MATH 102 - MIDTERM TEST 1

University of British Columbia

Name (print): Solution key

ID number:

Section number:

Date: *October 3, 2013*
Time: *6:00 p.m. to 7:00 p.m.*
Number of pages: *8 (including cover page)*
Exam type: *Closed book*
Aids: *No calculators or other electronic aids*

Rules governing formal examinations:

Each candidate must be prepared to produce, upon request, a UBC card for identification.

Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:

- *Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/record-ers/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;*
- *Speaking or communicating with other candidates;*
- *Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.*

Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

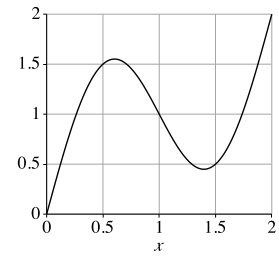
Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

For examiners' use only		
Section	Mark	Possible marks
MC		10
SAP		5
LAP 1		5
LAP 2		10
LAP 3		10
Total		40

Multiple choice

No partial points will be given for work shown.

1. **Rate of change:** Shown in the figure below is the graph of a function. You are told that the average rate of change of this function over the interval $a \leq x \leq b$ is $r_{avg} = 1$. If $a = 0$, which of the following is possible?



(A) $b = 0.5$ (B) $b = 1$ (C) $b = 1.25$

(D) $b = 1.5$ (E) $b = 1.75$

2. **The derivative:** Which of the following describes the derivative of a function $f(x)$?

   (A) It is defined as $\dfrac{f(x+h) - f(x)}{h}$.

   (B) It is the line we see when we zoom into the graph of $f(x)$.

   (C) It is the average rate of change of $f(x)$ over the interval $0 < x < h$.

   (D) More than one of the above answers are correct.

   (E) None of the above are correct.

3. **Critical and inflection points:** Which ONE of the following statements is always true for all differentiable functions f that satisfies the stated condition.

   (A) When $f''(a) = 0$ the function $y = f(x)$ has an inflection point at $x = a$.

   (B) If $f(x)$ has a local maximum at a then $f'(a) < 0$ and $f''(a) = 0$.

   (C) If $f(x)$ has a local minimum at a then $f'(a) = 0$ and $f''(a) < 0$.

   (D) If $f(x)$ is increasing and concave up at a then $f'(a) > 0$ and $f''(a) > 0$.

   (E) Both the functions $f(x) = x^3$ and $f(x) = x^6$ have inflection points at $x = 0$.

Enter your answers to these four questions here:

MC.1 [**2 pts**]	MC.2 [**2 pts**]	MC.3 [**2 pts**]
B	E	D

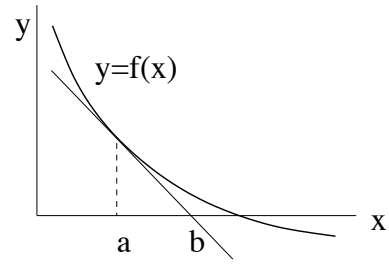
Multiple choice (continued)

4. **Tangent lines:** As shown in the figure below, the tangent line to the graph of $f(x)$ at $x = a$ intersects the x-axis at $x = b$. Which of the following expressions gives the value of b?

(A) $b = a - \dfrac{f(a)}{f'(a)}$, (B) $b = a + \dfrac{f(a)}{f'(a)}$,

(C) $b = a + \dfrac{f'(b)}{f(b)}$, (D) $b = f(a) - f'(a)a$,

(E) $b = f(a) + f'(a)(x - a)$.



5. **Critical points:** In order for the function $f(x) = \frac{1}{3}x^3 + 2x^2 + qx + 2$ to have any critical points, we require that the constant q satisfy which of the following statements?

(A) $0 \leq q \leq 16$ (B) $q \geq 2$ (C) $4 \leq q \leq 16$

(D) $q \leq -4$ or $q \geq 4$ (E) $q \leq 4$

Enter your answers to these two questions here:

MC.4 [**2 pts**]	MC.5 [**2 pts**]
A	E

Short-answer problems

A correct answer in the box will get full points. Work shown may get partial points.

1. **Derivative rules:** Given two functions $f(x)$ and $g(x)$, suppose we know

$$f(5) = 1, \quad f'(5) = -3, \quad g(5) = 2, \quad g'(5) = 10.$$

(a) Compute $h'(5)$ where $h(x) = f(x)g(x)$.

Answer: 4 **[1 pt]**

(b) Compute $k'(5)$ where $k(x) = \dfrac{f(x)}{g(x)}$.

Answer: -4 **[1 pt]**

2. **Limits:** Calculate the following limits.

(a) $\displaystyle\lim_{x\to 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2}$.

$$\frac{\sqrt{x}-\sqrt{2}}{x-2} \cdot \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}} = \frac{x-2}{(x-2)(\sqrt{x}+\sqrt{2})} = \frac{1}{\sqrt{x}+\sqrt{2}}$$

Answer: $\frac{1}{2\sqrt{2}}$ or $\frac{\sqrt{2}}{4}$ **[1 pt]**

(b) $\displaystyle\lim_{x\to\infty} \frac{x}{x+1}$.

Answer: 1 **[1 pt]**

(c) $\displaystyle\lim_{x\to-\infty} \frac{x^2}{x^3+1}$.

Answer: 0 **[1 pt]**

Long-Answer Problem #1

Use the definition of the derivative to compute the derivative of $f(x) = -4x^2 + 5x - 1$. Show your work. No points will be given for using the power rule. **[5 pts]**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{1pt}$$

1pt

$$= \lim_{h \to 0} \frac{-4(x+h)^2 + 5(x+h) - 1 - (-4x^2 + 5x - 1)}{h}$$

$$= \lim_{h \to 0} \frac{-4x^2 - 8xh - 4h^2 + 5x + 5h - 1 + 4x^2 - 5x + 1}{h}$$

$$= \lim_{h \to 0} \frac{-8xh - 4h^2 + 5h}{h} \quad \text{1pt} \quad = \lim_{h \to 0} -8x - 4h + 5 \quad \text{1pt}$$

$$\boxed{f'(x) = -8x + 5} \quad \text{1pt}$$

Long-Answer Problem #2

Show your work and reasoning.

(a) Find all minima, maxima and inflection points of $f(x) = x^4 - x^2$. Sketch the graph of f. **[8 pts]**

$f'(x) = 4x^3 - 2x = 2x(2x^2 - 1) = 0$

$\Rightarrow x = 0, \quad x = \pm\frac{1}{\sqrt{2}}$

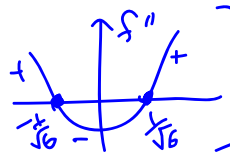
1pt 1pt

$f''(x) = 12x^2 - 2$

$f''(0) = -2 < 0$ so $\boxed{x = 0 \text{ is a max}}$ 1pt

$f''(\pm\frac{1}{\sqrt{2}}) = 12\,\frac{1}{2} - 2 = 6 - 2 = 4 > 0$ so $\boxed{x = \pm\frac{1}{\sqrt{2}} \text{ are mins.}}$ 1pt

$f''(x) = 12x^2 - 2 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{6}}$ 1pt



so f'' changes sign at both $\boxed{x = \pm\frac{1}{\sqrt{6}}}$ and so these are $\boxed{\text{inflection pts}}$

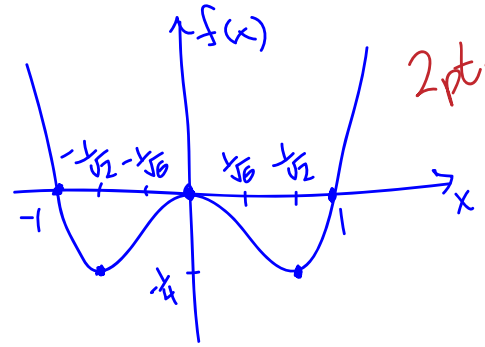
1pt

$f(x) = x^4 - x^2 = 0$

$x^2(x^2 - 1) = 0$

$x = 0, \pm 1$

$\frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$



2pts

(b) Consider the function $f(x) = x^4 + ax^3 - x^2$. Find all values of a for which f has no inflection points or show that no such values exist. **[2 pts]**

$f'(x) = 4x^3 + 3ax^2 - 2x$

$f''(x) = 12x^2 + 6ax - 2 = 0 \Rightarrow 6x^2 + 3ax - 1 = 0$

f has no inflection pts when $9a^2 + 24 < 0$ 1pt

but there are $\boxed{\text{no } a \text{ values}}$ for which that happens.

1pt

Long-Answer Problem #3

Find the point or points on the graph of f whose tangent line to f goes through the point $(0, 1/2)$ where

$$f(x) = \frac{1}{1+x^2}.$$

Sketch your tangent lines on the graph provided at the bottom of this page. Label the relevant points. Show your work and reasoning. **[10 pts]**

$$f'(x) = \frac{0\cdot(1+x^2) - 1\cdot 2x}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2} \quad \text{2pts}$$

Tangent line at $(a, f(a))$: $y = \frac{-2a}{(1+a^2)^2}(x-a) + \frac{1}{1+a^2}$

Goes through $(0, \frac{1}{2})$: $\frac{1}{2} = \frac{-2a}{(1+a^2)^2}(-a) + \frac{1}{1+a^2}$ 2pts

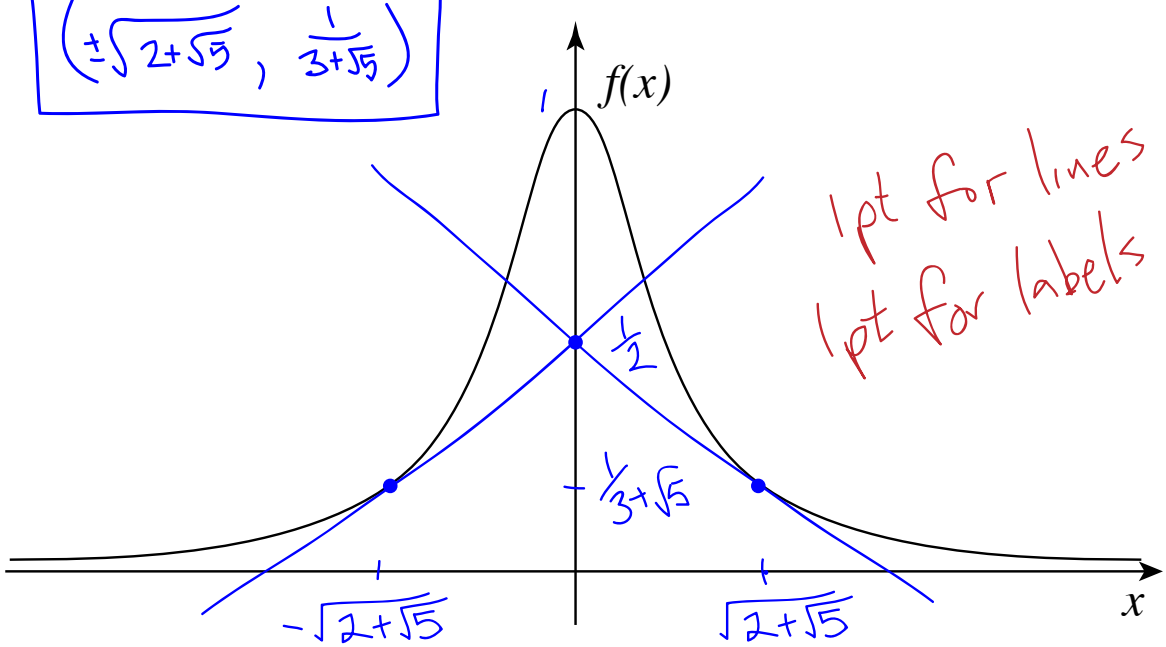
$$\frac{1}{2}(1+a^2)^2 = 2a^2 + 1 + a^2 = 3a^2 + 1$$

$$1 + 2a^2 + a^4 = 6a^2 + 2 \Rightarrow a^4 - 4a^2 - 1 = 0 \quad \text{1pt}$$

$$a^2 = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5} \quad \text{1pt} \qquad (2-\sqrt{5} < 0)$$

$$a = \pm\sqrt{2+\sqrt{5}} \ \text{1pt}, \quad f(a) = \frac{1}{1+2+\sqrt{5}} = \frac{1}{3+\sqrt{5}} \ \text{1pt}$$

$$\boxed{\left(\pm\sqrt{2+\sqrt{5}}, \frac{1}{3+\sqrt{5}}\right)}$$



1pt for lines
1pt for labels

This page may be used for rough work. It will not be marked.