Lecture 19 (Oct. 21, 2013)

Learning Goals: ① Properties of exponential function

② introduction of Differential Equation

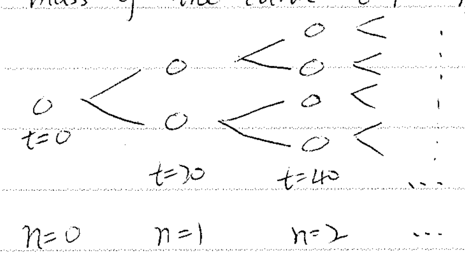
• Exponential Growth:

e.g. A single cell of the bacterium E. Coli would divide every 20 minutes.

One cell of E. Coli can produce a super-colony equal in weight to the entire earth in a short time.

mass of one cell: 10^{-12} kg

mass of the earth: 6×10^{24} kg



use n - # of times a cell divides

N - # of cells after n-th division

n	N(n)
n=0	$1=2^0$
n=1	$2=2^1$
n=2	$4=2\times2=2^2$
⋮	⋮
n=m	2^m

m - positive integer

$\Rightarrow N(n)=2^n$

* To find how much time for the weight of the colony equal to the weight of earth,

we have $10^{-12}\times N(n)=6\times10^{24}$

$\Rightarrow 10^{-12}\times2^n=6\times10^{24} \Rightarrow 2^n=6\times10^{36} \Rightarrow n=122.17$

$\Rightarrow T=\frac{122.17}{3}=40.72$ (hours) (a cell doubles three times after 1 hour)

* Make the function more general, we replace the base 2 by a positive constant a and the power n by real number x

• Exponential Function: $f(x)=a^x$, $a>0$

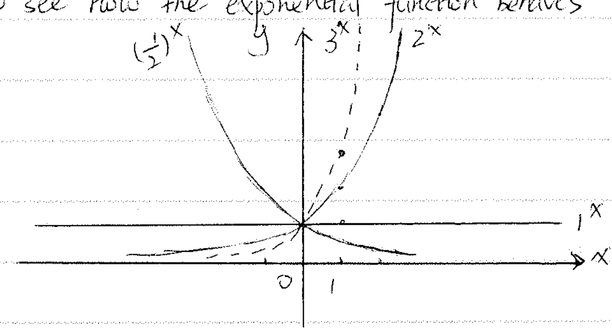
* sketch $f(x)=2^x$, $f(x)=1^x$, $f(x)=3^x$, $f(x)=(\frac{1}{2})^x=2^{-x}$ to see how the exponential function behaves

$a=2$. $f(0)=1$, $f(1)=2$, $f(2)=4$, $f(-1)=\frac{1}{2}$, $f(-2)=\frac{1}{4}$ ...

$a=3$. $f(0)=1$. $f(1)=3$ $f(2)=9$. $f(-1)=\frac{1}{3}$. $f(-2)=\frac{1}{9}$

$a=1$ $f(x)=1^x=1$.

$a=\frac{1}{2}$ $f(0)=1$. $f(1)=\frac{1}{2}$. $f(2)=\frac{1}{4}$, $f(-1)=2$. $f(-2)=4$



Properties: ① $a^b \cdot a^c = a^{b+c}$, $b, c \in \mathbb{R}$

② $(a^b)^c = a^{b\cdot c}$,

③ $a^x > 0$ for all values of x

④ $a^0 = 1$ and $a^1 = a$. All the exponential functions go through the point $(0,1)$

⑤ $a > 1$, $f(x) \to 0$ for $x \to -\infty$; $f(x) \to +\infty$ for $x \to +\infty$

$a < 1$, $f(x) \to 0$ for $x \to +\infty$. $f(x) \to 0$ for $x \to -\infty$

⑥ $f(x) = a^x$ is defined, continuous and differentiable for all real numbers of x

Derivative: By definition. $f(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} = a^x \cdot \lim_{h\to 0} \frac{a^h - 1}{h}$

By $f'(0) = \lim_{h\to 0} \frac{a^{0+h} - a^0}{h} = \lim_{h\to 0} \frac{a^h - 1}{h}$, we have $f'(x) = a^x \cdot f'(0)$

varies for different a

* Sketch the tangent line at $(0,1)$ for a^x with different a

Notice that the slope of the tangent line is in $(-\infty, +\infty)$

There exists a special base to make $f'(0) = 1$, we call it "e".

• Natural exponential function: $f(x) = e^x$, $e \approx 2.71826$

e is defined by $\lim_{h \to 0} (1+h)^{\frac{1}{h}} = e$

Property: $f'(0) = \lim_{h\to 0} \frac{e^h - 1}{h} = 1$

Derivative: $f'(x) = e^x \cdot f'(0) = e^x$

or assume $y = f(x) = e^x$, we have $\frac{dy}{dx} = e^x = y$

differential equation

(equation that connects the function and its derivative)