

# Today

- Power rule for  $y=x^{m/n}$
- Exponential review (brief - 2 CQs)
- The derivative of exponential functions
- Reminders:
  - Midterm 2 is in less than 2 weeks.
  - OSH 5 due Monday

For  $y=x^n$ ,  $y'=nx^{n-1}$  when  $n$  is an integer. What if  $n$  is rational?

For  $y = x^{1/m}$  with  $m$  an integer,  
what is  $y'$ ?

Rewrite the equation as  $y^m = x$  and take  
derivatives:

(A)  $m y^{m-1} = 1$

(B)  $m y^{m-1} y' = 1$

(C)  $m y^{m-1} = x$

(D)  $y'^m = 1$

For  $y = x^{1/m}$  with  $m$  an integer,  
what is  $y'$ ?

Rewrite the equation as  $y^m = x$  and take  
derivatives:

(A)  $m y^{m-1} = 1$

(B)  $m y^{m-1} y' = 1$

(C)  $m y^{m-1} = x$

(D)  $y'^m = 1$

For  $y = x^{1/m}$  with  $m$  an integer,  
what is  $y'$ ?

Solve  $m y^{m-1} y' = 1$  for  $y'$ :

(A)  $y' = 1/(m y^{m-1})$

(B)  $y' = 1/(m x^{m-1})$

(C)  $y' = 1/(m x^{(m-1)/m})$

(D)  $y' = (1/m) x^{1-1/m}$

For  $y = x^{1/m}$  with  $m$  an integer,  
what is  $y'$ ?

Solve  $m y^{m-1} y' = 1$  for  $y'$ :

(A)  $y' = 1/(m y^{m-1})$  <--- ok but we know  $y$ !

(B)  $y' = 1/(m x^{m-1})$

(C)  $y' = 1/(m x^{(m-1)/m}) = (1/m) x^{(1/m)-1}$

(D)  $y' = (1/m) x^{1-1/m}$

# Power rule for differentiation – summary

- When  $p$  is an integer and  $y = x^p$ ,  $y' = px^{p-1}$ .
  - We showed this using the def. of deriv.
- When  $p = 1/n$  and  $y = x^p$ ,  $y' = px^{p-1}$ .
  - We just used implicit diff. to show this.
- When  $p = m/n$  and  $y = x^p$ ,  $y' = px^{p-1}$ .
  - You do this (exercise in course notes).

# Exponential functions

Which of the following is an exponential function?

(A)  $x^n$

(B)  $2^x$

(C)  $e^2$

(D)  $\ln(x)$



# Exponential functions

Which of the following is an exponential function?

(A)  $x^n$       <--power function

(B)  $2^x$       <--variable in the exponent!

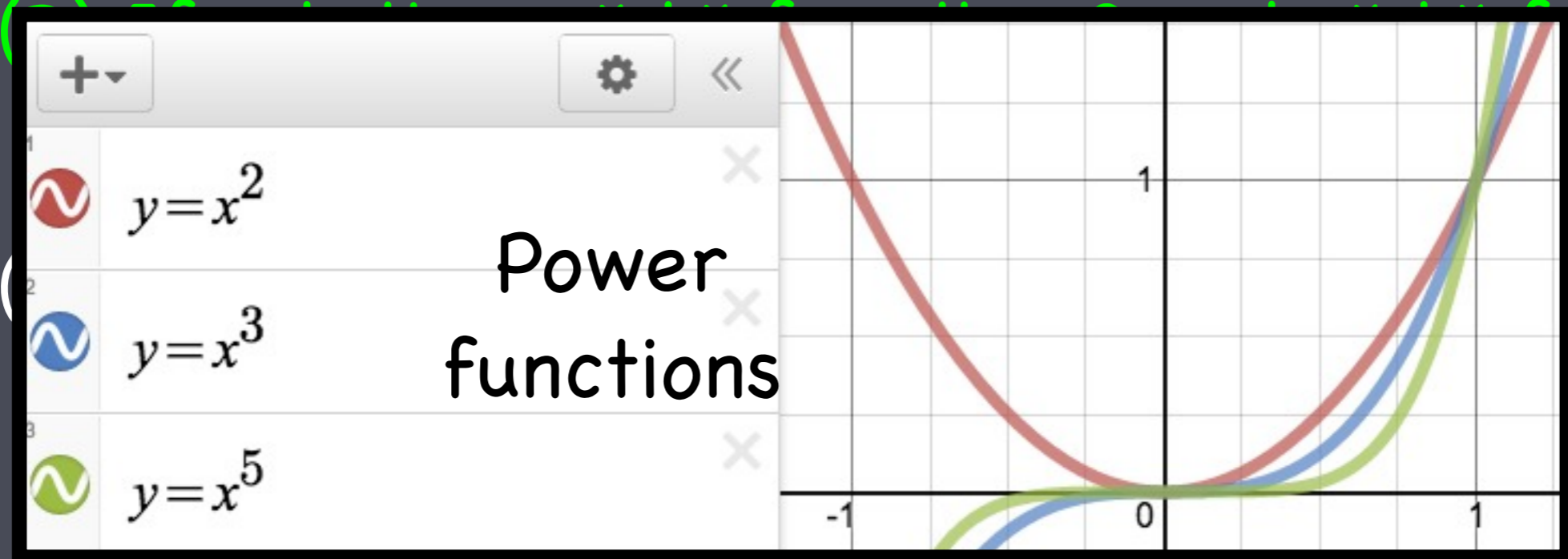
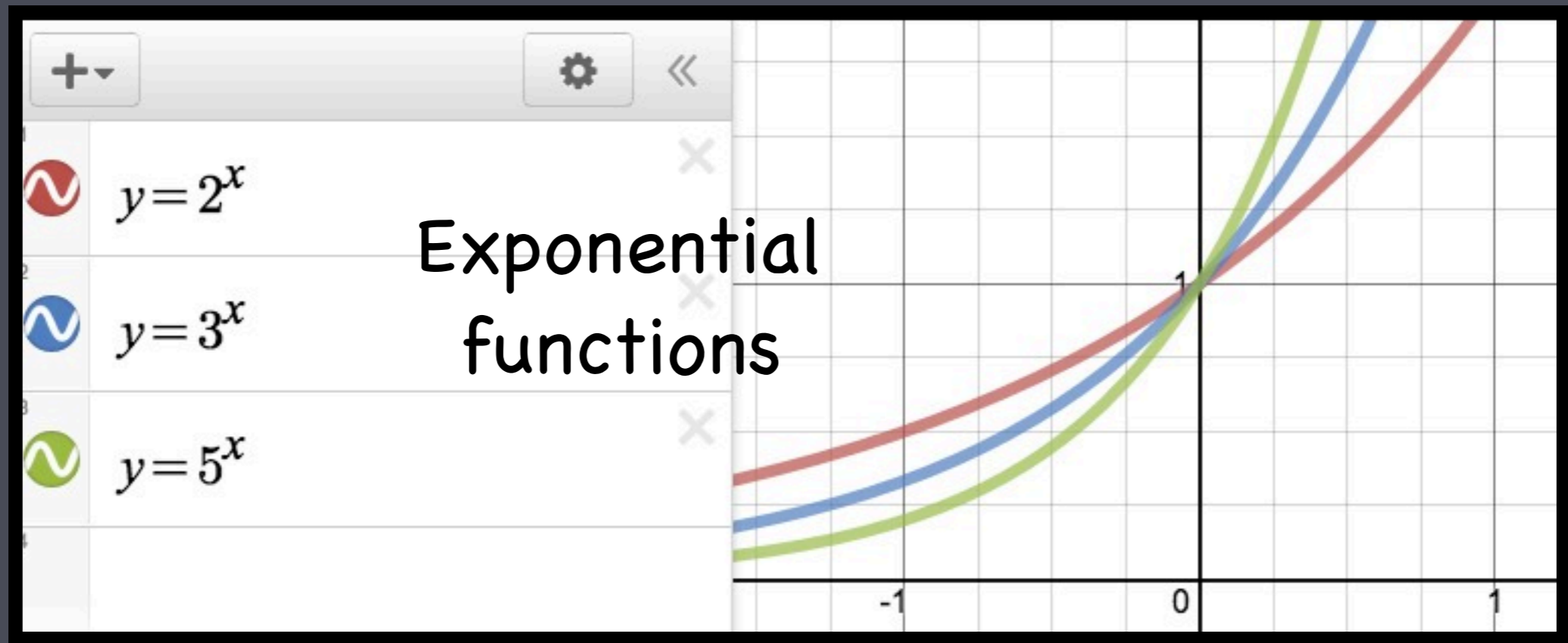
(C)  $e^2$       <--an exp. calculation but not an exp. function

(D)  $\ln(x)$       <--the INVERSE of an exp. function

# Exponential functions $a^x$ where $a > 1$ ...

- (A) All go through the point (1,1).
- (B) All go through the point (0,0).
- (C) All go through the point (1,0).
- (D) If  $a < b$  then  $a^x < b^x$  for all  $x > 0$  and  $a^x > b^x$  for all  $x < 0$ .
- (E) If  $a < b$  then  $a^x < b^x$  for all  $x > 1$  and  $a^x > b^x$  for all  $x < 1$ .

# Exponential functions $a^x$



The derivative of  $f(x)=a^x$  is...

(A)  $f'(x) = xa^{x-1}$ .

(B)  $f'(x) = ax^{a-1}$ .

(C)  $f'(x) = (a^{x+h}-a^x)/h$

(D)  $f'(x) = a^x$ .

(E)  $f'(x) = Ca^x$ .

Checking your high school memory here.

# Derivative of $f(x)=2^x$

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{2^{x+h} - 2^x}{h} = \lim_{h \rightarrow 0} \frac{2^x(2^h - 1)}{h} \\&= 2^x \lim_{h \rightarrow 0} \frac{2^h - 1}{h} = C_2 2^x \\&\quad \lim_{h \rightarrow 0} \frac{2^h - 1}{h} \approx 0.69338746258 = C_2\end{aligned}$$

# Derivative of $f(x)=a^x$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\ &= a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h} = C_a a^x \\ &\quad \lim_{h \rightarrow 0} \frac{a^h - 1}{h} \approx ?? = C_a \end{aligned}$$

# Find a special value of $a$ .

• When is  $C_a = \lim_{h \rightarrow 0} \frac{a^h - 1}{h} = 1$ ?

Want  $\frac{a^h - 1}{h} \approx 1$  (for  $h$  small)

With  $h=0.1$ ,  $a \approx 2.5937424601$ .

With  $h=0.01$ ,  $a \approx 2.70481382942$ .

With  $h=0.001$ ,  $a \approx 2.71692393224$ .

With  $h=0.0001$ ,  $a \approx 2.71814592682$ .

With  $h=0.00001$ ,  $a \approx 2.71826823719$ .

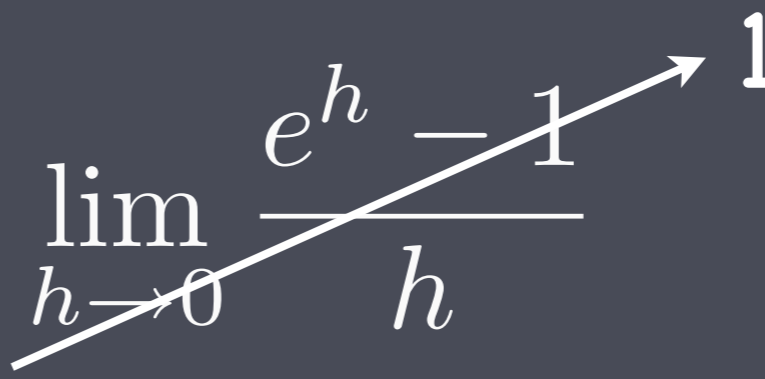
$a^h - 1 \approx h$

$a^h \approx h + 1$

$a \approx (h + 1)^{\frac{1}{h}}$

What is this special  $a$  value?  $a=e!$

We just found a function that is its own derivative!  $f(x)=e^x$ .

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} \\ &= e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h} \\ &= e^x \end{aligned}$$


This is precisely how  $e$  is defined - the number whose exponential function is its own derivative.



# Differential equations

- What real number is the same as its own square?
  - Equivalent to asking “what  $x$  satisfies the equation  $x=x^2$ ?”
  - Call this an algebraic equation.
- What function is equal to its own derivative?
  - Equivalent to asking “what  $f(x)$  satisfies  $f'(x)=f(x)$ ?”
  - Call this a “differential equation”. (DE)

DE example: Which of the following satisfies  $f'(x)=f(x)$ ?

(A)  $f(x) = 2^x$

(B)  $f(x) = e^x$

(C)  $f(x) = x^{-1}$

(D)  $f(x) = -x^{-1}$

DE example: Which of the following satisfies  $f'(x)=f(x)$ ?

(A)  $f(x) = 2^x$

(B)  $f(x) = e^x$

(C)  $f(x) = x^{-1}$

(D)  $f(x) = -x^{-1}$

DE example: Which of the following satisfies  $f'(x)=f(x)^2$ ?

(A)  $f(x) = 2^x$

(B)  $f(x) = e^x$

(C)  $f(x) = x^{-1}$

(D)  $f(x) = -x^{-1}$

DE example: Which of the following satisfies  $f'(x)=f(x)^2$ ?

(A)  $f(x) = 2^x$

(B)  $f(x) = e^x$

(C)  $f(x) = x^{-1}$

(D)  $f(x) = -x^{-1}$