Lecture 17 (Oct. 16, 2013)

Learning Goals: ① Chain Rule

② Implicit Differentiation

- Composite function: Suppose $f(x)$, $g(x)$ are two functions, their composite is defined by

$$(f\circ g)(x) = f(g(x))$$
$$(g\circ f)(x) = g(f(x))$$

- Chain Rule: $y=f(u)$ is differentiable at $u=g(x)$, and $g(x)$ is differentiable at x

Then $(f\circ g)(x)$ is differentiable at x and

$$(f\circ g)'(x) = f'(g(x))\cdot g'(x)$$

or $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

↑ take the derivative of y with respect to u

Example 1: Find the derivative of $y=(3x^2+1)^2$

#1. Power Rule / derivative of a polynomial

#2 Product Rule

#3 Chain Rule by assuming $y=f(u)=u^2$ with $u=3x^2+1$

then $\frac{dy}{du}=2u$; $\frac{du}{dx}=6x$

final answer shouldn't include "u"

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 2u\cdot 6x = 12ux = 12x(3x^2+1) = 36x^3+12x$$

Example 2: Find the derivative of $y=\sqrt{x+\sqrt{x}}$

Recall that $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$, assume the composite function $y=\sqrt{u}$ with $u=x+\sqrt{x}$

Then $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$; $\frac{du}{dx} = 1+\frac{1}{2\sqrt{x}}$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{2\sqrt{u}}\cdot\left(1+\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x+\sqrt{x}}}\cdot\frac{2\sqrt{x}+1}{2\sqrt{x}} = \frac{2\sqrt{x}+1}{4\sqrt{x^2+x\sqrt{x}}}$$

Example 3: Find the derivative and second derivative of $y=\frac{(x+1)^2}{1+x^2}$

use quotient rule first,

$$y' = \frac{[(x+1)^2]'(1+x^2) - (x+1)^2\cdot[(1+x^2)]'}{(1+x^2)^2}$$

to find $[(x+1)^2]'$, we assume $f(u)=u^2$ and $u=x+1$

then $[(x+1)^2]' = 2u \cdot 1 = 2(x+1)$

$$\Rightarrow y' = \frac{2(x+1)(1+x^2) - (x+1)^2 \cdot (2x)}{(1+x^2)^2} = \frac{2(x+1)(1+x^2-x^2-x)}{(1+x^2)^2} = \frac{2(x+1)(1-x)}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

use quotient rule again

$$y'' = \frac{[2(1-x^2)]' \cdot (1+x^2)^2 - 2(1-x^2) \cdot [(1+x^2)^2]'}{(1+x^2)^4}$$

to find $[(1+x^2)^2]'$, we assume $f(u) = u^2$ with $u = 1+x^2$

then $[(1+x^2)^2]' = 2u \cdot 2x = 4x(1+x^2)$

$$\Rightarrow y'' = \frac{2 \cdot (-2x)(1+x^2)^2 - 2(1-x^2) \cdot 4x(1+x^2)}{(1+x^2)^4} = \frac{-4x(1+x^2) - 8x(1-x^2)}{(1+x^2)^3}$$

$$= \frac{-12x + 4x^3}{(1+x^2)^3} = \frac{4x(x^2-3)}{(1+x^2)^3}$$

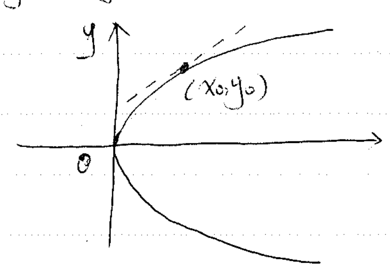
* Biology Pointview

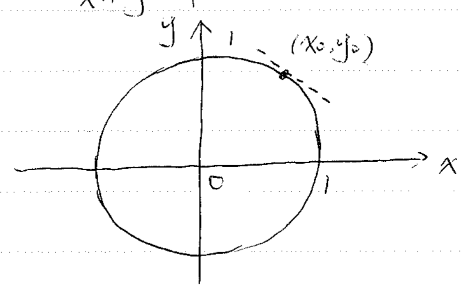
e.g. Population of large carnivores $C = f(P)$ ← Population of preys $P = g(V)$ ← Abundance of vegetation $V = h(r)$ ← rainfall

then $\frac{dC}{dr} = \frac{dC}{dP} \cdot \frac{dP}{dV} \cdot \frac{dV}{dr}$

- Implicit Differentiation

e.g. $y^2 - x = 0$ $\qquad$ $x^2 + y^2 = 1$





#1 $y_0 > 0$, $y = \sqrt{x}$ $\qquad$ $y_0 > 0$, $y = \sqrt{1-x^2}$

#2 treat y as a function of x and take the derivative of both sides of the equation.

$(y^2 - x)' = (0)' \Rightarrow 2y \cdot \frac{dy}{dx} - 1 = 0 \Rightarrow \frac{dy}{dx} = \frac{1}{2y} \Rightarrow \left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{2y_0}$