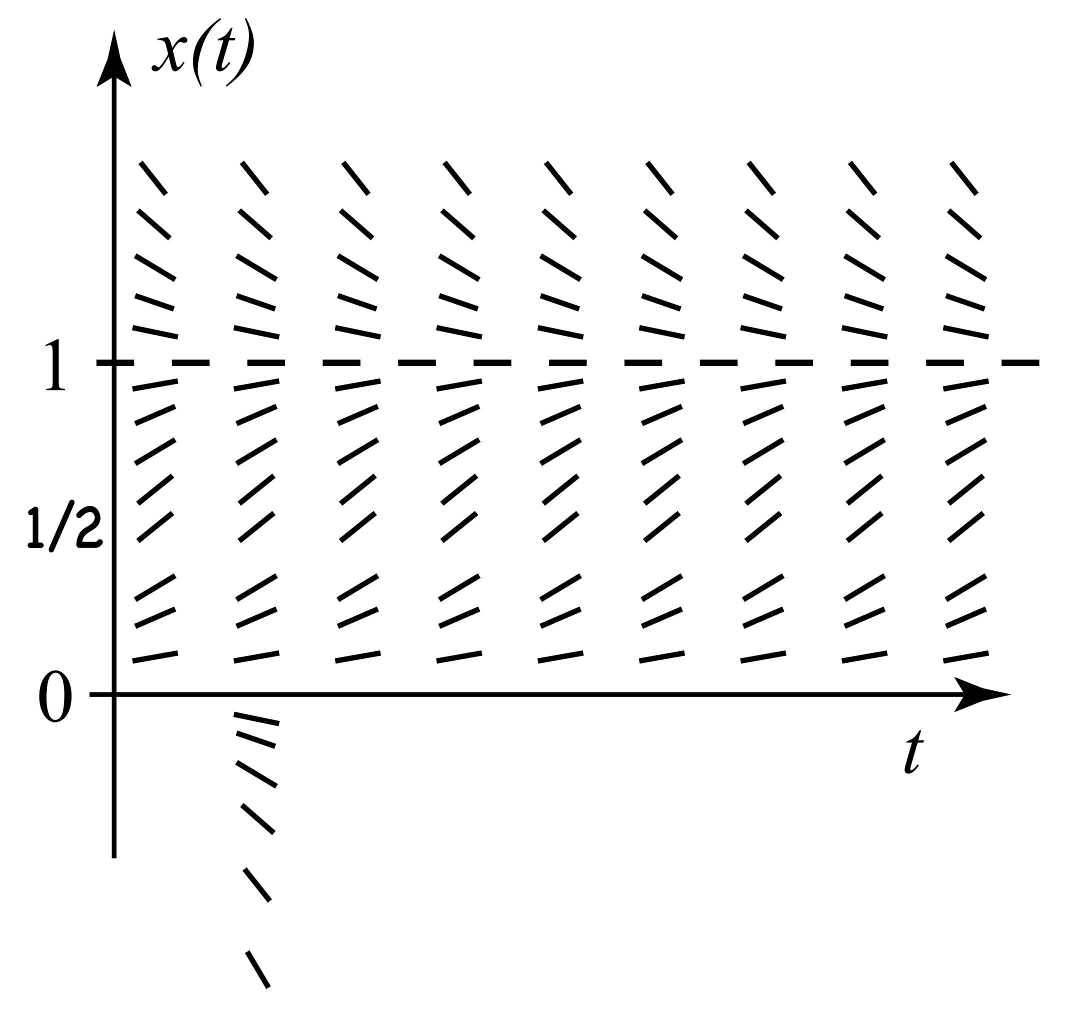

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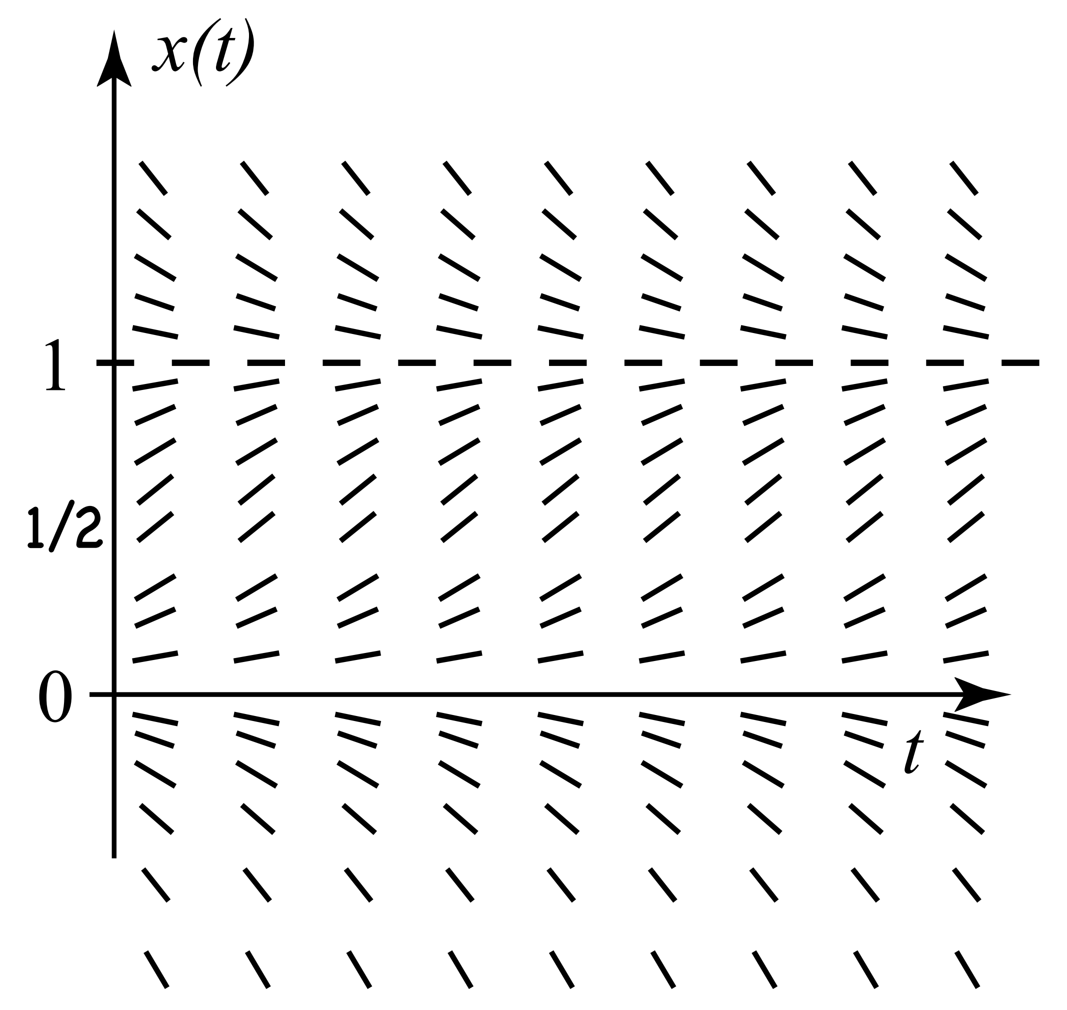

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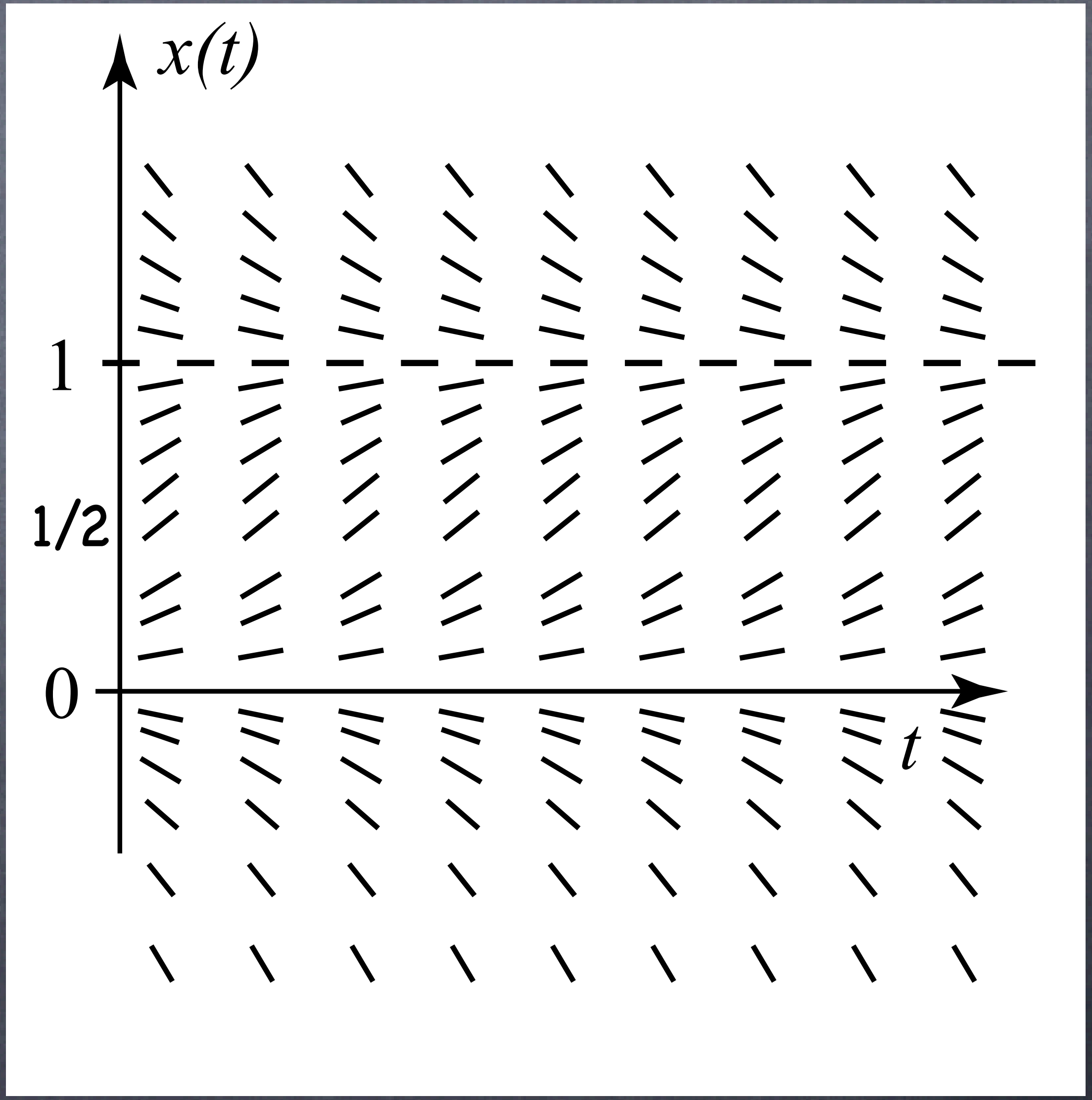

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- Solution curves must be tangent to slope field everywhere.

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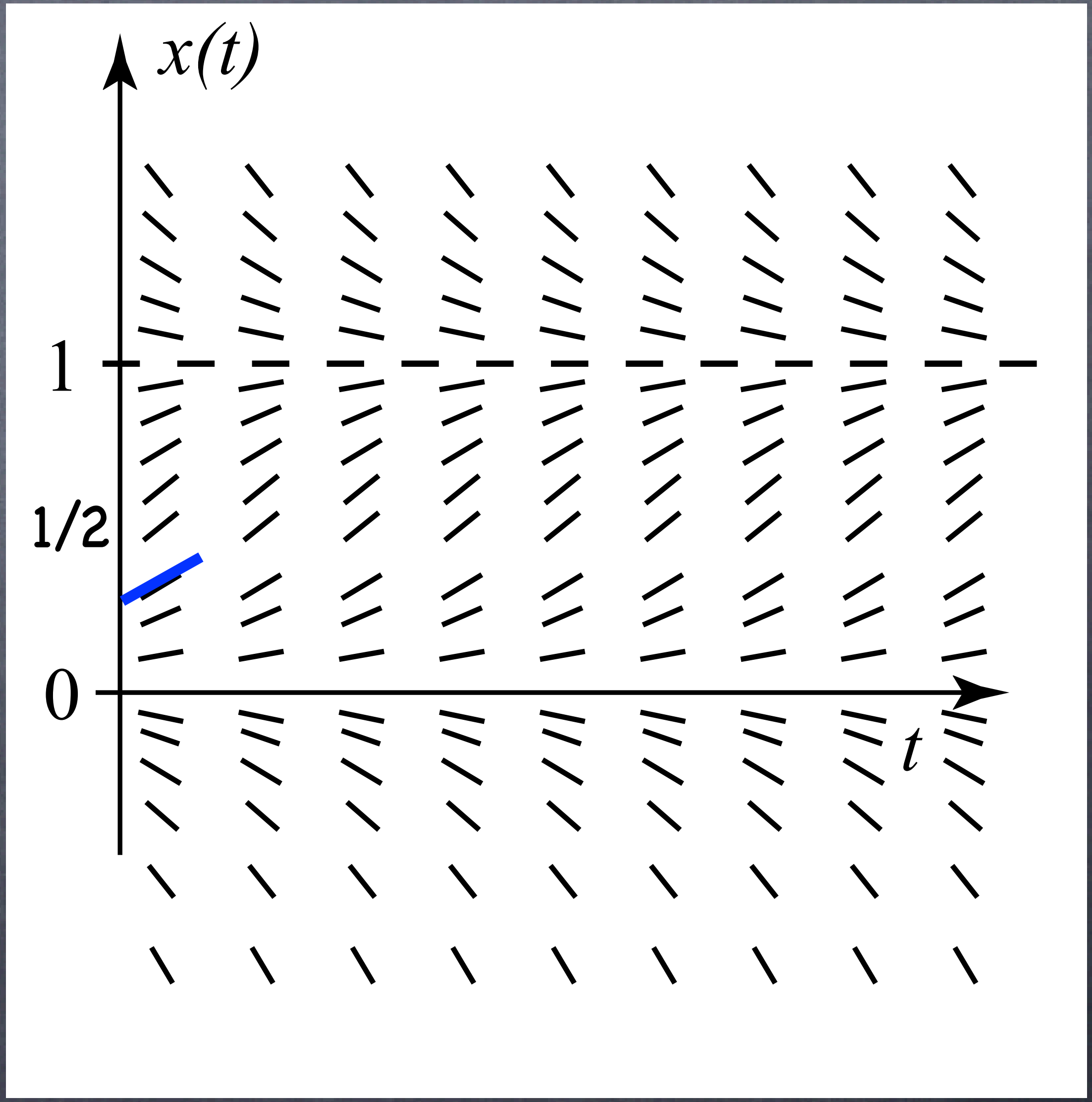

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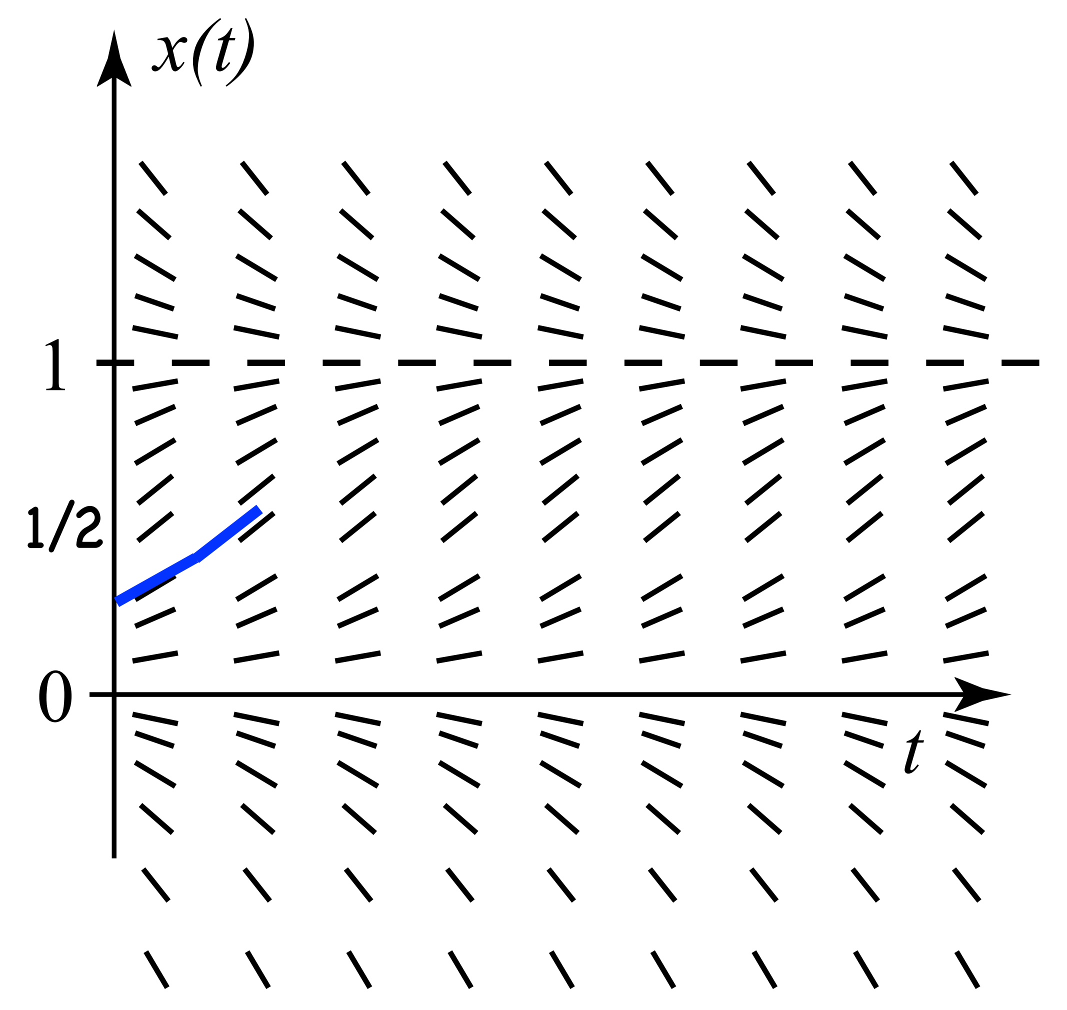

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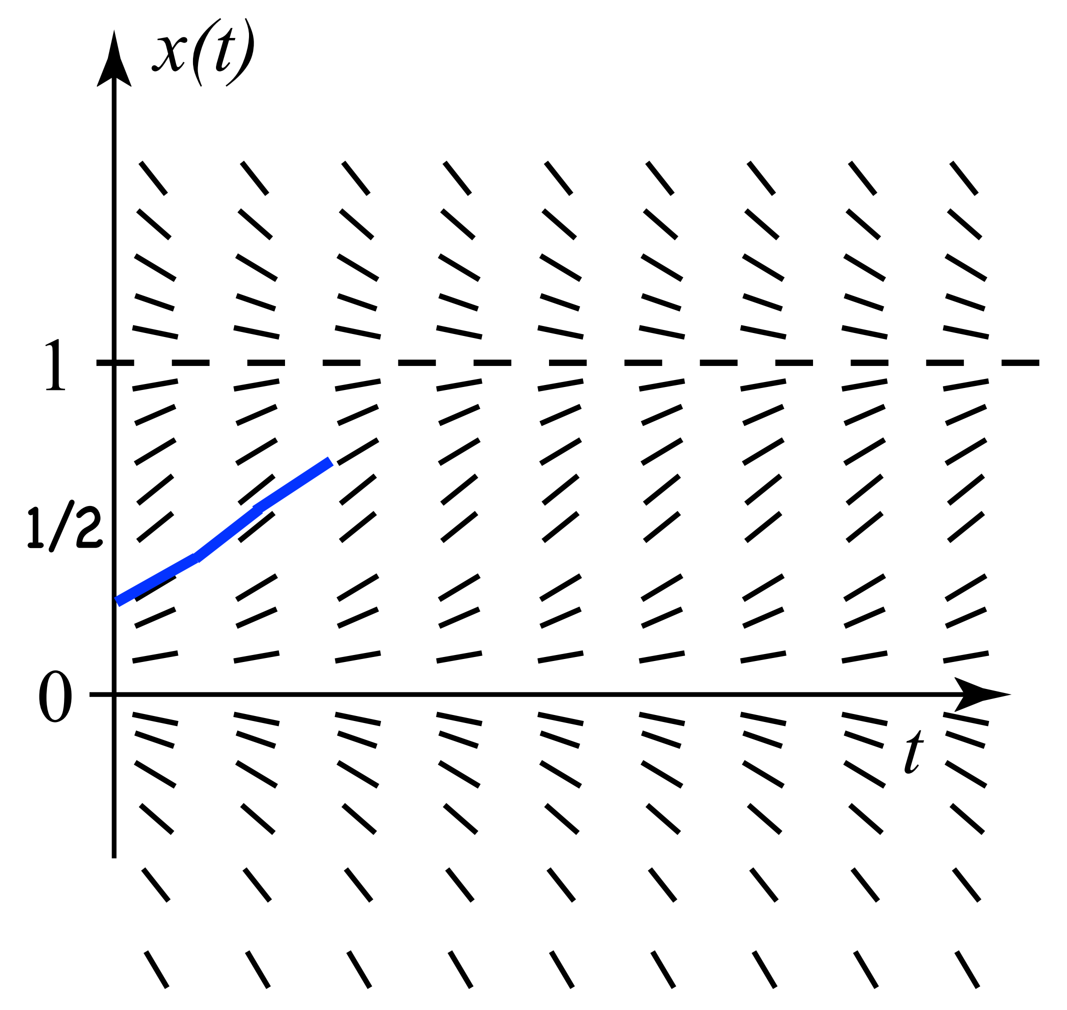

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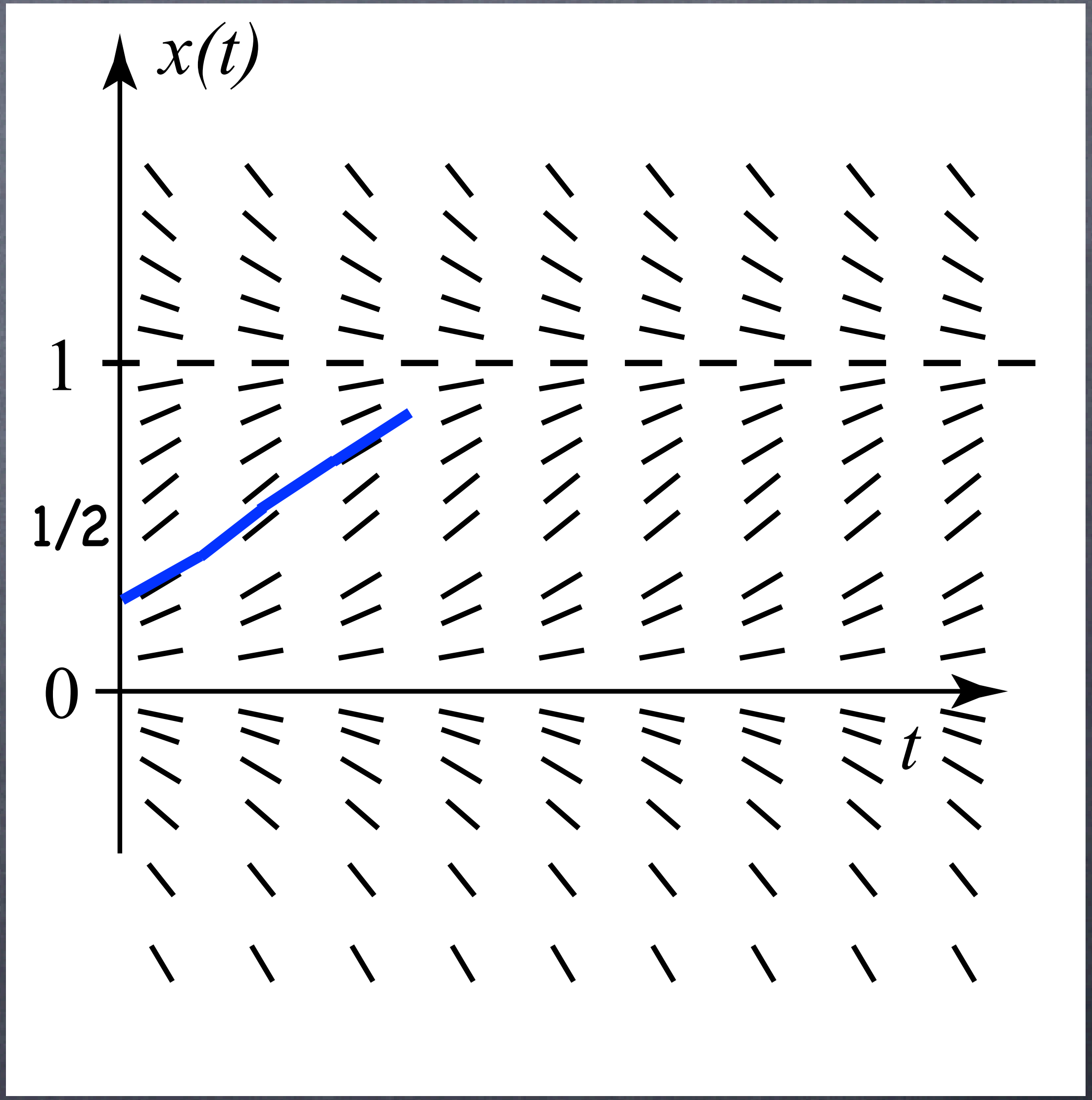

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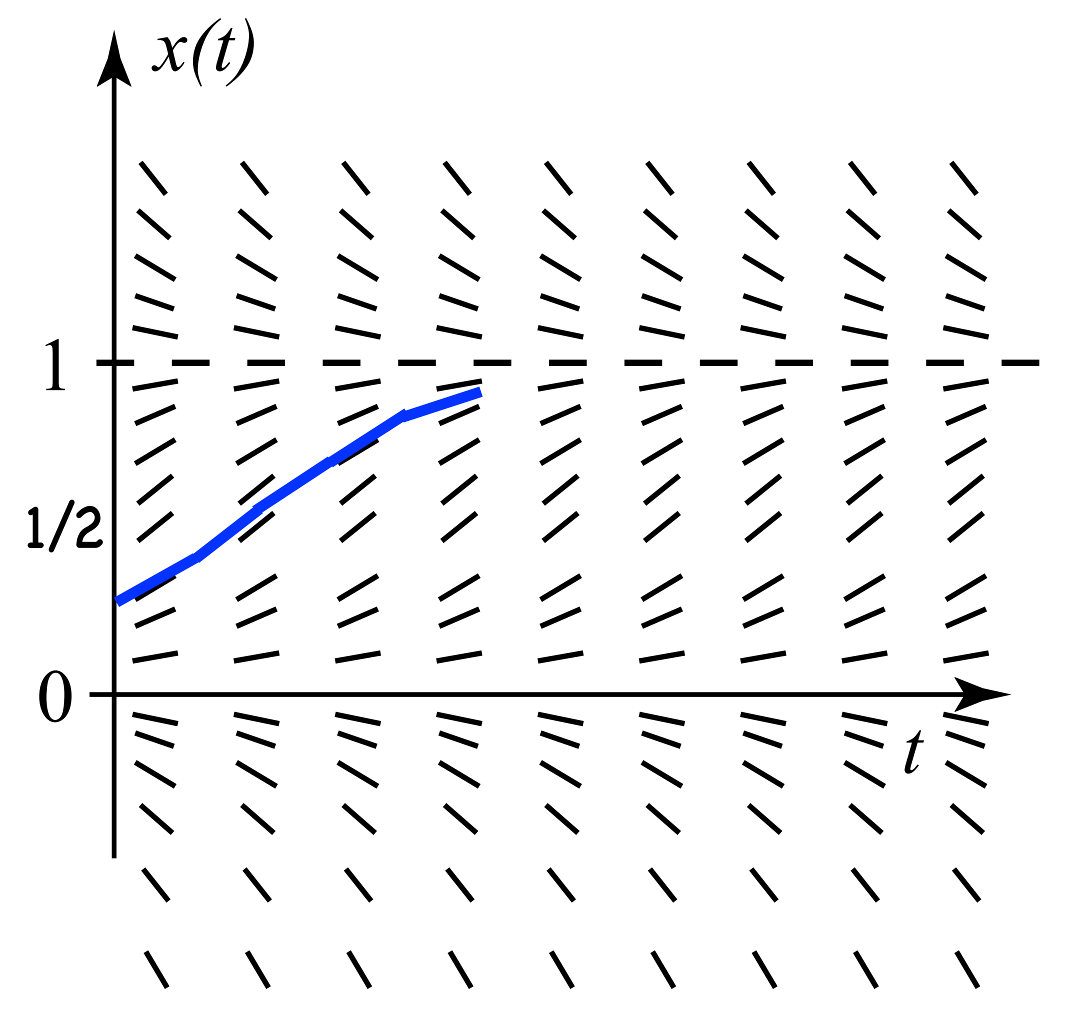

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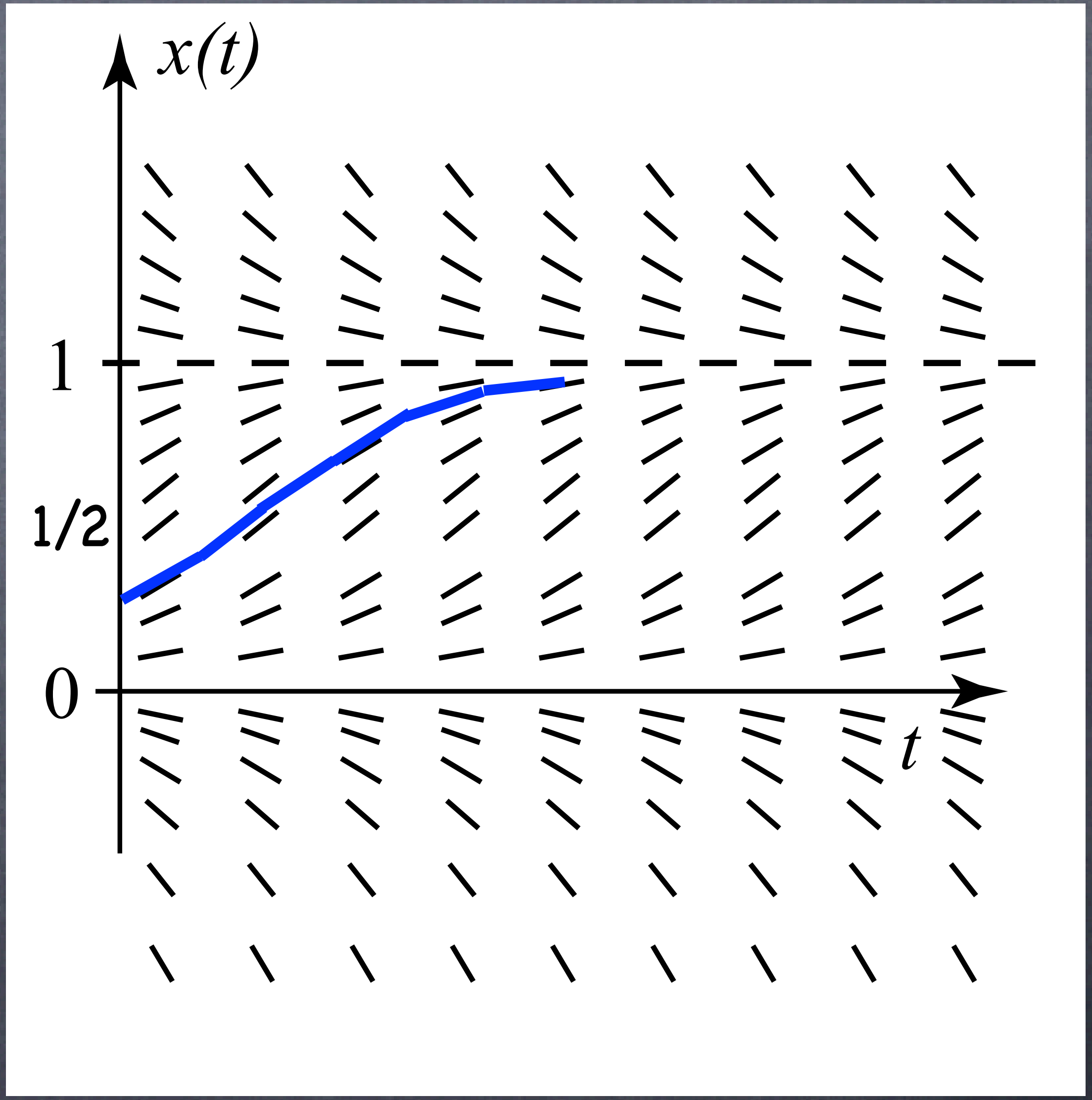

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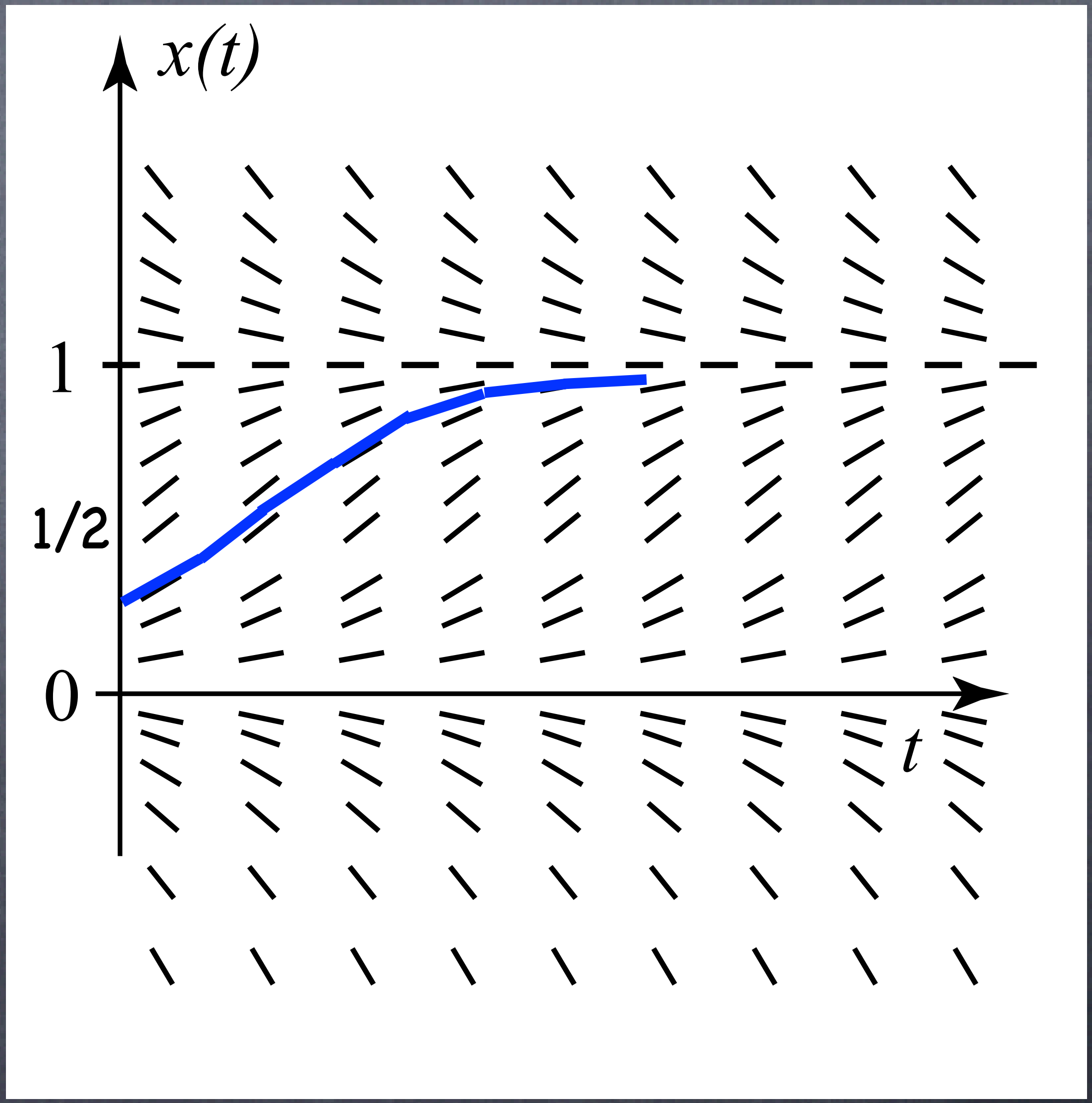
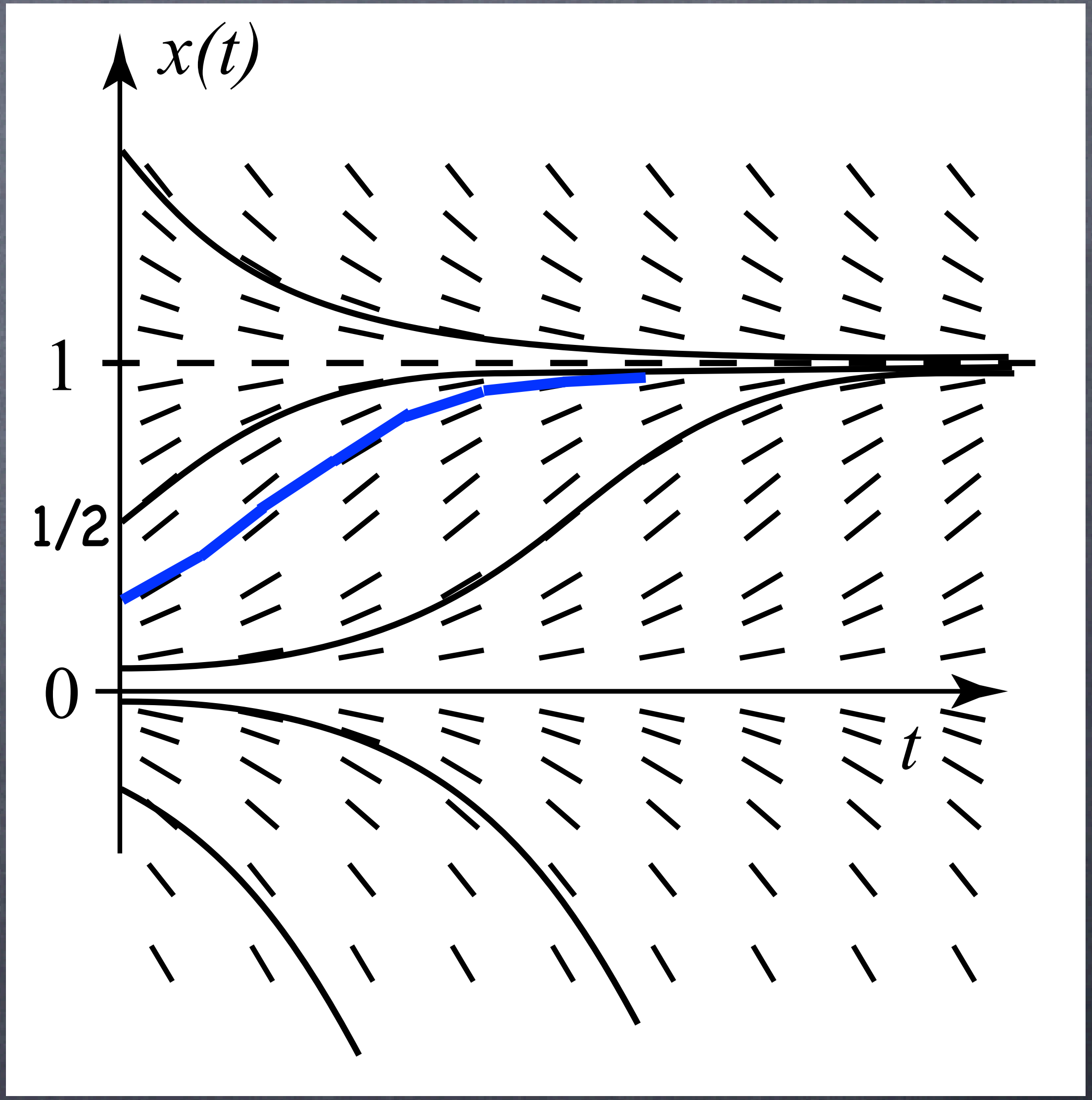


$$x' = x(1 - x)$$

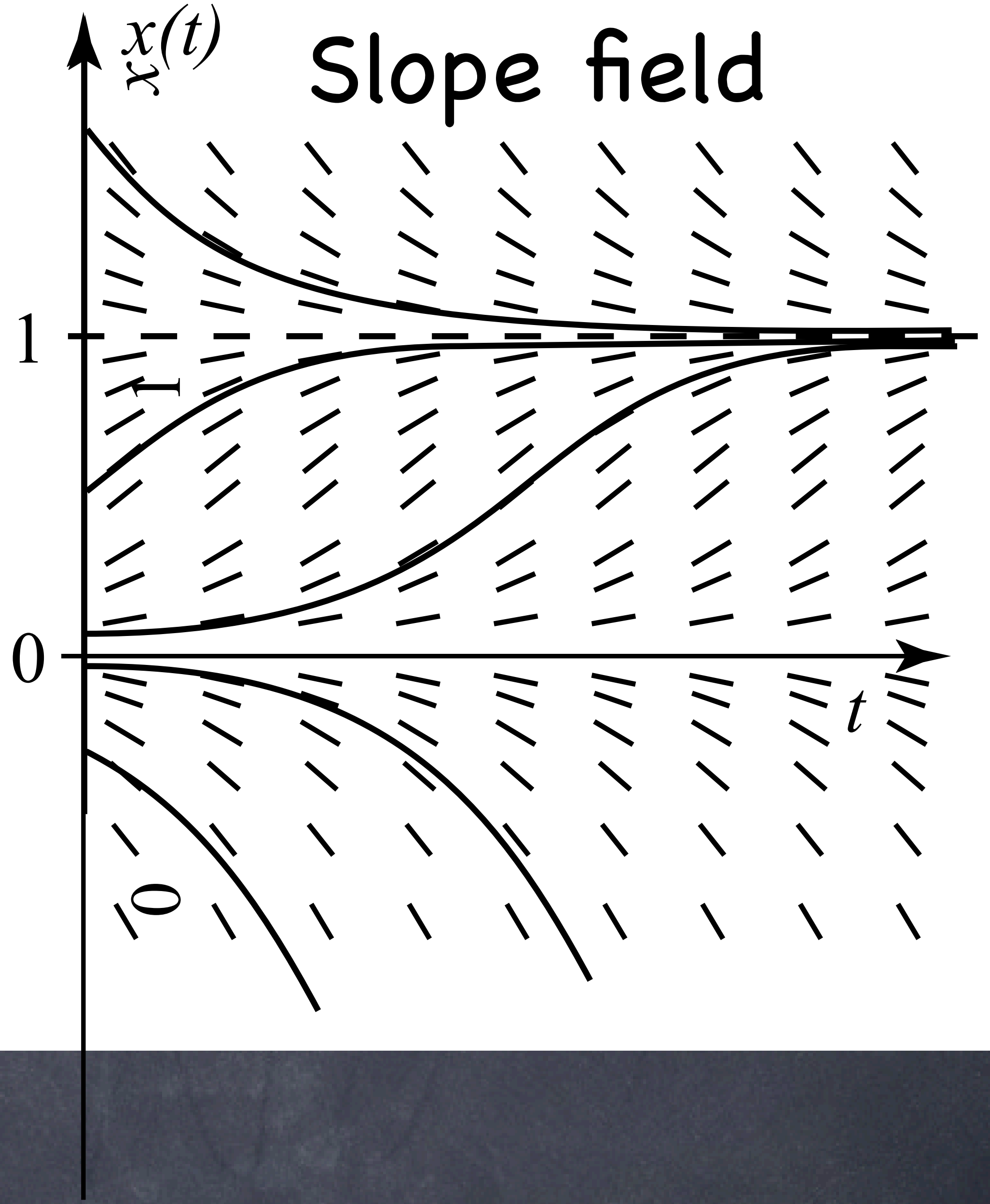
Slope field



- Slope field.
- At any t, don't know x yet so plot all possible x' values
- Now draw them for all t.
- Solution curves must be tangent to slope field everywhere.

# Velocity versus position

Velocity (x′) vs. position (x)

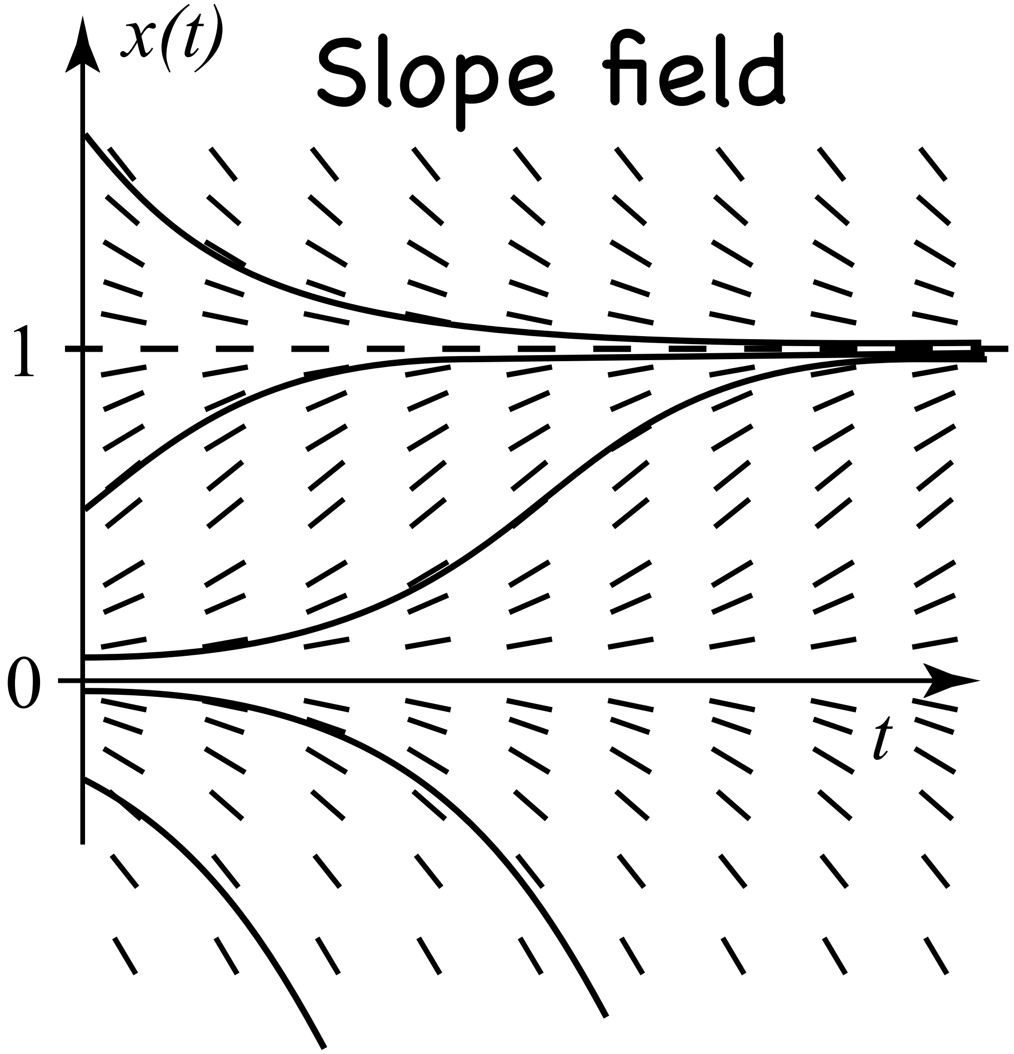


Slope field

# Velocity versus position

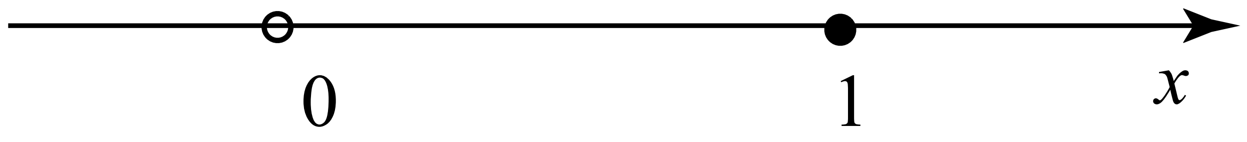
Velocity (x′) vs. position (x)

Slope field



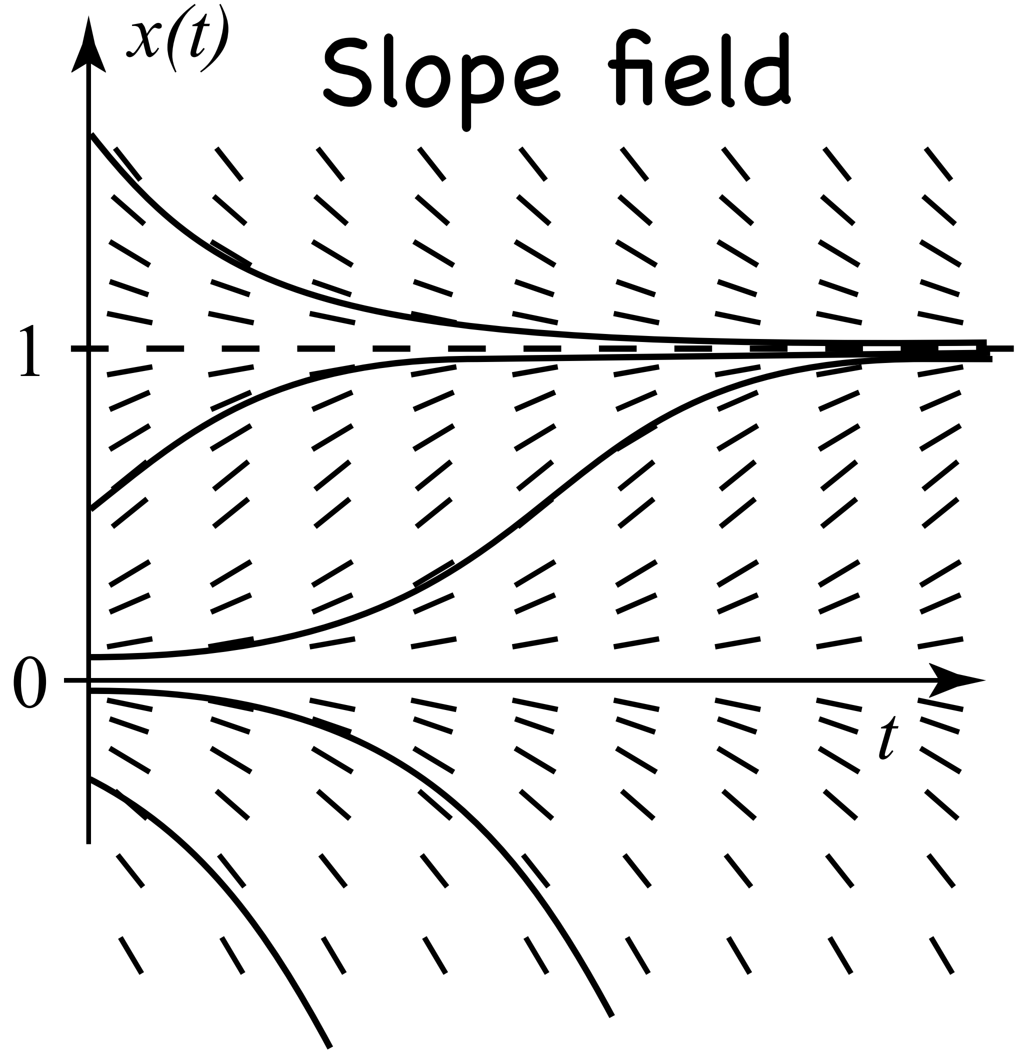
# Velocity versus position

Velocity (x′) vs. position (x)



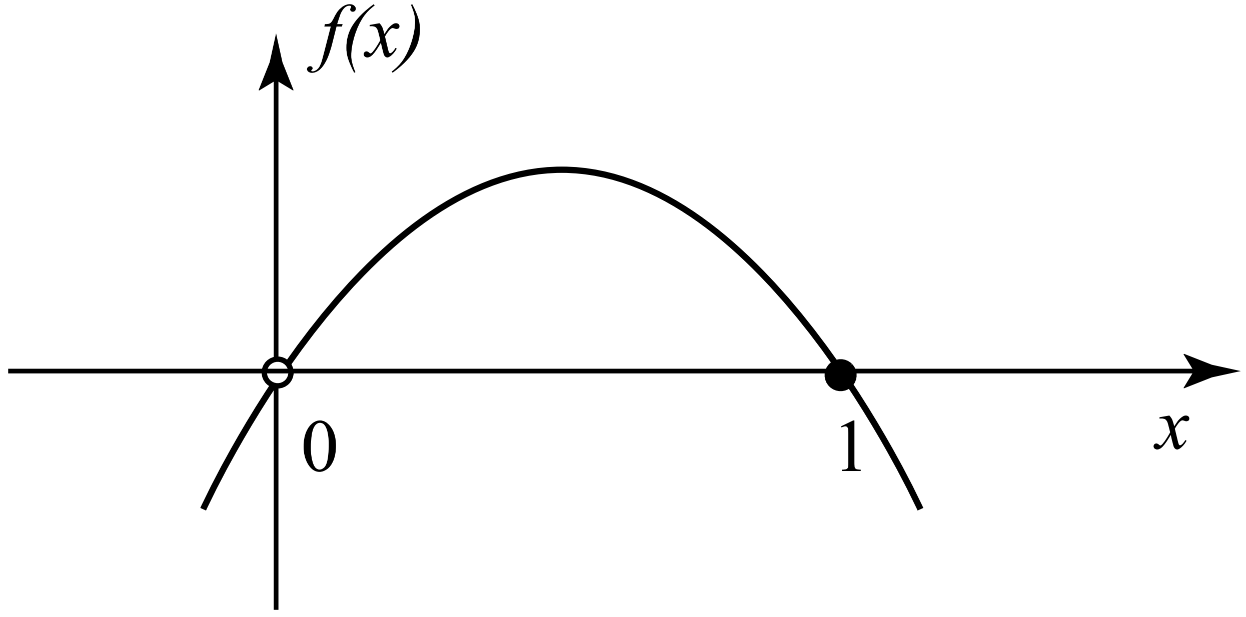
$x' = f(x) = x(1-x)$

Slope field

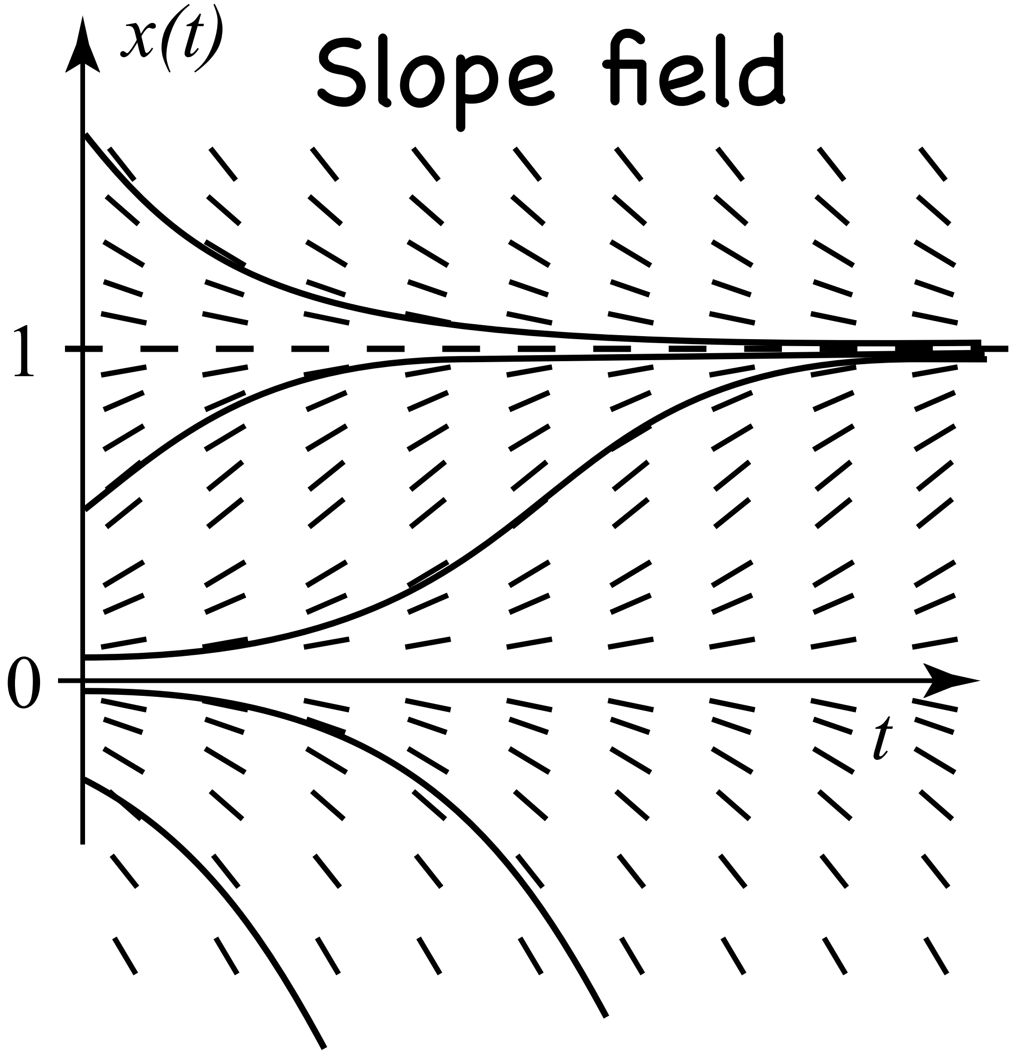


# Velocity versus position

Velocity ($x'$) vs. position ($x$)



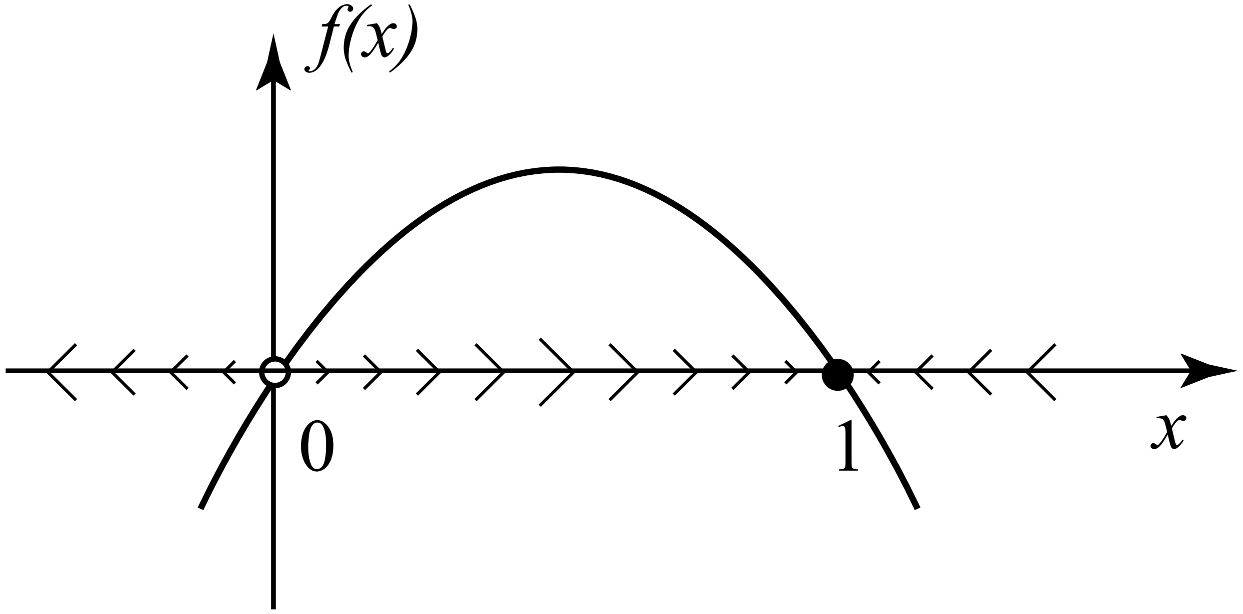
$x' = f(x) = x(1-x)$



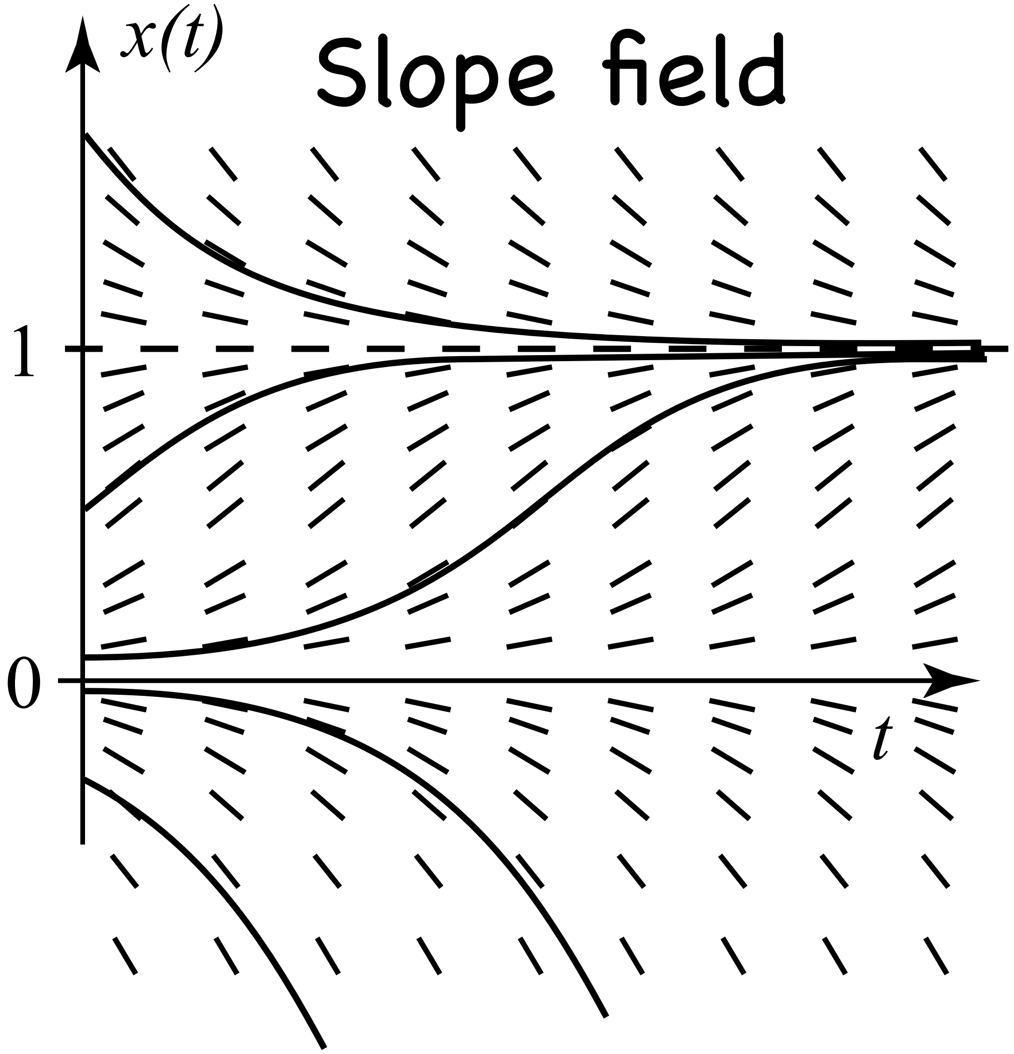
Slope field

# Velocity versus position

Velocity ($x'$) vs. position ($x$)

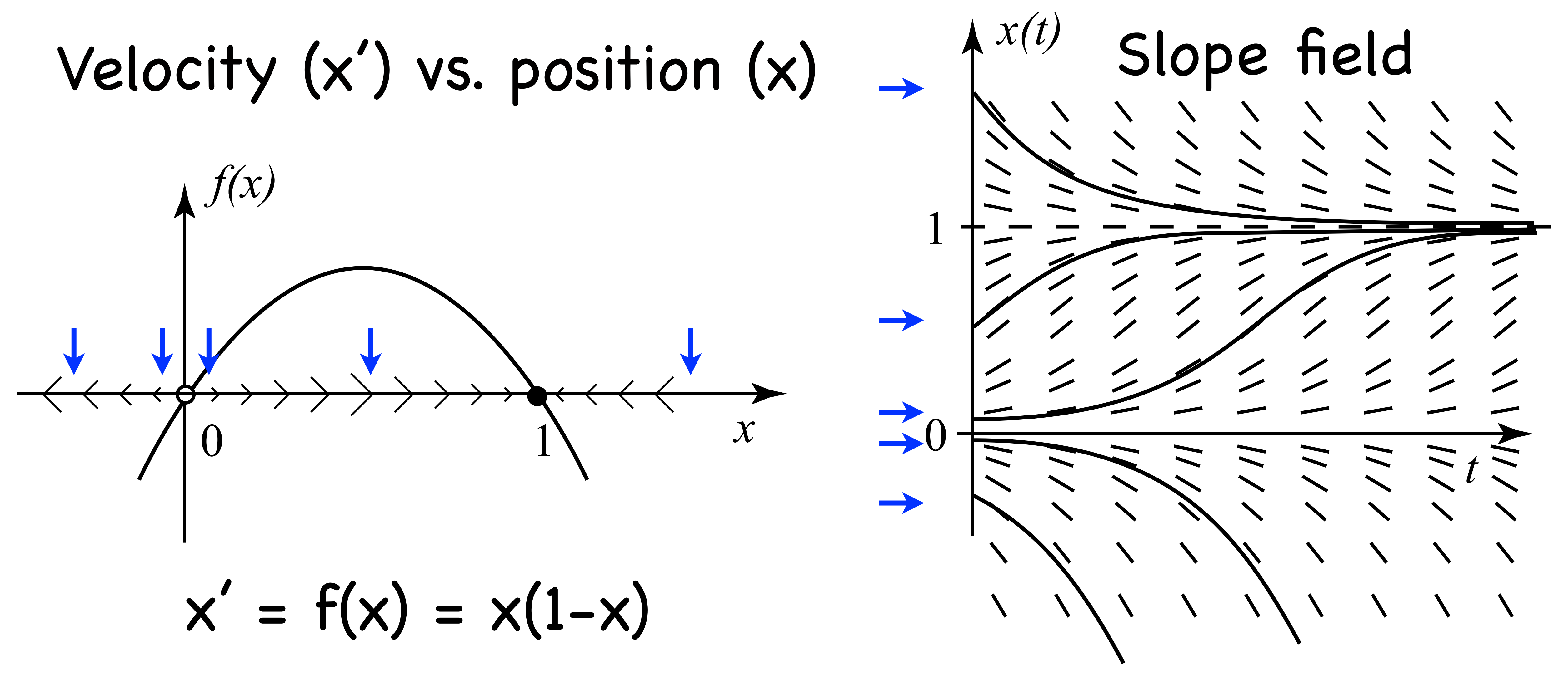


$$x' = f(x) = x(1-x)$$



Slope field

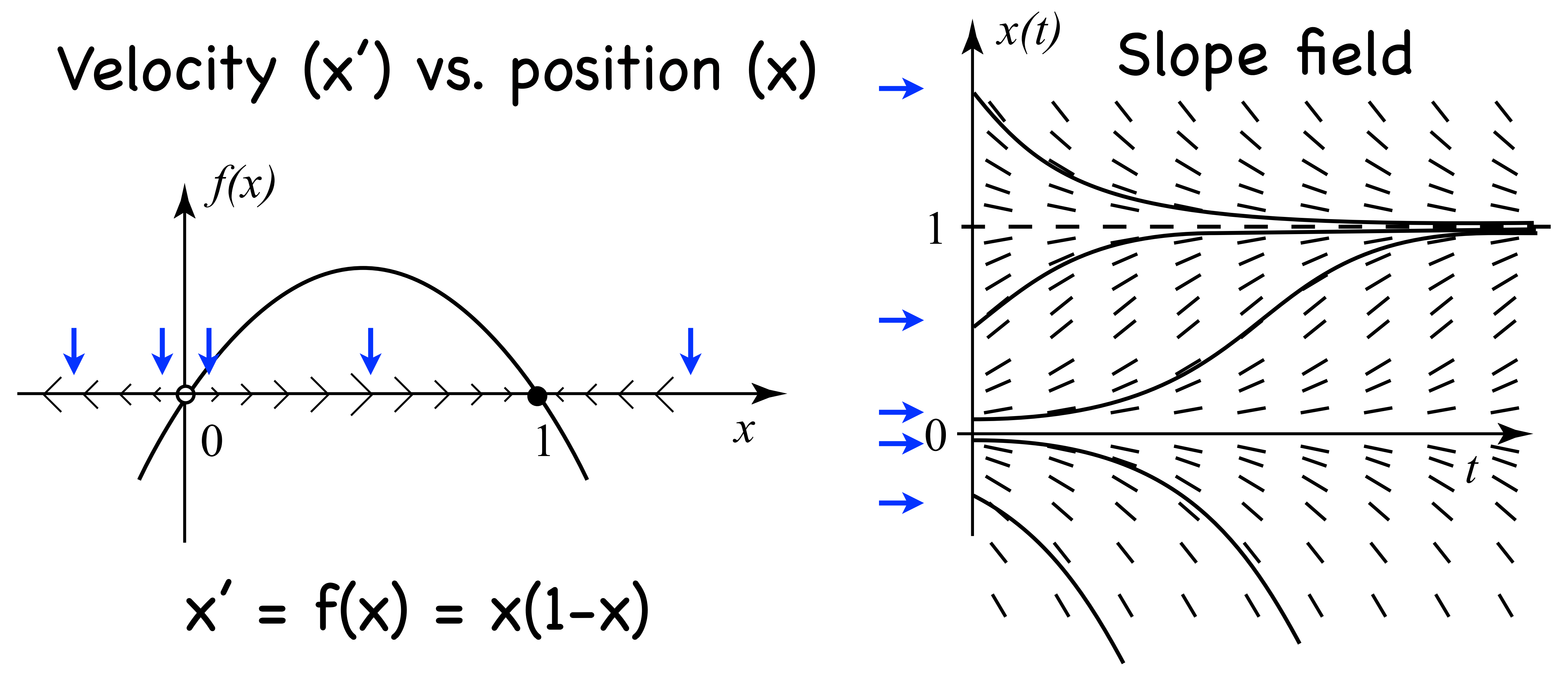
# Velocity versus position



Velocity (x′) vs. position (x)

$x' = f(x) = x(1-x)$

# Velocity versus position

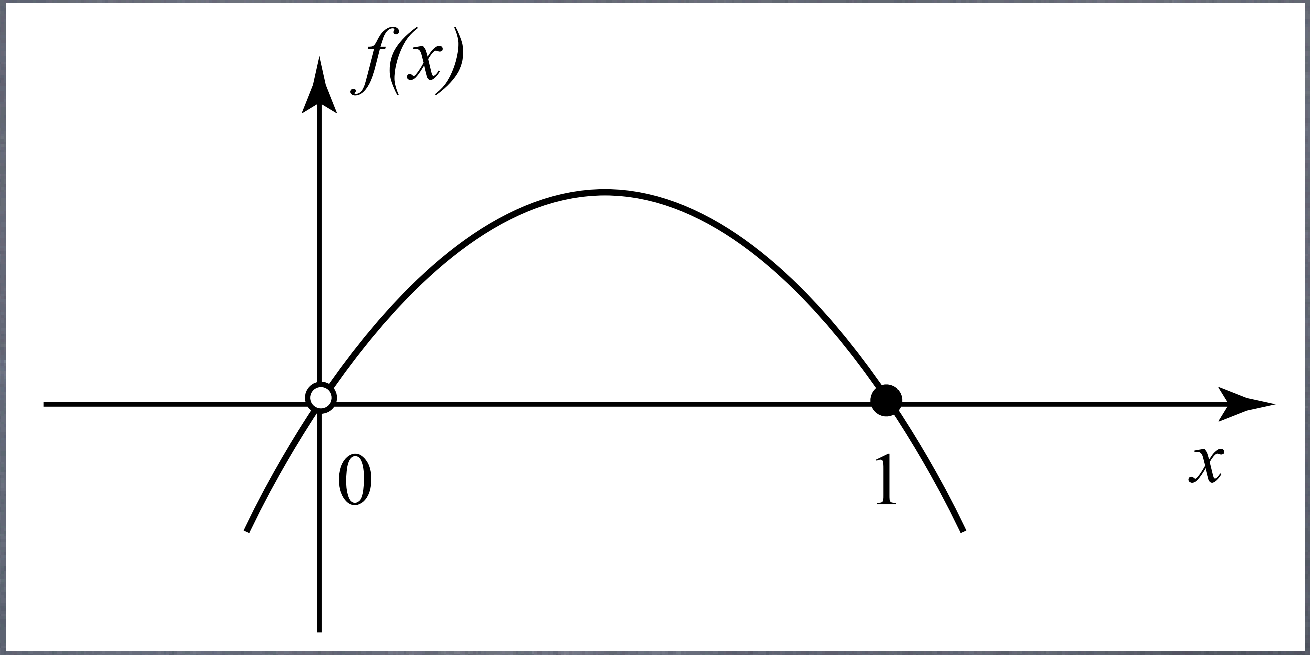


Stable steady state- all nearby solutions approach

Unstable steady state - not stable

# Determine stability

$$x' = x(1 - x)$$



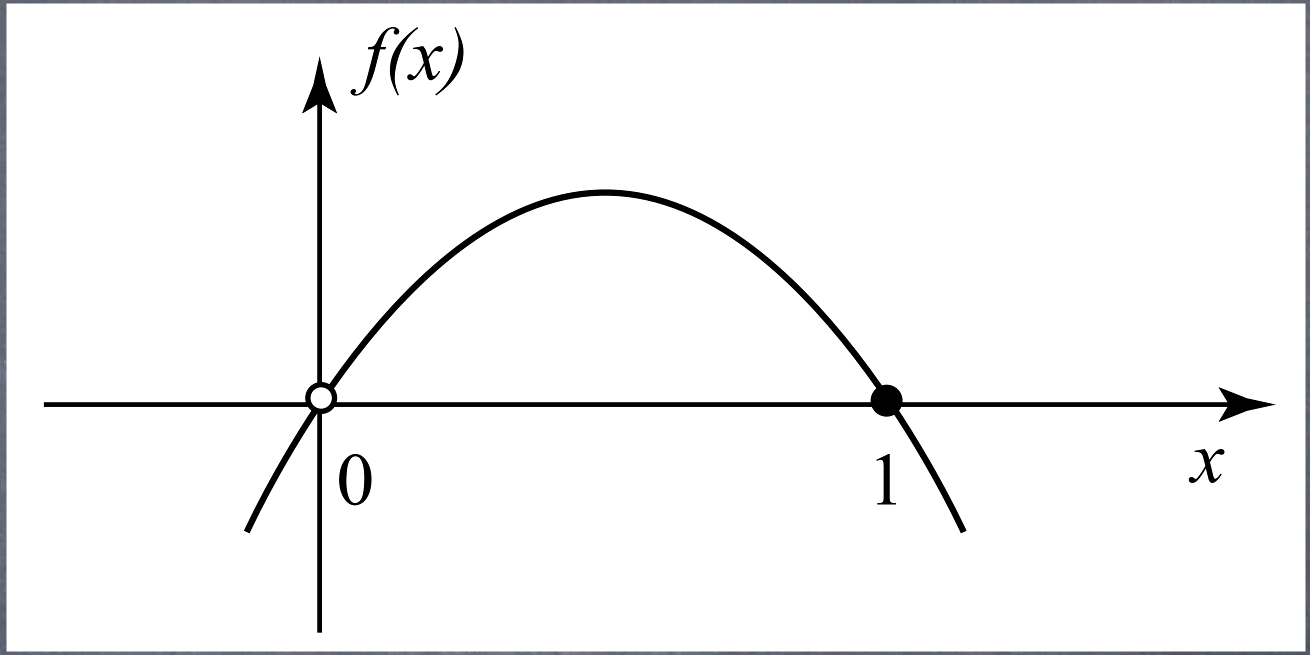
- If you start at x(0)=-0.01, the solution

(A) increases

(B) decreases

# Determine stability

$$x' = x(1-x)$$



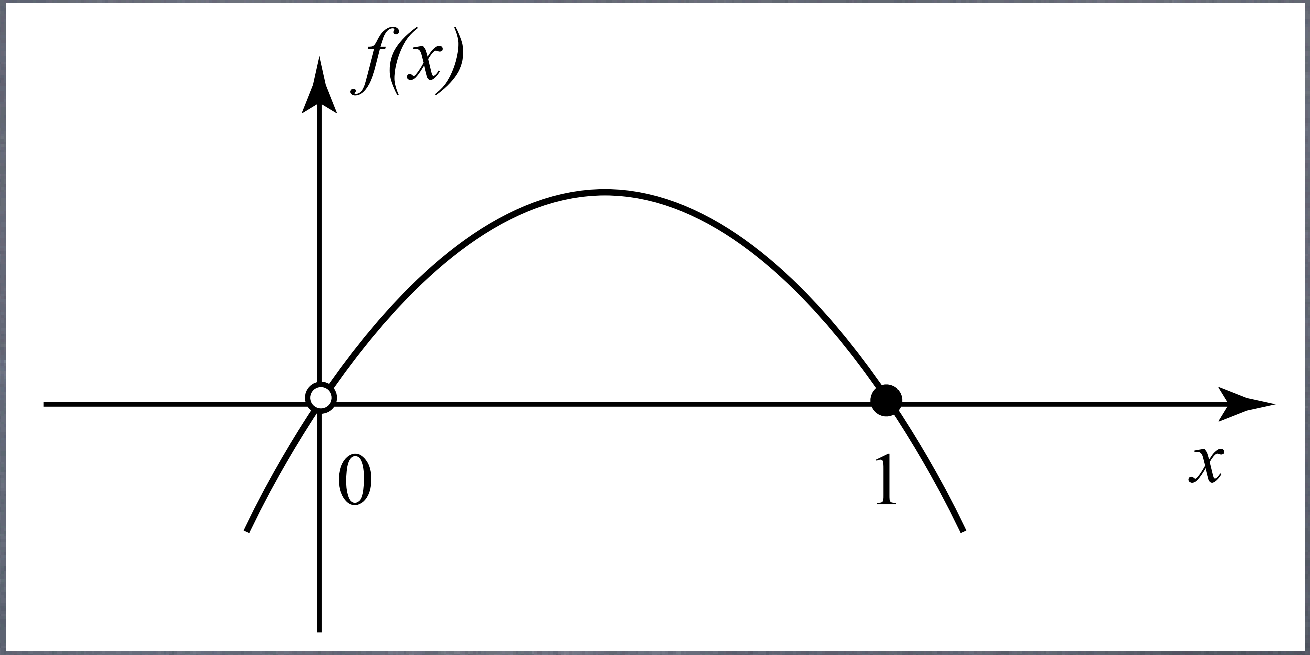
- If you start at x(0)=0.01, the solution

(A) increases

(B) decreases

# Determine stability

$$x' = x(1-x)$$



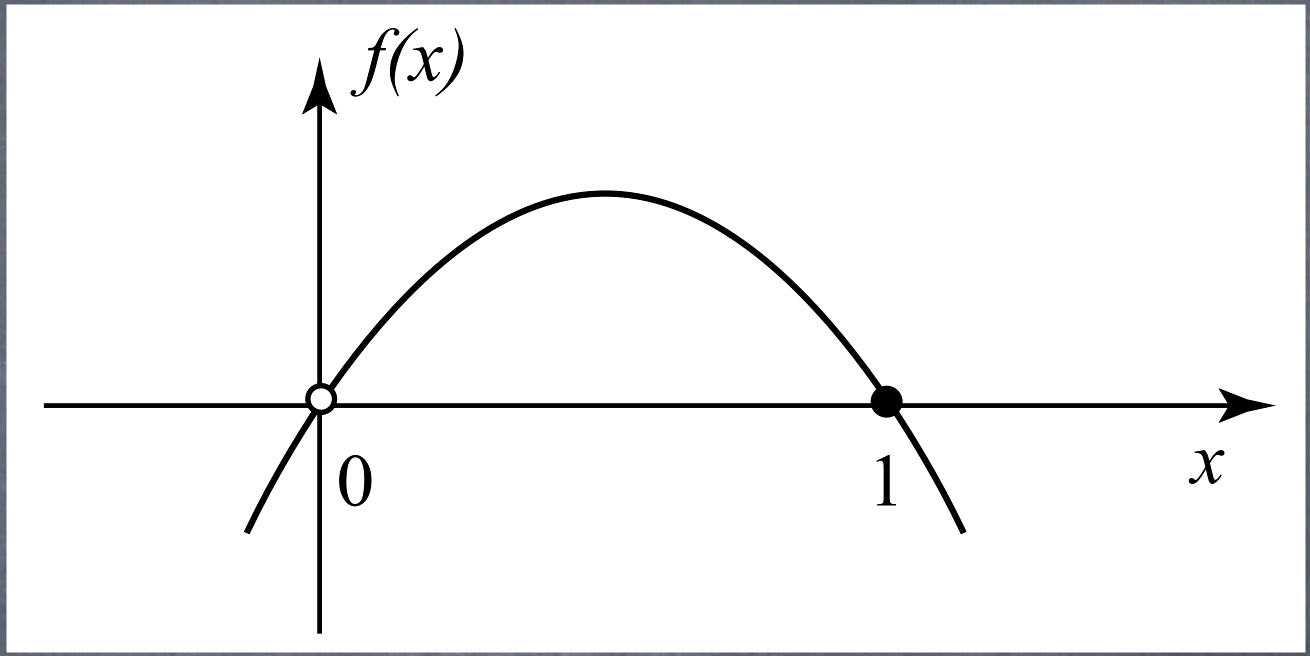
- If you start at x(0)=0.99, the solution

(A) increases

(B) decreases

# Determine stability

$$x' = x(1-x)$$



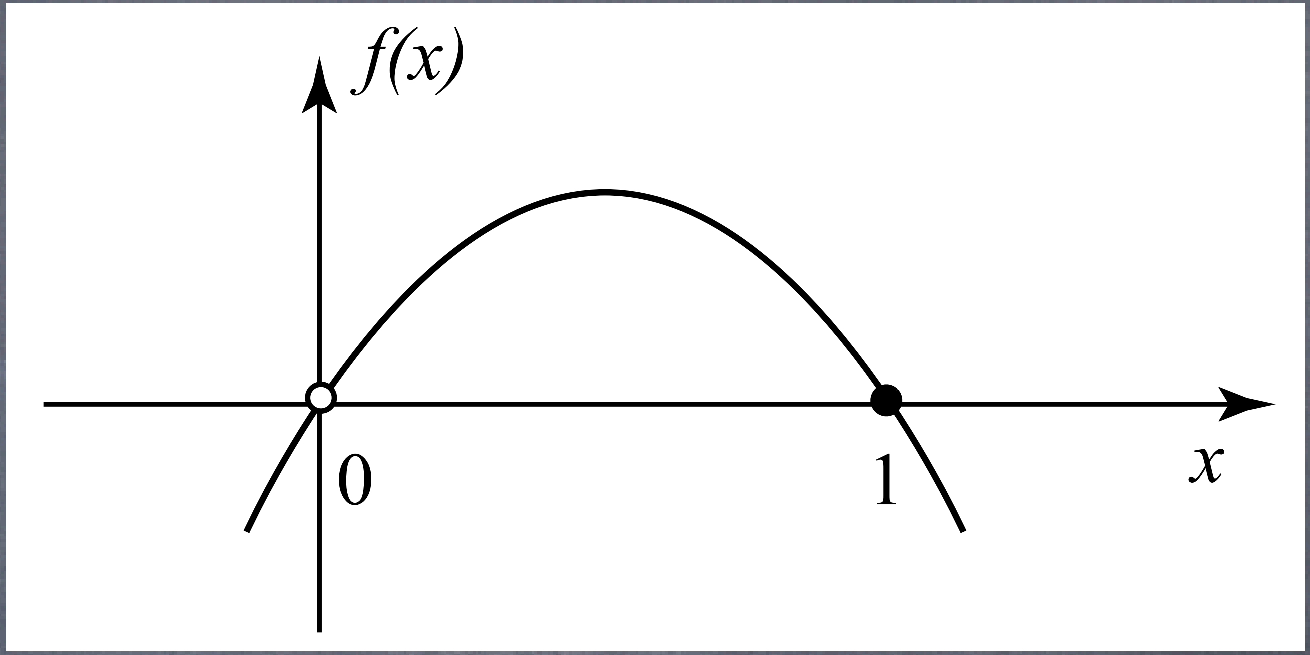
- If you start at x(0)=1.01, the solution

(A) increases

(B) decreases

# Determine stability

$$x' = x(1-x)$$



(A) Both x(t)=0 and x(t)=1 are stable steady states.

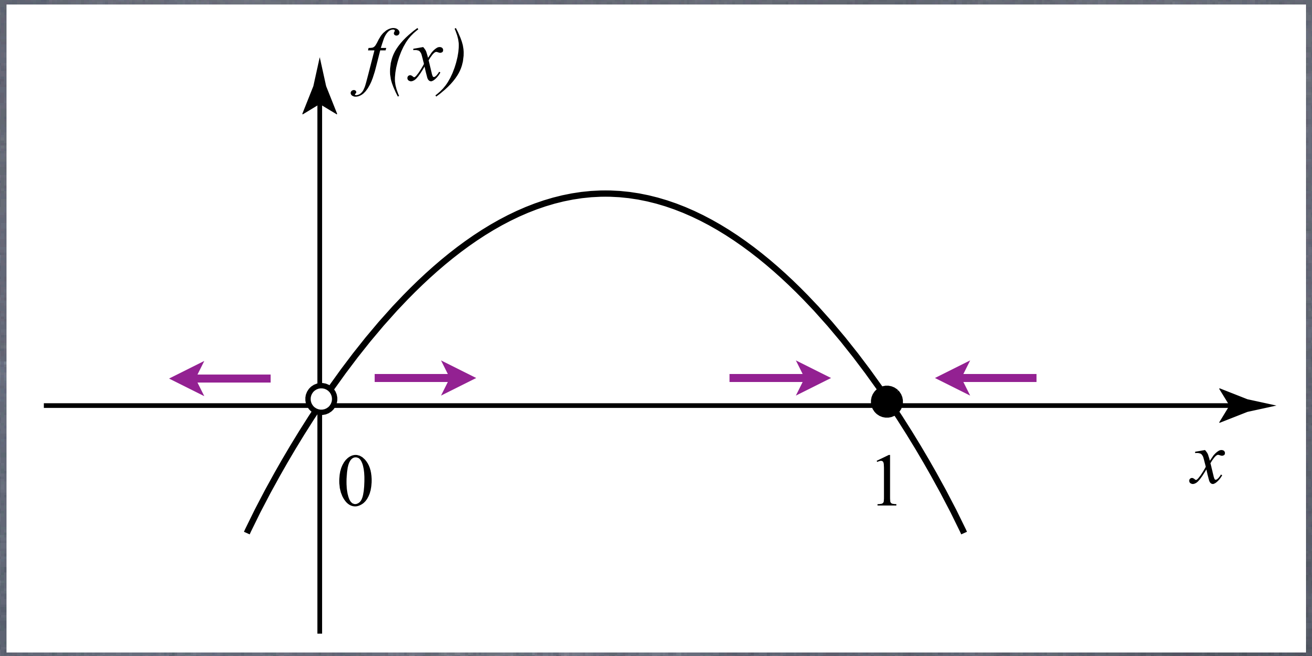
(B) x(t)=0 is stable and x(t)=1 is unstable.

(C) x(t)=0 is unstable and x(t)=1 is stable.

(D) Both x(t)=0 and x(t)=1 are unstable steady states.

# Determine stability

$$x' = x(1 - x)$$



(A) Both x(t)=0 and x(t)=1 are stable steady states.

(B) x(t)=0 is stable and x(t)=1 is unstable.

(C) x(t)=0 is unstable and x(t)=1 is stable.

(D) Both x(t)=0 and x(t)=1 are unstable steady states.

Stable - solid dot. Unstable - empty dot.