Lecture 24 (Nov. 01, 2013)

Learning Goals: Application of DE $\frac{dy}{dx} = ay + b$

• Why DE ?

Our goal is to use a function to describe two things and to analyze the behaviour of function

eg. optimization problem, related rates problem

find $y = f(x)$ then find $f(a)$, $f'(x)$, ...

But sometimes we can only build up a relationship between the rate of change and function itself

e.g. DE $\frac{dy}{dt} = f(y)$

then the question becomes, can we find the solution $y(t)$ and analyze it.

• DE $\frac{dy}{dx} = ay + b$, a, b - nonzero constants, $y(x=0) = y_0$

$\Rightarrow \frac{dz}{dx} = az$ with $z = y + \frac{b}{a}$

$\Rightarrow z = z_0 e^{at}$ with $z_0 = y_0 + \frac{b}{a}$ $\Rightarrow y = -\frac{b}{a} + (y_0 + \frac{b}{a}) e^{at}$

* be able to use this method to solve the DE for any given a, b.

• Application I: Newton's Cooling Law

Change of the temperature of an object is proportional to the difference between the temperature of the environment and the temperature of the object.

Assume $T(t)$ - temperature of the object at any time

E - temperature of the environment, constant

k - constant of proportionality

Change of temperature over time: $\frac{dT}{dt}$

order matters

Difference between the temperature of the environment and the temperature of the object: $E - T$

$\Rightarrow \frac{dT}{dt} = k \cdot (E - T)$ assume $y = T - E$ $\Rightarrow \frac{dy}{dt} = -ky$ $\Rightarrow y = y_0 e^{-kt}$

$\Rightarrow T(t) = E + (T_0 - E) e^{-kt}$, $T_0 = T(t=0)$

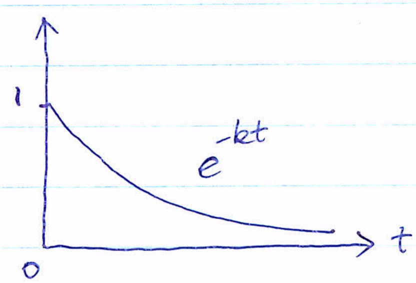
* You can get the same $T(t)$ by substituting $a = -k$, $b = k \cdot E$ in the solution of $\frac{dy}{dx} = ay + b$

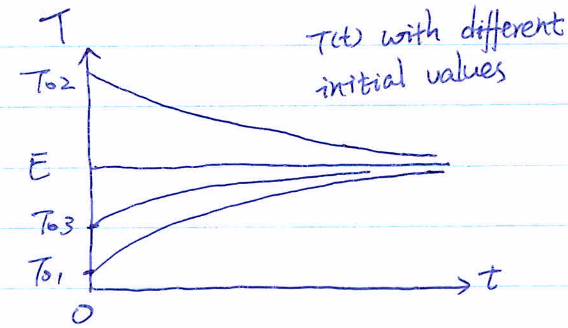
Long-term behavior of $T(t)$: $(k > 0)$

(1) if $T_0 = E$, then $T(t) = E$

(2) if $T_0 > E$, then $T_0 - E > 0$ and e^{-kt} decreases $\Rightarrow$ $T(t)$ approaches to E for $t \to +\infty$

(3) if $T_0 < E$, then $T_0 - E < 0$ and e^{-kt} decreases $\Rightarrow$ $T(t)$ approaches to E for $t \to +\infty$





• Application 2: Free falling object with friction

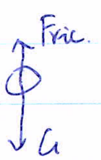
Recall that a free falling object without considering the friction has

position function $y(t) = \frac{1}{2}gt^2$ with $y(0)=0$, $v(0)=0$

velocity function $v(t) = \frac{dy}{dt} = g\cdot t$, $v(0)=0$

acceleration function $a(t) = \frac{dv}{dt} = g$, g - gravity constant

Given that the $\boxed{\text{friction}}$ is $\boxed{\text{proportional}}$ to the $\boxed{\text{velocity}}$ of the object, what is $v(t)$?



$m\cdot a(t) = m\cdot g - k\cdot v(t)$, k - proportional constant.

$\Rightarrow m\cdot \frac{dv}{dt} = m\cdot g - k\cdot v \Rightarrow \frac{dv}{dt} = -\frac{k}{m}v + g$

To solve the DE, assume $u = v - \frac{mg}{k}$ then $\frac{du}{dt} = -\frac{k}{m}\cdot u$

$\Rightarrow u(t) = u_0 e^{-\frac{k}{m}\cdot t}$ with $u_0 = u(t=0) = v(t=0) - \frac{mg}{k} = -\frac{mg}{k}$

$\Rightarrow v(t) = \frac{mg}{k} - \frac{mg}{k}\cdot e^{-\frac{k}{m}\cdot t} = \frac{mg}{k}(1 - e^{-\frac{k}{m}\cdot t})$.