Lecture 26 (Nov. 06, 2013)

Learning Goal: Euler's method (estimate the solution to the DE numerically)

· Initial value problem: $\begin{cases} \frac{dy}{dt} = f(y) \\ y(t=0) = y_0 \end{cases}$

e.g. $f(y) = ay + b \Rightarrow y(t) = -\frac{b}{a} + (y_0 + \frac{b}{a})e^{at}$

$f(y) = y^2 \Rightarrow y(t) = ?$

$f(y) = \sqrt[3]{y} \Rightarrow y(t) = ?$

} we can estimate $y(t)$ at any time t based on using slope of the secant line as the approximation of the slope of the tangent line

Assume Δt - small time increment

$y(0+\Delta t) = y(\Delta t)$ is the function value we want to estimate

slope of the secant line passing through $(0, y_0)$ and $(\Delta t, y(\Delta t))$ is given by

$\frac{y(\Delta t) - y_0}{\Delta t - 0}$, which is the approximation of $\left.\frac{dy}{dt}\right|_{t=0} = f(y_0)$

$\Rightarrow \frac{y(\Delta t) - y_0}{\Delta t} \approx f(y_0) \Rightarrow y(\Delta t) \approx y_1 = y_0 + f(y_0)\cdot\Delta t$

Similarly, to estimate $y(2\Delta t)$, we have $\frac{y(2\Delta t) - y(\Delta t)}{2\Delta t - \Delta t} \approx \left.\frac{dy}{dt}\right|_{t=\Delta t} = f(y(\Delta t))$

replace $y(\Delta t)$ by its approximation y_1, we have $y(2\cdot\Delta t) \approx y_2 = y_1 + f(y_1)\cdot\Delta t$

· Euler's method: Given the initial value problem $\frac{dy}{dt} = f(y)$ with $y(t=0) = y_0$

to estimate the solution value at $t=T$

we subdivide the time length into n steps (n - positive integer), $\Delta t = \frac{T}{n}$

assume t_k - the time point at the end of k-th step

$y(t_k)$ - actural value of the solution at $t = t_k$

y_k - approximation of the solution at $t = t_k$, $k = 1, 2, \cdots, n$

we have $y(t=0) = y_0$

$y(t=t_1) \approx y_1 = y_0 + f(y_0)\cdot\Delta t$

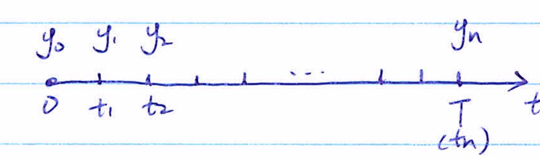
$y(t=t_2) \approx y_2 = y_1 + f(y_1)\cdot\Delta t$

$\vdots$

$y(t=t_k) \approx y_k = y_{k-1} + f(y_{k-1})\cdot\Delta t$

$\vdots$

$y(t=T) \approx y_n = y_{n-1} + f(y_{n-1})\cdot\Delta t$



Example 1: Newton's law of cooling $\frac{dT}{dt} = 0.2(10-T)$ has solution $T(t) = 10 + (T_0 - 10)e^{-0.2t}$

Given $T(0)=0$, estimate $T(0.4)$ with $\Delta t = 0.4$ and $\Delta t = 0.2$.

① $\Delta t = 0.4$. $n=1$. $T(0.4) \approx y_1 = y_0 + f(y_0)\cdot \Delta t = 0 + 0.2\times(10-0)\times 0.4 = 8$

② $\Delta t = 0.2$, $n=2$. $y_1 = y_0 + f(y_0)\cdot\Delta t = 0 + 0.2\times(10-0)\times 0.2 = 0.4$

$T(0.4) \approx y_2 = y_1 + f(y_1)\cdot \Delta t = 0.4 + 0.2\times(10-0.4)\times 0.2 = 0.784$

Notice: compare the two approximations with the actural solution value $T(0.4) = 0.7688$

Both of them overestimate the solution value

larger than the actural solution value

Smaller Δt provides a better approximation.