Lecture 21 (Oct. 25. 2013)

Learning Goals: ① Application of logarithm
② exponential growth
③ solutions of differential equation.

- Application of logarithm: attached at the end of lecture notes

- Exponential Growth

Recall the # of E. coli bacteria can be written as $N(n) = 2^n$

* start from 1 cell
* one cell divides into two every 20 minutes
* t (min) represents the time starting from 1 single cell, $n = \frac{t}{20}$

Write N as a function of t in the natural exponential form

$$N(t) = 2^{t/20} = (e^{\ln 2})^{t/20} = e^{\frac{\ln 2}{20} \cdot t} = e^{kt}$$

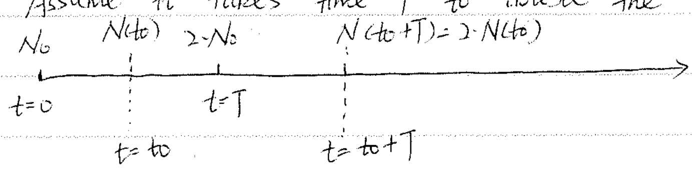
k - growth rate of E. coli bacteria $= \frac{\ln 2}{\text{doubling time}}$

20 (min) - doubling time (how long it takes to double the population)

$1/k$ - characteristic time (how long it takes to produce 1 cell)

Example 1 Suppose the population growth following $N(t) = N_0 e^{rt}$, N_0, r - constants. Find how long it takes to double the population.

Assume it takes time T to double the population



Solve $N(t_0+T) = 2 \cdot N(t_0)$ for T $\Rightarrow$ $N_0 e^{r(t_0+T)} = 2 \cdot N_0 e^{rt_0}$

$\Rightarrow$ $r(t_0+T) = \ln 2 + rt_0$

$\Rightarrow$ $T = \ln 2 / r$

* Given the exponential function with base 2 or e, be able to find the growth rate, doubling time, characteristic time, ..., meaning of the function

- Differential Equation:

e.g. $y = e^x \Rightarrow \frac{dy}{dx} = e^x = y$

e.g. $\frac{dy}{dx} = k \cdot y$, k - constant ; $\frac{dy}{dx} = k \cdot y + b$, b - constant ; $\frac{d^2y}{dx^2} = -y$

more types of DE: $\frac{dy}{dx} = f(y)$, $f(y)$ is nonlinear, $\frac{dy}{dx} = f(x,y)$ ...

Solutions of DE: function $y = f(x)$ that satisfies the differential equation.

Example 2: Given $\frac{dy}{dx} = r \cdot y$, which of the following is a solution?

① $y = 4e^{rx}$; ② $y = e^{rx} + 4$; ③ $y = e^{-rx}$

* To verify if the function is the solution of the DE, plug it to the LHS and RHS of the DE to see if they equal to each other.

① $y = 4e^{rx}$, then $\frac{dy}{dx} = 4e^{rx} \cdot r = 4re^{rx}$
$r \cdot y = r \cdot 4e^{rx} = 4re^{rx}$ $\Rightarrow y = 4e^{rx}$ is a solution

② $y = e^{rx} + 4$ then $LHS = \frac{dy}{dx} = e^{rx} \cdot r$
$RHS = r \cdot y = r(e^{rx} + 4)$ $\Rightarrow y = e^{rx} + 4$ is not a solution

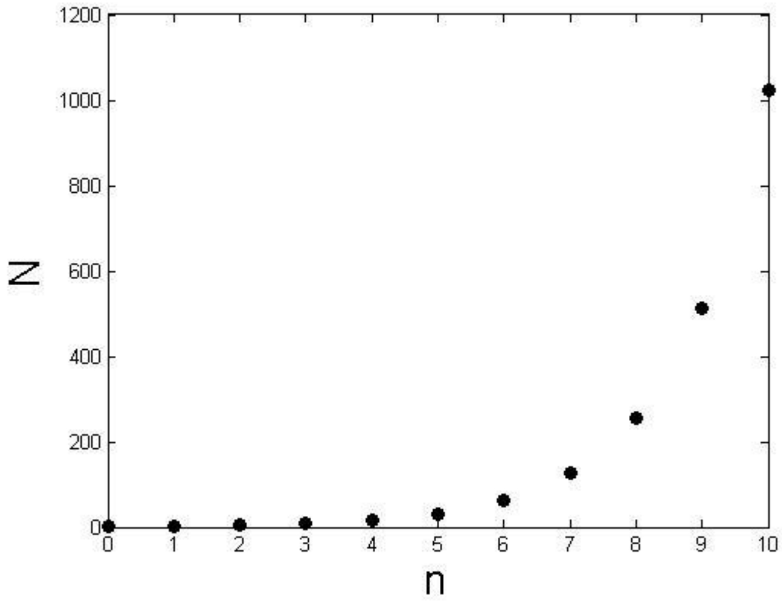
③ $y = e^{-rx}$, then $LHS = -re^{-rx}$
$RHS = re^{rx}$ $\Rightarrow y = e^{-rx}$ is not a solution

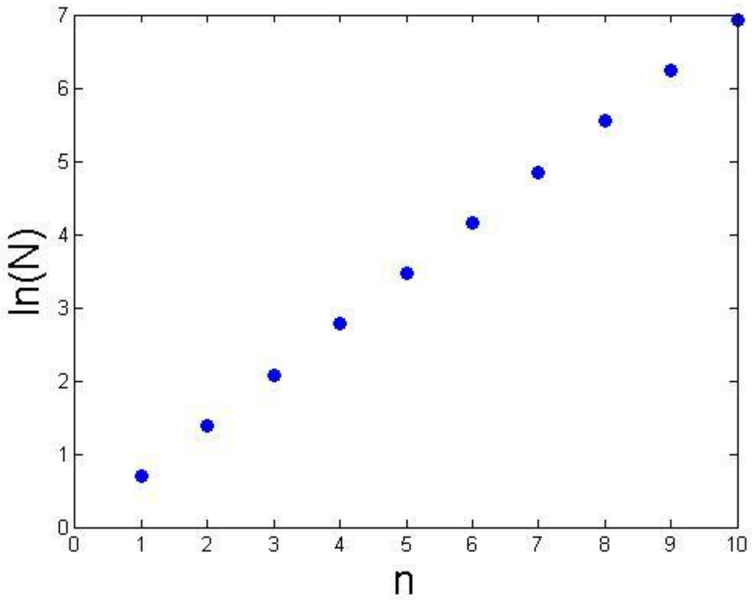
Application of Logarithm

- Plot data that varies on large scale

e.g. The number of E. Coli bacteria (N) is a function of how many times (n) the first cell doubles itself.

n	0	1	2	3	4	5	6	7	8	9	10
N	1	2	4	8	16	32	64	128	256	512	1024
ln(N)	0	ln(2)	2ln(2)	3ln(2)	4ln(2)	5ln(2)	6ln(2)	7ln(2)	8ln(2)	9ln(2)	10ln(2)



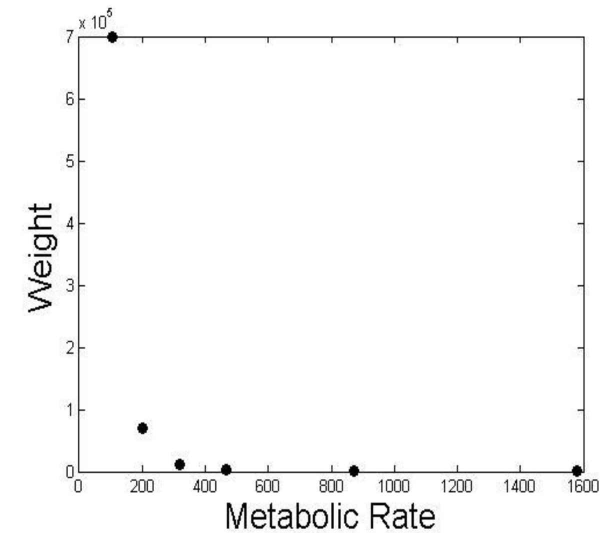


Allometry

- Mass of the animal (y) is a function of their metabolic rate (x) in the form

$$y = ax^b, \quad a > 0$$

	mouse	rat	rabbit	dog	man	horse
MR(x)	1580	873	466	318	202	106
Mass(y)	25	226	2200	11700	70000	700000



- Use logarithm scaling both variables

$$\ln(y) = \ln(a) + b\ln(x), \quad a > 0$$

	mouse	rat	rabbit	dog	man	horse
ln(x)	7.37	6.77	6.14	5.76	5.31	4.66
ln(y)	3.21	5.42	7.70	9.37	11.15	13.45