Midterm Practice:

1. Exactly one of the following statement about the derivative is true:
   
   (a) When \( f''(a) = 0 \), \( f(x) \) has an inflection point at \( x = a \).
   
   (b) When \( f''(a) = 0 \), \( f(x) \) has a saddle point at \( x = a \).
   
   (c) If \( f(x) \) has both a critical point and an inflection point at \( a \), then \( f(x) \) has a saddle point at \( x = a \).
   
   (d) If \( f'(x) < 0 \) for all points around \( x = a \), and \( f'(a) = 0 \), then \( f(x) \) has a local minimum.
   
   (e) If \( f(x) \) has a local maximum at \( x = a \), then \( f'(a) = 0 \) and \( f''(a) < 0 \).

2. Use the definition of the derivative to compute \( f'(x) \) where \( f(x) = 3x^2 - 6x + 3 \):

3. Find the all the tangent lines of

   \[ f(x) = x^3 - x \]

   that pass through the points \((0,2)\).

4. When \( x = 1000 \), the function \( h(x) = \frac{8x^3 - 12x^2 + x - 1}{2x^3 - 3x^4 + 4x^2 - x + 7} \) is closest to

   (a) 4   (b) 4000000   (c) 4000   (d) 0.000004   (e) .004.

5. Bonus: Suppose that at \( x = a \), \( f(x) \) changes from concave up to concave down, does \( f'(x) \) has a local minimum at \( x = a \)?
Partial Solutions:

1. (c) is the only true statement about derivatives.
   For (a), note that for \( f(x) = x^4 \), \( f''(0) = 0 \) but \( f(x) \) has a local minimum at \( x = 0 \), indeed \( f(x) \) is concave up everywhere.
   For (e), while it is true that “if \( f'(a) = 0 \) and \( f''(a) < 0 \) than \( f(x) \) has a local max,” it is not true that “if \( f(x) \) has a local maximum at \( x = a \), then \( f'(a) = 0 \) and \( f''(a) < 0 \).” The counterexample again is \( f(x) = -x^4 \), which has a local maximum at \( x = 0 \) even though \( f''(0) = 0 \).
   In general, when being asked about the classification of critical point via the second derivative, \( x^4 \) is a great test case to keep in mind.

2. 
   \[
   \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x + h)^2 - 6(x + h) + 3 - 3x^2 + 6x - 3}{h}
   = \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 6x - 6h + 3 - 3x^2 + 6x - 3}{h}
   = \lim_{h \to 0} \frac{6xh + 3h^2 - 6h}{h}
   = \lim_{h \to 0} 6x + 3h - 6
   = 6x - 6
   \]

3. Note that the point \((0, 2)\) is not a point on the graph of the function \( y = f(x) \). SO we need to find a general equation for the tangent line at any point \( x = a \) and figure out which of those tangent lines passes though \((0, 2)\). The equation of the line through the point \((a, f(a))\) is 
   \[
   y = f(a) + f'(a)(x - a)
   = a^3 - a + (3a^2 - 1)(x - a)
   \]
   To see for what value of \( a \) (if any) this line passes through \((0, 2)\), we simply plug the point into the equation for the line and solve for \( a \)
   
   \[
   2 = a^3 - a + (3a^2 - 1)(-a)
   \Rightarrow
   2 = -2a^3
   \Rightarrow
   -1 = a
   \]
   Therefore, the tangent line through \((-1, 0)\) passes through the point \((0, 2)\). The equation of the tangent line at this point it 
   \[
   y = 2(x + 1)
   \]

Try throwing both into Desmos to see this explicitly.

4. (d)