MATH 102 - MIDTERM TEST 2

University of British Columbia

Name (print):

ID number:

Section number:

Date: *November 5, 2013*
Time: *6:00 p.m. to 7:00 p.m.*
Number of pages: *7 (including cover page)*
Exam type: *Closed book*
Aids: *No calculators or other electronic aids*

Rules governing formal examinations:

Each candidate must be prepared to produce, upon request, a UBC card for identification.

Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:

- *Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/record-ers/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;*
- *Speaking or communicating with other candidates;*
- *Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.*

Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

For examiners' use only		
Section	Mark	Possible marks
MC		10
SAP		10
LAP 1		8
LAP 2		7
Total		35

Multiple choice

No partial points will be given.

1. Let $a > 0$ be some constant. Then the derivative of $y = a^x \cdot e^x$ is

   (A) $xa^{x-1} \cdot e^x$ (B) $(xa^{x-1} + a^x)e^x$ (C) $(1 + \ln(a))a^x e^x$

   (D) $(a + e)^x$ (E) $\ln(a)a^x e^x$

$y' = \ln a \, a^x \cdot e^x + a^x e^x = (1 + \ln a) a^x e^x$

2. Shown in the figure below is a function $y = f(x)$ and its inverse. Also shown are two lines. One of these lines is tangent to $f(x)$ at the point (1,2) with a slope 2/3. The other line is tangent to the inverse function and goes through the point (2,1). At what point do the two tangent lines intersect?

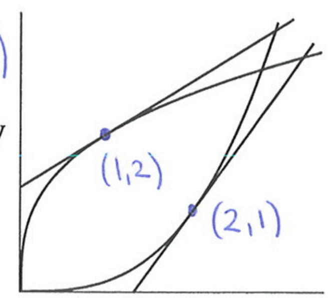
   (A) (3,3) (B) (5,4) (C) (4,5) (D) (4,4) (E) (5,5)

The original line is (using pt.-slope formula)

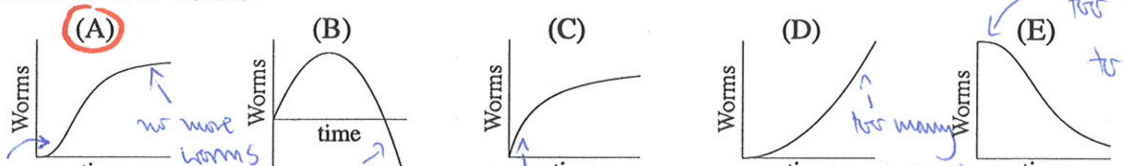
$y - 2 = \frac{2}{3}(x-1)$.

For the inverse we swap x and y so the other line is $y - 1 = \frac{3}{2}(x-2)$.

So: $\frac{2}{3}(x-1) + 2 = \frac{3}{2}(x-2) + 1 \iff \frac{5}{6}x = \frac{10}{3} \iff x = 4$ and $y = 4$



3. A robin arrives at a new garden plot at time $t = 0$ to feed on earthworms. Which of the following graphs represents the quantity of worms eaten by time t given that, at first, it takes time to dig until worms are found. Then after feeding for some time, the feeding rate slows down as it gets harder to find worms.



too many worms to start with

no more worms

time to dig in

cannot have negative worms

no time to dig

too many worms

Enter your answers to these three questions here:

MC.1 [2 pts]	MC.2 [2 pts]	MC.3 [2 pts]
C	D	A

Multiple choice (continued)

4. Which of the following functions satisfies the differential equation $\frac{dy}{dt} = y^2$?

(A) $y = (e^t)^2$ (B) $y = \frac{1}{3}t^3$ (C) $y = \frac{1}{1-t}$ (D) $y = \frac{1}{1+t}$ (E) $y = 2t$

$$\frac{dy}{dt} = \frac{-1}{(1-t)^2} \cdot (-1) = \frac{1}{(1-t)^2}$$

$$y^2 = \frac{1}{(1-t)^2}$$

So $\frac{dy}{dt} = y^2$.

Or: use integration: (not required for MATH 102)

$$\int \frac{1}{y^2}\,dy = \int dt$$

$$\Leftrightarrow \quad -\frac{1}{y} = t + C$$

$$\Leftrightarrow \quad y = \frac{1}{\hat{C} - t}$$ where $\hat{C}$ is some constant.

5. Two researchers are arguing over which model best fits the data set consisting of the points (1,2), (2,4), and (3,5). Researcher A thinks the best model is $y = 2x$ (Model A). Researcher B thinks the best model is $y = \frac{3}{2}x + \frac{1}{2}$ (Model B). Which one of the following is a reasonable claim?

(a) The sum of squared residuals for Model A is 1 while the sum of squared residuals for Model B is 1/2 so Model A is better than Model B.

(b) The sum of squared residuals for Model A is 1 while the sum of squared residuals for Model B is 1/2 so Model B is better than Model A.

(c) The sum of squared residuals for Model A is 1 while the sum of squared residuals for Model B is 1/4 so Model A is better than Model B.

(d) The sum of squared residuals for Model A is 1 while the sum of squared residuals for Model B is 1/4 so Model B is better than Model A.

Model A: $\text{Error} = \sum_{i=1}^{3} (y_i - 2x_i)^2 = (2-2)^2 + (4-4)^2 + (5-6)^2 = 1$

Model B: $\text{Error} = \sum_{i=1}^{3} \left(y_i - \left(\frac{3}{2}x_i + \frac{1}{2}\right)\right)^2 = (2-2)^2 + \left(4 - \frac{7}{2}\right)^2 + (5-5)^2 = \frac{1}{4}$

$\Rightarrow$ Error in Model B is less so that it is "better".

Enter your answers to these two questions here:

MC.4 [2 pts]	MC.5 [2 pts]
C	d

Short-answer problems

A correct answer in the box will get full points.

1. Find the tangent line of $y^2 = e^{x^2} + 5x$ at the point $(0, -1)$.

Implicitely: $\frac{d}{dx}(y^2) = \frac{d}{dx}(e^{x^2} + 5x)$

$\Leftrightarrow 2yy' = 2xe^{x^2} + 5$ +2

Plug in $(0,-1)$ to get $-2y' = 5 \Rightarrow y' = -\frac{5}{2}$ +1

So line is $y + 1 = -\frac{5}{2}(x - 0)$ +1

Answer: $y = -\frac{5}{2}x - 1$ **[4 pt]**

2. Two populations are known to satisfy the differential equations

$$\frac{dy_1}{dt} = k_1 y_1, \quad \frac{dy_2}{dt} = k_2 y_2$$

with $k_1 > k_2$ If they start out at equal levels, when will y_1 be twice as large as y_2? Your answer should be in terms of k_1 and k_2.

The solutions are $y_1 = Ae^{k_1 t}$ and $y_2 = Be^{k_2 t}$.

We want to find t such that $y_1 = 2y_2$ +1, i.e.,

$Ae^{k_1 t} = 2Be^{k_2 t}$. But $A = y_1(0) = y_2(0) = B$.

So $e^{(k_1 - k_2)t} = 2$. +1

So $t = \frac{\ln(2)}{k_1 - k_2}$. +1

Answer: **[3 pt]**

3. Psychophysics experiments indicate that the perceived intensity P of an electric shock is related to the actual intensity A by a power law. That is, $P = CA^n$ where C and n are constants. When the data is plotted with $\ln(P)$ on the vertical axis and $\ln(A)$ on the horizontal axis (a log-log plot), it lies on a line with slope 3.5.

   (a) Which of the two constants C and n can you determine from this result and what is its value?

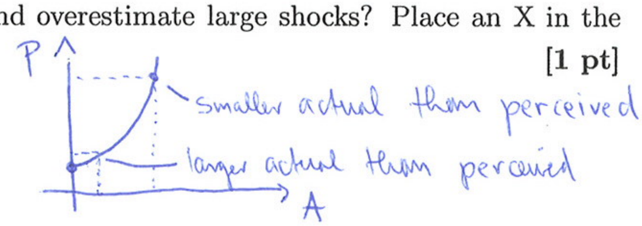
$\ln P = \ln C + n \cdot \ln(A)$ +1

In a log-log plot this is like

$y = \ln C + nX$. We know slope is 3.5. So

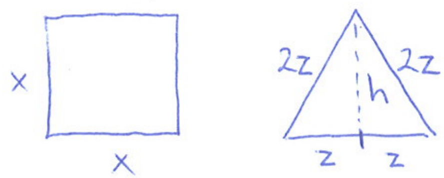
Answer: $n = 3.5$ +1 **[2 pt]**

   (b) Does this mean that humans ☐ overestimate small shocks and underestimate large shocks or that they ☒ underestimate small shocks and overestimate large shocks? Place an X in the box in front of the correct interpretation. **[1 pt]**



Long-Answer Problem #1

A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle (all sides having equal length). How much of the wire should go to the square to maximize the total area enclosed by both figures? (Area of a triangle = $\frac{1}{2}$base×height.) **[8 pt]**



$h = \sqrt{4z^2 - z^2} = \sqrt{3}\, z$ +1

Want to maximize $A = x^2 + zh = x^2 + \sqrt{3}\, z^2$ +1

Constraint: perimeter $P = 4x + 6z = 10$ +1

So $z = \dfrac{10 - 4x}{6} = \dfrac{5}{3} - \dfrac{2}{3}x$.

So $A = x^2 + \sqrt{3}\left(\frac{5}{3} - \frac{2}{3}x\right)^2$.

So $A' = 2x + 2\sqrt{3}\left(\frac{5}{3} - \frac{2}{3}x\right)\cdot\left(-\frac{2}{3}\right)$

$A'' = 2 + 2\sqrt{3}\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) > 0$ +3

So A is concave up and absolute max./min. occur at endpoints which are $x = 0$ or $x = \frac{10}{4} = \frac{5}{2}$.

$A(0) = \sqrt{3}\left(\frac{5}{3}\right)^2 = 25 \cdot \frac{\sqrt{3}}{3 \cdot 3} = 25 \cdot \frac{1}{3\sqrt{3}}$

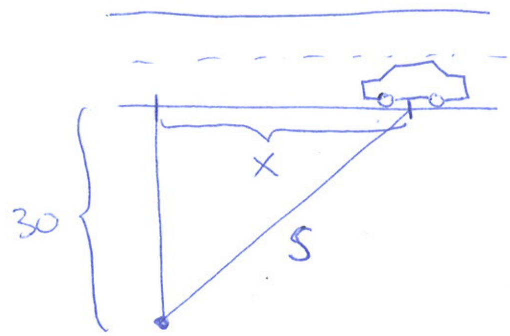
$A\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 = 25 \cdot \frac{1}{2 \cdot 2}$ +1

Since $2 \cdot 2 < 3 \cdot \sqrt{3}$ we have $A\left(\frac{5}{2}\right) > A(0)$.

This means all the wire should go to the square. +1

Long-Answer Problem #2

A police officer is standing 30 m away from a highway with a radar gun pointed at passing cars. One of the passing cars is traveling at a speed of 40 m/s. What is the rate of change of the distance between the car and the radar gun if the measurement is made one second before the car passes the point on the highway closest to the officer? **[7 pts]**
Show your work and reasoning.



Know: $\frac{dx}{dt} = -40$

Want: $\frac{ds}{dt}$ when

$x = 1s \cdot 40 \frac{m}{s}$

$= 40m$ +1

+1

$s^2 = 30^2 + x^2$

so $2ss' = 2xx'$ +1 (here $'$ means $\frac{d}{dt}$)

$\Rightarrow s' = \frac{xx'}{s}$ +1

+1

$\Rightarrow s'\big|_{x=40} = \frac{40 \cdot (-40)}{\sqrt{30^2+40^2}} = \frac{-40^2}{\sqrt{2500}} = \frac{-40^2}{50} = \frac{-160}{5}$

+1

$= -32 \frac{m}{s}$

correct sign: +1