Lecture 16 (Oct. 11, 2013)

Learning Goals: (1) Least Squares

(2) Optimal Foraging

- Least Squares:

Data points: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

(1) Given the function $y = ax$, a - unknown and needs to be determined

Sum of Squared Residual: $S = \sum_{i=1}^{n} [y_i - f(x_i)]^2$

$$= (y_1 - ax_1)^2 + (y_2 - ax_2)^2 + \cdots + (y_n - ax_n)^2$$

* in order to find the optimal function, we want to find the minimum of S so that the optimal function is "close enough" to all the data points

Formula for optimal a: $$a = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

(2) Given the function $y = M$, M - unknown constant

Apply the idea of Least Squares, the optimal M should also minimize the sum of squared residual

Then $S = (y_1 - M)^2 + (y_2 - M)^2 + \cdots + (y_n - M)^2$

$$\Rightarrow S' = \sum_{i=1}^{n} 2(y_i - M)\cdot(-1) = -2\sum_{i=1}^{n} y_i + 2n\cdot M = 0 \Rightarrow M = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y} \text{ (average/mean)}$$

(always check the boundary $\lim_{M\to\infty} S(M) = +\infty$

(3) Given the function $y = ax + b$, a, b - unknown.

This requires knowledge of multi-variable calculus

Optimal a and b are given by: $a = \dfrac{P_{avg} - \bar{x}\bar{y}}{X^2_{avg} - \bar{x}^2}$; $b = \bar{y} - a\bar{x}$;

$P_{avg} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i$; $X^2_{avg} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

- Optimal Foraging: maximize the efficiency of collecting food

define the average rate of gaining food as

$$R(t) = \frac{\text{total energy gained}}{\text{total time spent}} = \frac{n\cdot f(t)}{t_0 + n\cdot t}$$

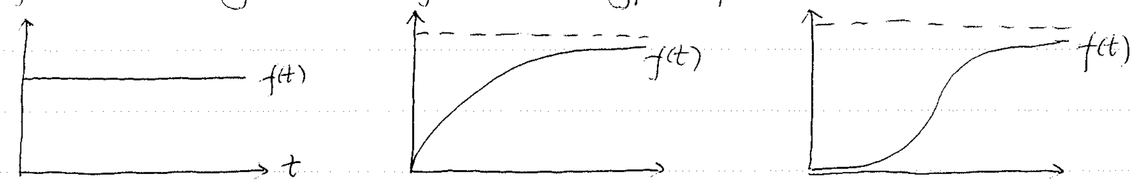
n - # of patches where the animal searches food per day

t - foraging time in one patch

$f(t)$ - energy gained in one patch depending on the time spent

t_0 - transit time between the nest and food patches

$f(t)$ can be very different for different types of patches



Example I: Given $f(t) = t\cdot(2-t)$, find t that maximize $R(t)$

$$R(t) = \frac{nt(2-t)}{t_0+nt} = n\cdot\frac{-t^2+2t}{t_0+nt}$$

$$\Rightarrow R'(t) = n\cdot\frac{(-2t+2)(t_0+nt)-(-t^2+2t)\cdot n}{(t_0+nt)^2} = n\cdot\frac{-nt^2-2t_0t+2t_0}{(t_0+nt)^2} = 0$$

$$\Rightarrow nt^2+2t_0t-2t_0=0 \qquad \Rightarrow t = \frac{-t_0 \pm\sqrt{t_0^2+2nt_0}}{n}$$ (ignore the negative value)

Compare $f(t)$ at the critical point with the boundary of the domain

$[0,2]$ or $[0,+\infty)$

$f(0)=f(2)=0$ $\qquad \lim_{t\to+\infty} f(t) = -\infty$

Confirm that $f(t)$ arrives maximum at the critical point

- More about Optimization Problem
  1) Compare the function value at the critical points and the boundary of the domain
  2) Find the domain
  3) Be aware of which quantity needs to be optimized