Name (print): Solution key

ID number:

Section number:

Date: October 3, 2013
Time: 6:00 p.m. to 7:00 p.m.
Number of pages: 8 (including cover page)
Exam type: Closed book
Aids: No calculators or other electronic aids

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- Speaking or communicating with other candidates;
- PURPOSELY EXPOSING WRITTEN PAPERS TO THE VIEW OF OTHER CANDIDATES OR IMAGING DEVICES. THE PLEA OF ACCIDENT OR FORGETFULNESS SHALL NOT BE RECEIVED.

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Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

For examiners’ use only

<table>
<thead>
<tr>
<th>Section</th>
<th>Mark</th>
<th>Possible marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>SAP</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>LAP 1</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>LAP 2</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>LAP 3</td>
<td></td>
<td>10</td>
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<tr>
<td>Total</td>
<td></td>
<td>40</td>
</tr>
</tbody>
</table>
Multiple choice

No partial points will be given for work shown.

1. **Rate of change:** Shown in the figure below is the graph of a function. You are told that the average rate of change of this function over the interval \( a \leq x \leq b \) is \( r_{avg} = 1 \). If \( a = 0 \), which of the following is possible?

(A) \( b = 0.5 \)  
(B) \( b = 1 \)  
(C) \( b = 1.25 \)  
(D) \( b = 1.5 \)  
(E) \( b = 1.75 \)

2. **The derivative:** Which of the following describes the derivative of a function \( f(x) \)?

(A) It is defined as \( \frac{f(x + h) - f(x)}{h} \).  
(B) It is the line we see when we zoom into the graph of \( f(x) \).  
(C) It is the average rate of change of \( f(x) \) over the interval \( 0 < x < h \).  
(D) More than one of the above answers are correct.  
(E) None of the above are correct.

3. **Critical and inflection points:** Which ONE of the following statements is always true for all differentiable functions \( f \) that satisfies the stated condition.

(A) When \( f''(a) = 0 \) the function \( y = f(x) \) has an inflection point at \( x = a \).  
(B) If \( f(x) \) has a local maximum at \( a \) then \( f'(a) < 0 \) and \( f''(a) = 0 \).  
(C) If \( f(x) \) has a local minimum at \( a \) then \( f'(a) = 0 \) and \( f''(a) < 0 \).  
(D) If \( f(x) \) is increasing and concave up at \( a \) then \( f'(a) > 0 \) and \( f''(a) > 0 \).  
(E) Both the functions \( f(x) = x^3 \) and \( f(x) = x^6 \) have inflection points at \( x = 0 \).

Enter your answers to these four questions here:

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<tbody>
<tr>
<td>B</td>
<td>E</td>
<td>D</td>
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</tbody>
</table>
Multiple choice (continued)

4. **Tangent lines:** As shown in the figure below, the tangent line to the graph of \( f(x) \) at \( x = a \) intersects the x-axis at \( x = b \). Which of the following expressions gives the value of \( b \)?

\[
\begin{align*}
(A) \quad & b = a - \frac{f(a)}{f'(a)}, \\
(B) \quad & b = a + \frac{f(a)}{f'(a)}, \\
(C) \quad & b = a + \frac{f'(b)}{f(b)}, \\
(D) \quad & b = f(a) - f'(a)a, \\
(E) \quad & b = f(a) + f'(a)(x - a).
\end{align*}
\]

5. **Critical points:** In order for the function \( f(x) = \frac{1}{3}x^3 + 2x^2 + qx + 2 \) to have any critical points, we require that the constant \( q \) satisfy which of the following statements?

\[
\begin{align*}
(A) \quad & 0 \leq q \leq 16 \\
(B) \quad & q \geq 2 \\
(C) \quad & 4 \leq q \leq 16 \\
(D) \quad & q \leq -4 \text{ or } q \geq 4 \\
(E) \quad & q \leq 4
\end{align*}
\]

Enter your answers to these two questions here:

<table>
<thead>
<tr>
<th>MC.4 [2 pts]</th>
<th>MC.5 [2 pts]</th>
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<tbody>
<tr>
<td>A</td>
<td>E</td>
</tr>
</tbody>
</table>
Short-answer problems

A correct answer in the box will get full points. Work shown may get partial points.

1. **Derivative rules:** Given two functions $f(x)$ and $g(x)$, suppose we know
   
   \[ f(5) = 1, \quad f'(5) = -3, \quad g(5) = 2, \quad g'(5) = 10. \]

   (a) Compute $h'(5)$ where $h(x) = f(x)g(x)$.

   Answer: \[4\] [1 pt]

   (b) Compute $k'(5)$ where $k(x) = \frac{f(x)}{g(x)}$.

   Answer: \[-4\] [1 pt]

2. **Limits:** Calculate the following limits.

   (a) \[ \lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2}. \]

   \[ \frac{\sqrt{x} - \sqrt{2}}{x - 2} \cdot \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}} = \frac{x - 2}{(x - 2)(\sqrt{x} + \sqrt{2})} = \frac{1}{\sqrt{x} + \sqrt{2}} \]

   Answer: \[\frac{1}{2\sqrt{2}}\] or \[\frac{\sqrt{2}}{4}\] [1 pt]

   (b) \[ \lim_{x \to \infty} \frac{x}{x + 1}. \]

   Answer: \[1\] [1 pt]

   (c) \[ \lim_{x \to -\infty} \frac{x^2}{x^3 + 1}. \]

   Answer: \[0\] [1 pt]
Long-Answer Problem #1

Use the definition of the derivative to compute the derivative of $f(x) = -4x^2 + 5x - 1$. Show your work. No points will be given for using the power rule. [5 pts]

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{-4(x+h)^2 + 5(x+h) - 1 - (-4x^2 + 5x - 1)}{h}$$

$$= \lim_{h \to 0} \frac{-4x^2 - 8xh - 4h^2 + 5x + 5h - 1 + 4x^2 - 5x + 1}{h}$$

$$= \lim_{h \to 0} \frac{-8xh - 4h^2 + 5h}{h}$$

$$= \lim_{h \to 0} -8x - 4h + 5$$

$$f'(x) = -8x + 5$$ [5 pts]
Long-Answer Problem #2

Show your work and reasoning.

(a) Find all minima, maxima and inflection points of \( f(x) = x^4 - x^2 \). Sketch the graph of \( f \). \([8 \text{ pts}]\)

\[ f'(x) = 4x^3 - 2x = 2x(2x^2 - 1) = 0 \]
\[ \Rightarrow x = 0, \quad x = \pm \frac{1}{\sqrt{2}} \]

\[ f''(x) = 12x^2 - 2 \]
\[ f''(0) = -2 < 0 \quad \text{so} \quad x = 0 \text{ is a max} \]
\[ f''(\pm \frac{1}{\sqrt{2}}) = 12 \cdot \frac{1}{2} - 2 = 6 - 2 = 4 > 0 \quad \text{so} \quad x = \pm \frac{1}{\sqrt{2}} \text{ are mins} \]

\[ f''(x) = 12x^2 - 2 = 0 \implies x = \pm \frac{1}{\sqrt{6}} \quad 1\text{ pt} \]

\[ \text{so } f'' \text{ changes sign at both } x = \pm \frac{1}{\sqrt{6}} \text{ and so these are inflection pts} \]

\[ f(x) = x^4 - x^2 = 0 \]
\[ x^2(1 - x^2) = 0 \]
\[ x = 0, \pm 1 \]
\[ \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \]

(b) Consider the function \( f(x) = x^4 + ax^3 - x^2 \). Find all values of \( a \) for which \( f \) has no inflection points or show that no such values exist. \([2 \text{ pts}]\)

\[ f'(x) = 4x^3 + 3ax^2 - 2x \]
\[ f''(x) = 12x^2 + 6ax - 2 = 0 \implies 6x^2 + 3ax - 1 = 0 \]
\[ \text{f has no inflection pts when } 9a^2 + 24 < 0 \quad 1\text{ pt} \]

but there are no \( a \) values for which that happens. \[1\text{ pt}\]
Long-Answer Problem #3

Find the point or points on the graph of \( f \) whose tangent line to \( f \) goes through the point \((0, 1/2)\) where

\[
f(x) = \frac{1}{1 + x^2}.
\]

Sketch your tangent lines on the graph provided at the bottom of this page. Label the relevant points. Show your work and reasoning. [10 pts]

\[
f'(x) = \frac{0 \cdot (1 + x^2) - 1 \cdot 2x}{(1 + x^2)^2} = \frac{-2x}{(1 + x^2)^2} \quad 2 \text{ pts}
\]

Tangent line at \((a, f(a))\):

\[
y = \frac{-2a}{(1 + a^2)^2} (x-a) + \frac{1+a^2}{1+a^2} \quad 2 \text{ pts}
\]

Goes through \((0, 1/2)\):

\[
\frac{1}{2} = \frac{-2a}{(1 + a^2)^2} (-a) + \frac{1}{1+a^2} \quad 2 \text{ pts}
\]

\[
\frac{1}{2} (1+a^2)^2 = 2a^2 + 1 + a^2 = 3a^2 + 1 \\
1 + 2a^2 + a^4 = 6a^2 + 2 \quad \Rightarrow \quad a^4 - 4a^2 - 1 = 0 \quad 1 \text{ pt}
\]

\[
a^2 = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5} \quad 1 \text{ pt} \quad (2 - \sqrt{5} < 0)
\]

\[
a = \pm \sqrt{2 + \sqrt{5}} \quad 1 \text{ pt} , \quad f(a) = \frac{1}{1 + 2 + \sqrt{5}} = \frac{1}{3 + \sqrt{5}} \quad 1 \text{ pt}
\]

\[
\left( \pm \sqrt{2 + \sqrt{5}}, \frac{1}{3 + \sqrt{5}} \right) \]

\[
1 \text{ pt for lines} \\
1 \text{ pt for labels}
\]
This page may be used for rough work. It will not be marked.