Today

- Introduction to Differential Equations
- Linear DE ($y' = ky$)
- Nonlinear DE (e.g. $y' = y\,(1-y)$)
- Qualitative analysis (phase line)

Differential equations (DE)

Carbon dating: The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write down a DE for the amount of C-14 at time t.

(A) $C'(t) = k\ C(t)$ where $k>0$.

(B) $C'(t) = k\ C(t)$ where $k<0$.

(C) $C(t) = C_0e^{kt}$.

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- DE + IC is called an Initial Value Problem (IVP)

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- Determine half-life/doubling time from data or k.

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object falling through air	pendulum under water	water draining from a vessel

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$$N' = bN-(cN)N = bN-cN^2$$

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where r=b and K=1/c. This is a nonlinear DE because of the N^2.

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- Qualitative analysis - extract information about the general solution without solving.
    - Steady states
    - Slope fields
    - Stability of steady states
    - Plotting y' versus y (state space/phase line)

$$\underset{\text{velocity}}{\underset{\uparrow}{x'}} = x(1 - \underset{\text{position}}{\underset{\uparrow}{x}})$$

- **Steady state.** Where can you stand so that the DE tells you not to move?

(A) x=-1

(B) x=0

(C) x=1/2

(D) x=1

This is the logistic eq with r=1, K=1.

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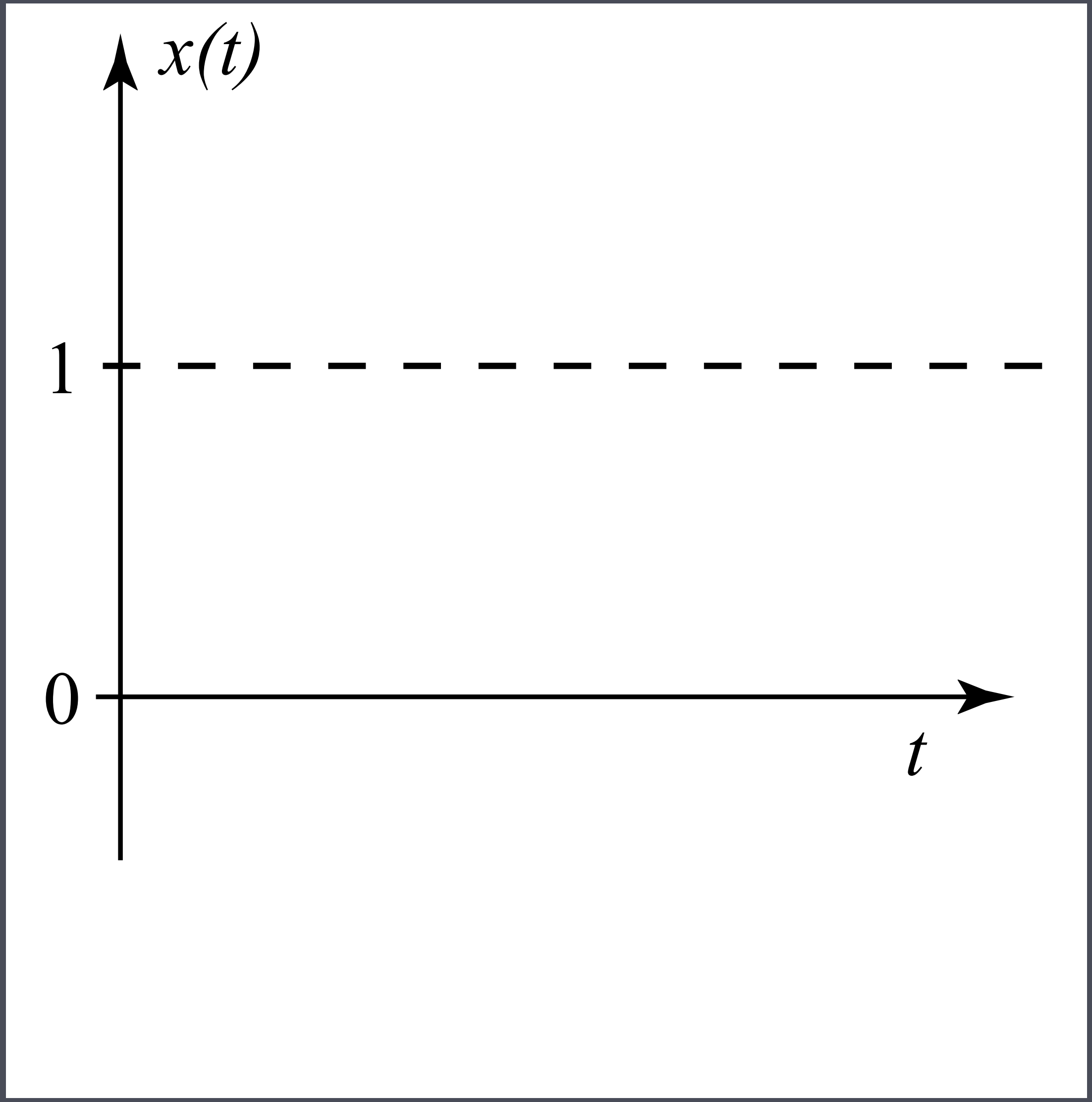


A steady state is a constant solution.

$$y' = -y(y-1)(y+1)$$

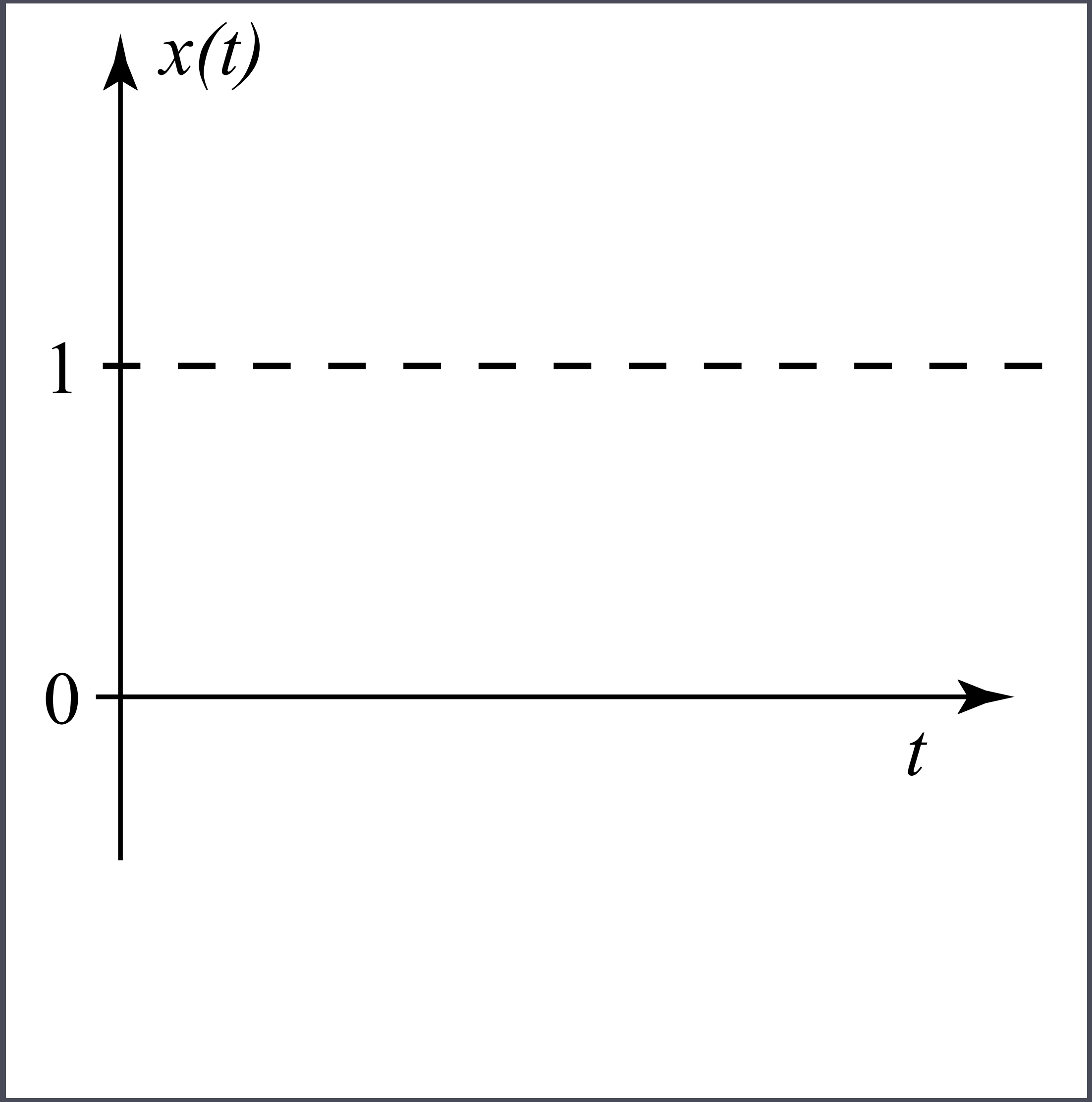
- What are the steady states of this equation?

$$x' = x(1 - x)$$

velocity

position

Slope field



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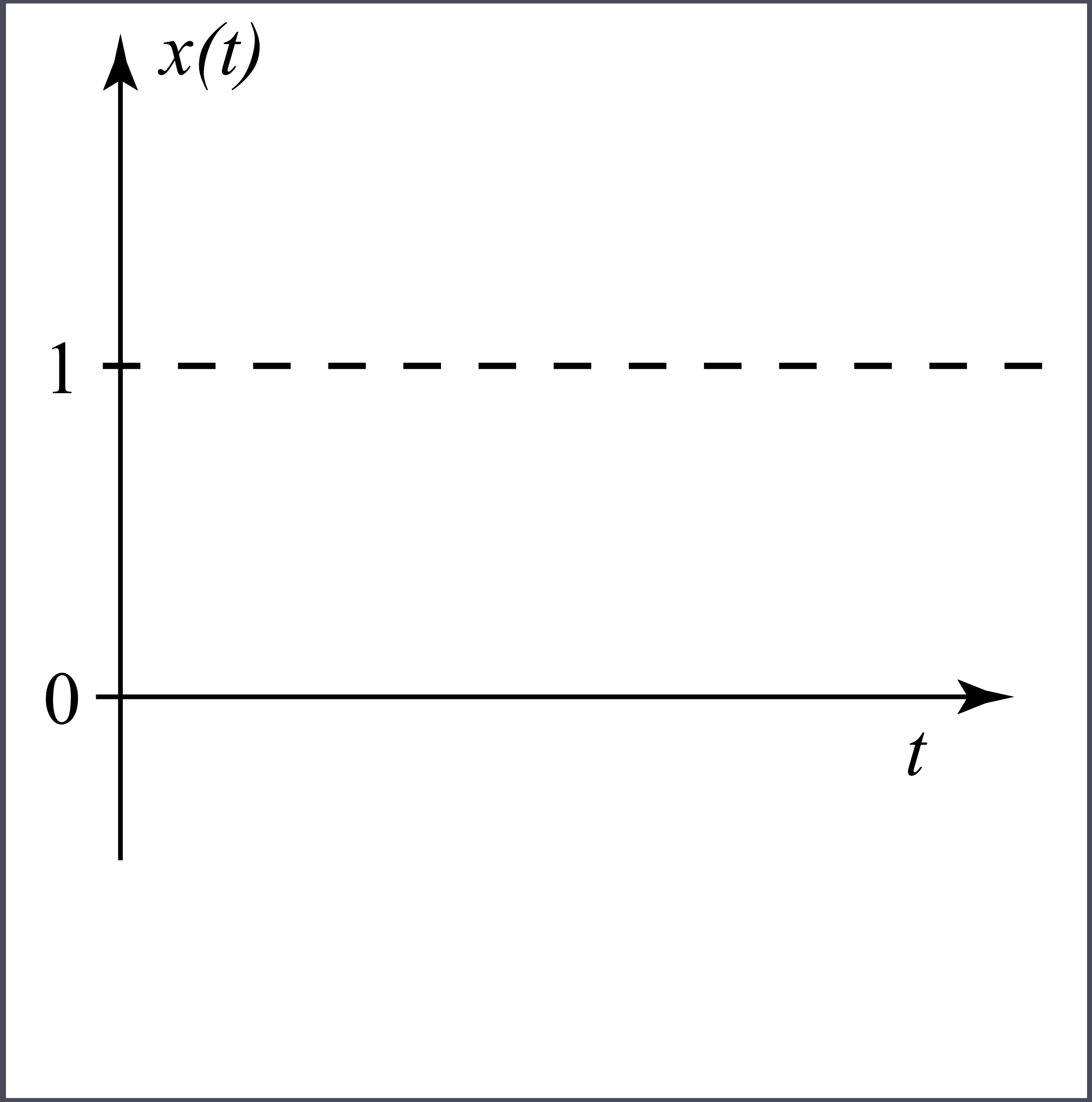

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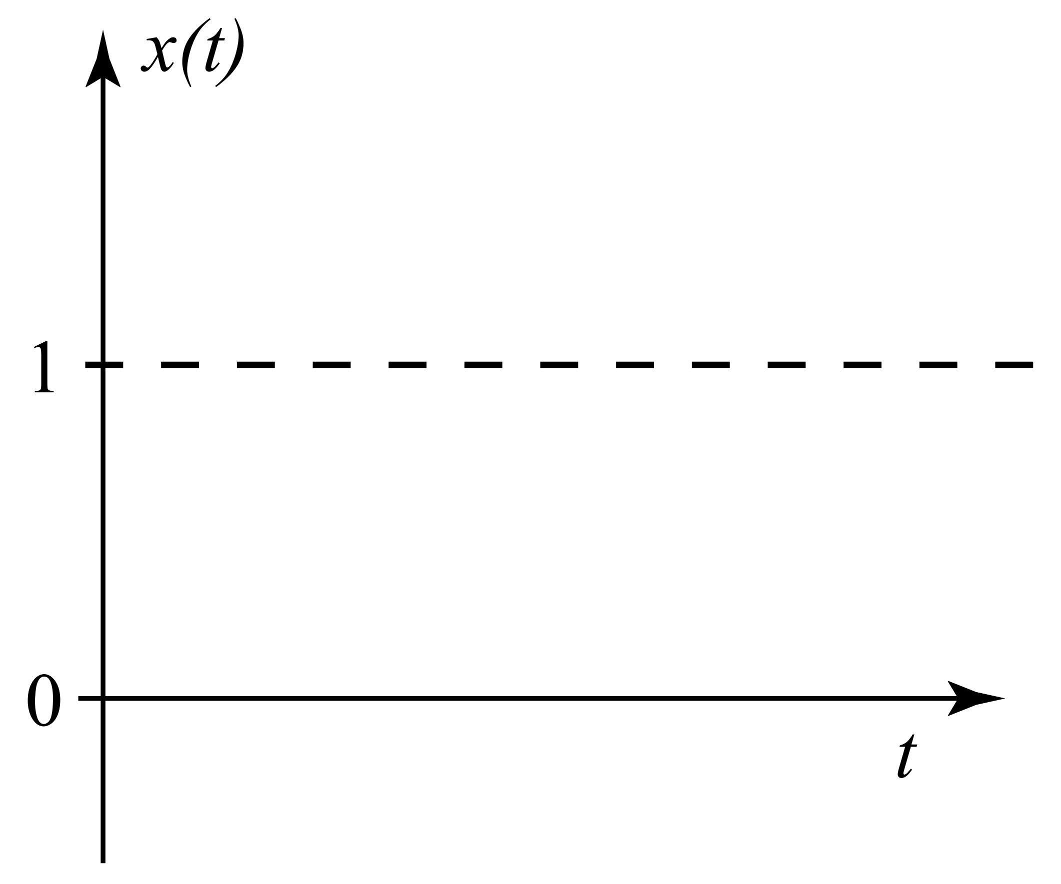

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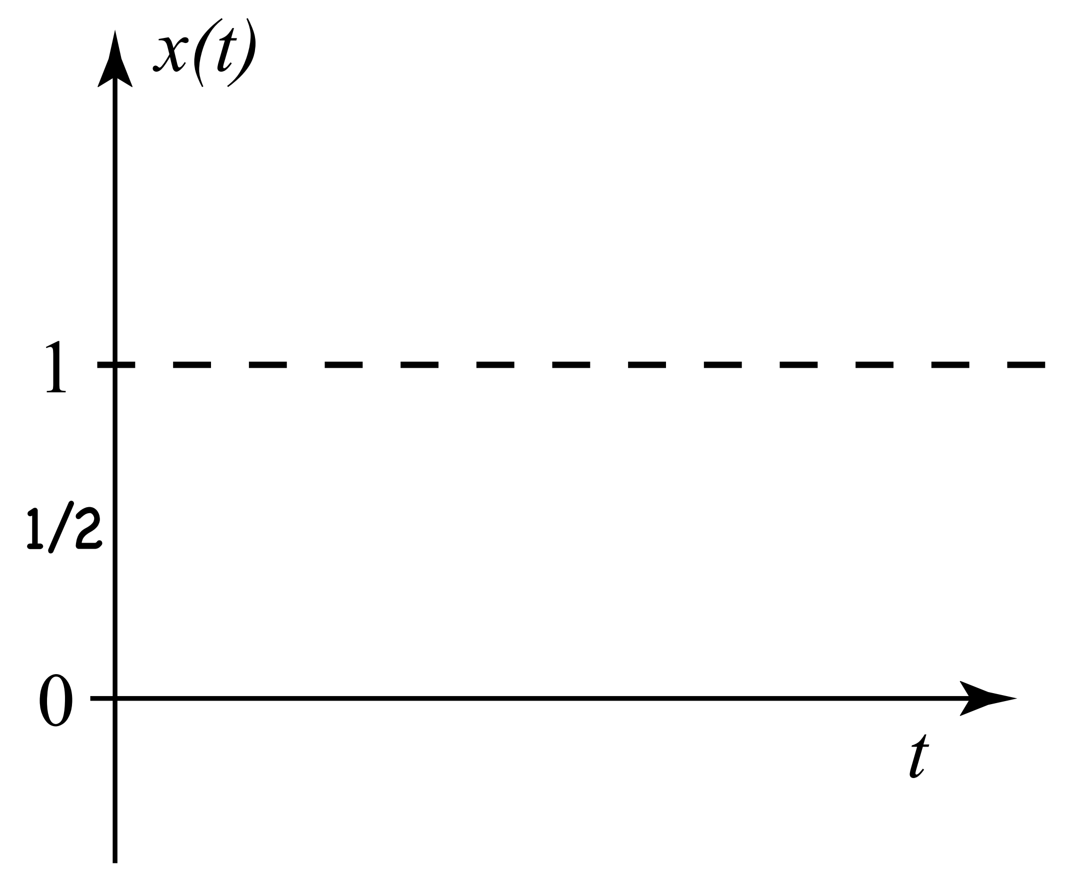

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  - When x(t)=1/2 what is x'?

(A) 0

(B) 1/4

(C) 1/2

(D) 1

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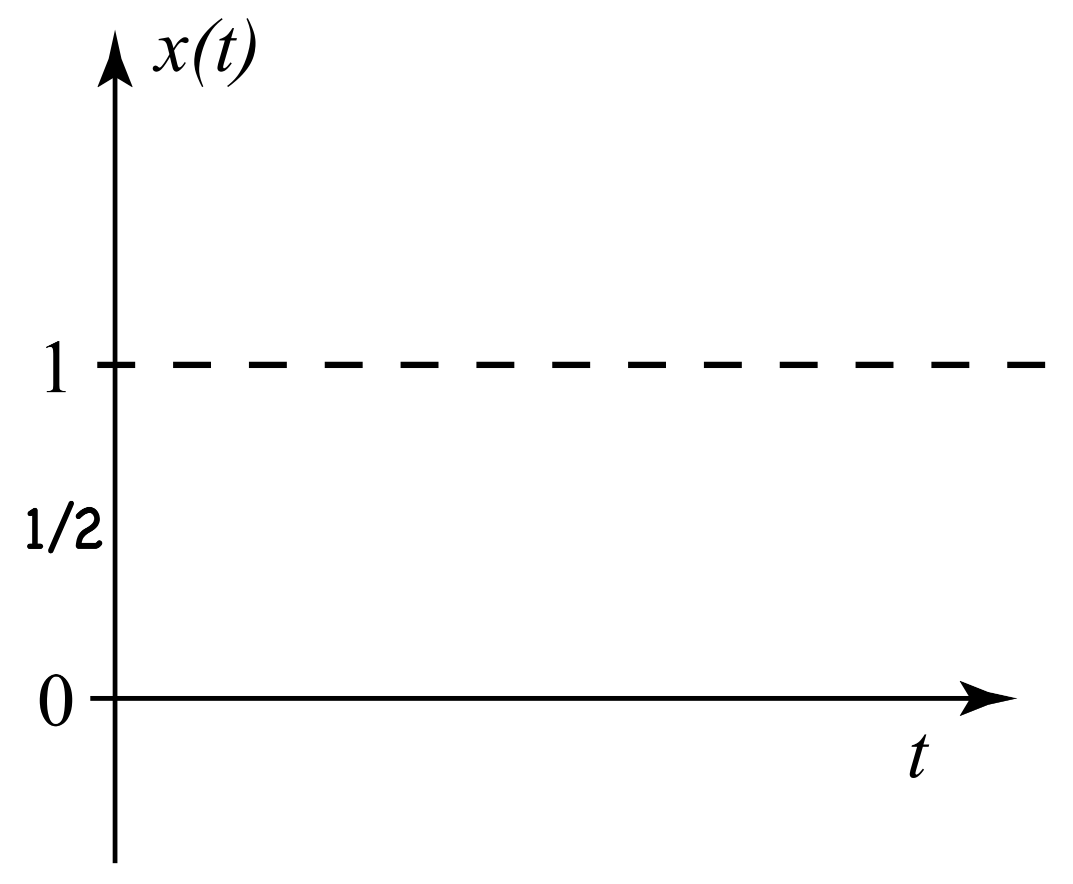

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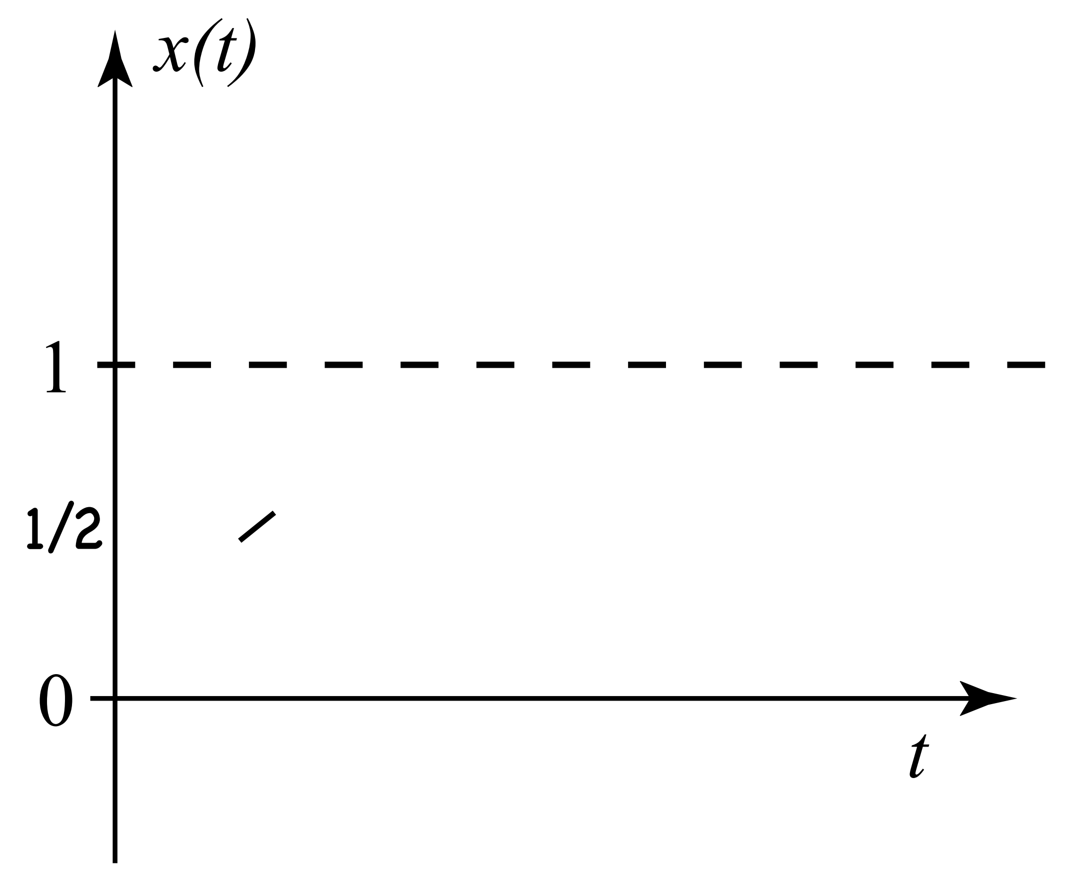

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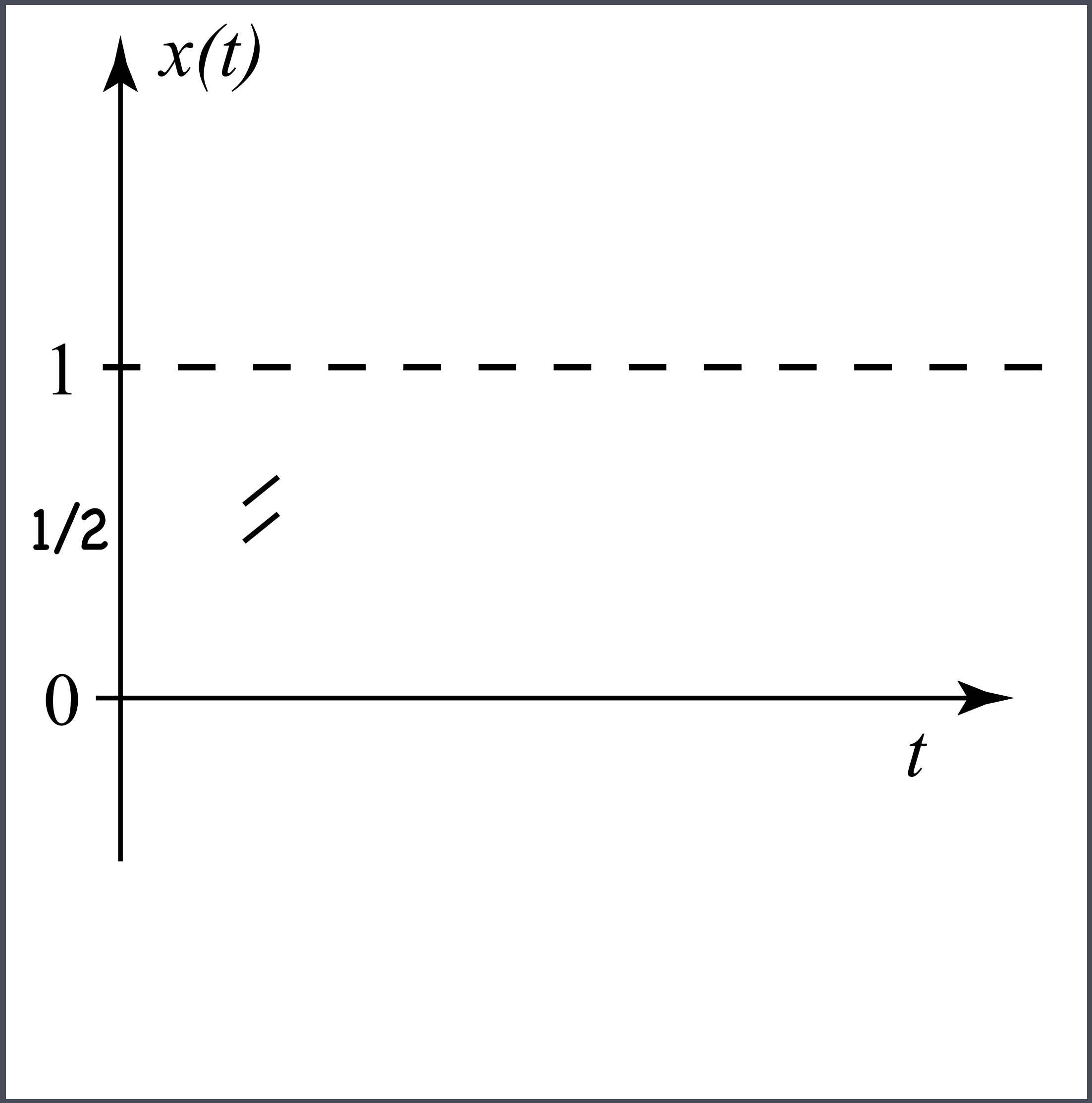

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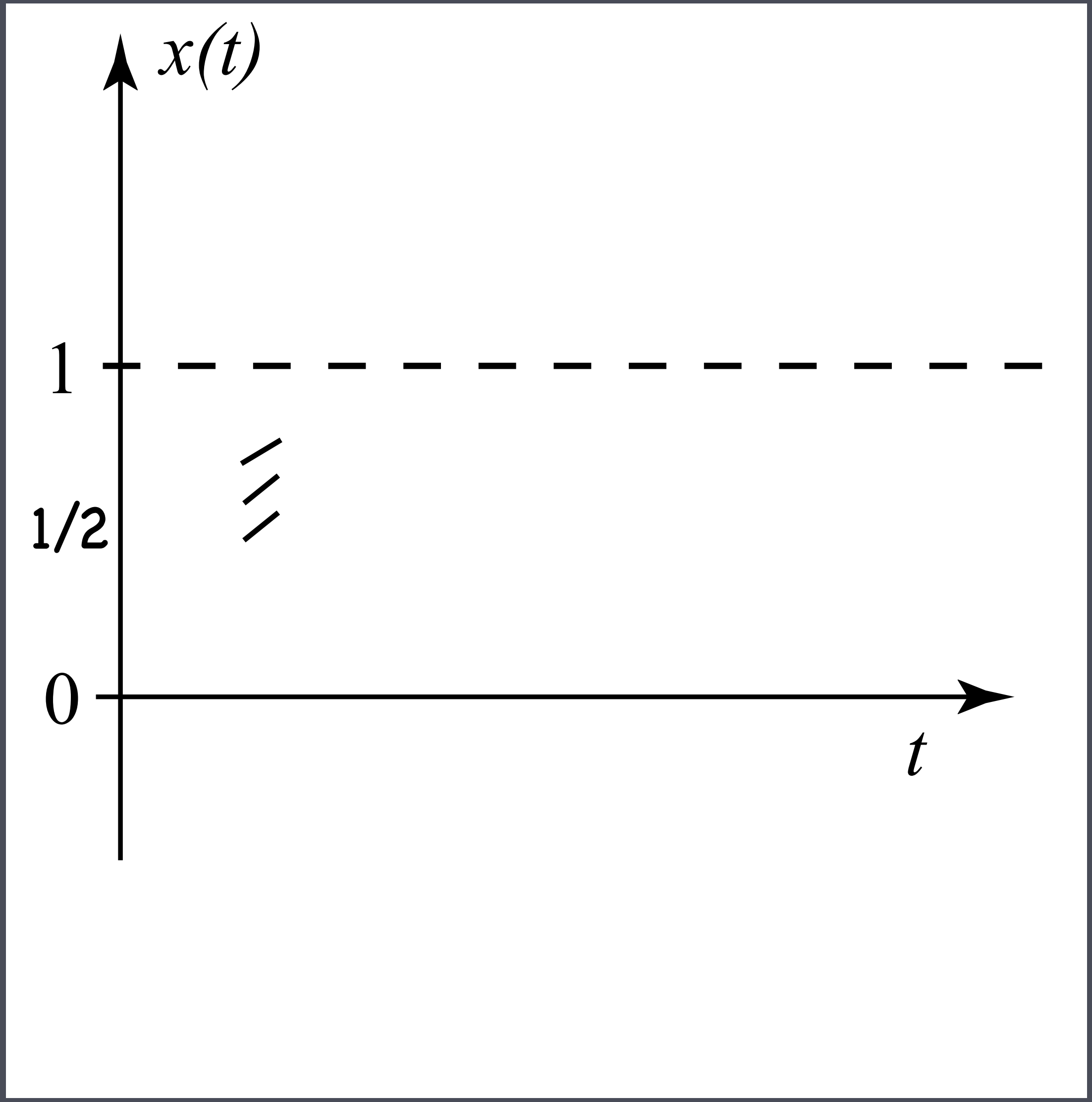

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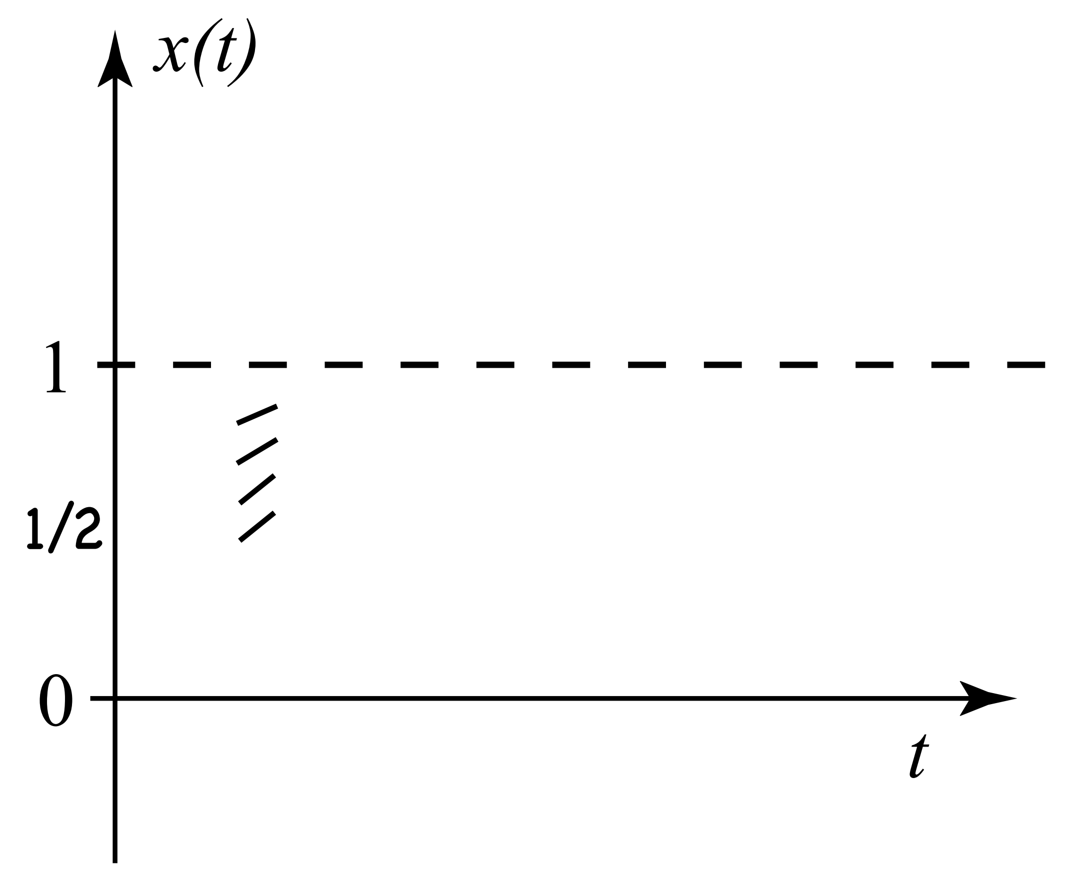

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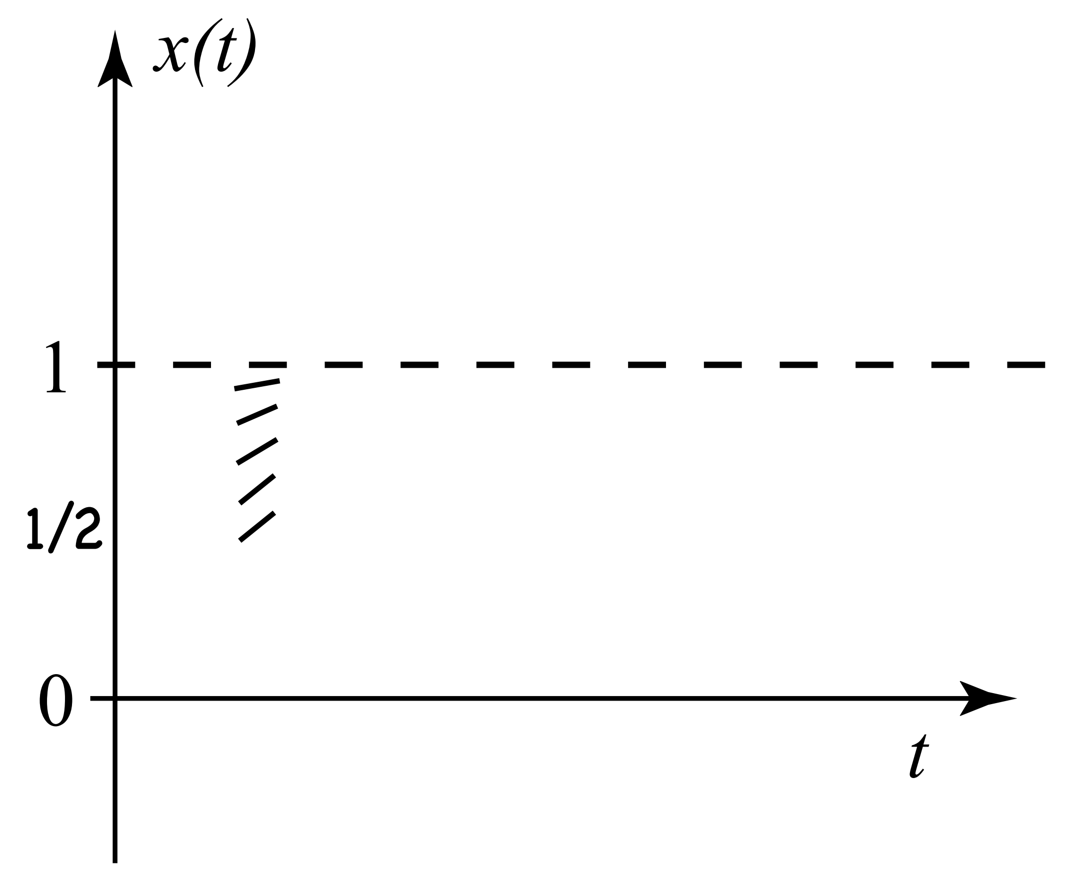

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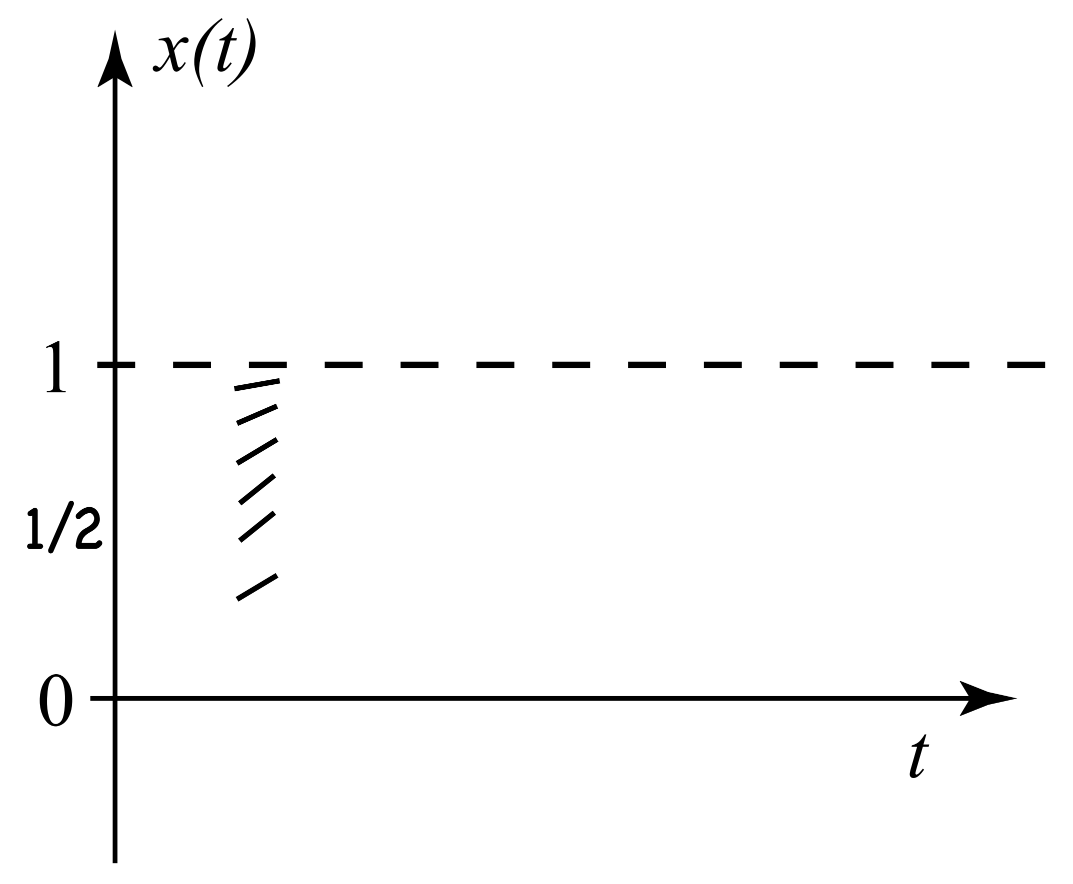

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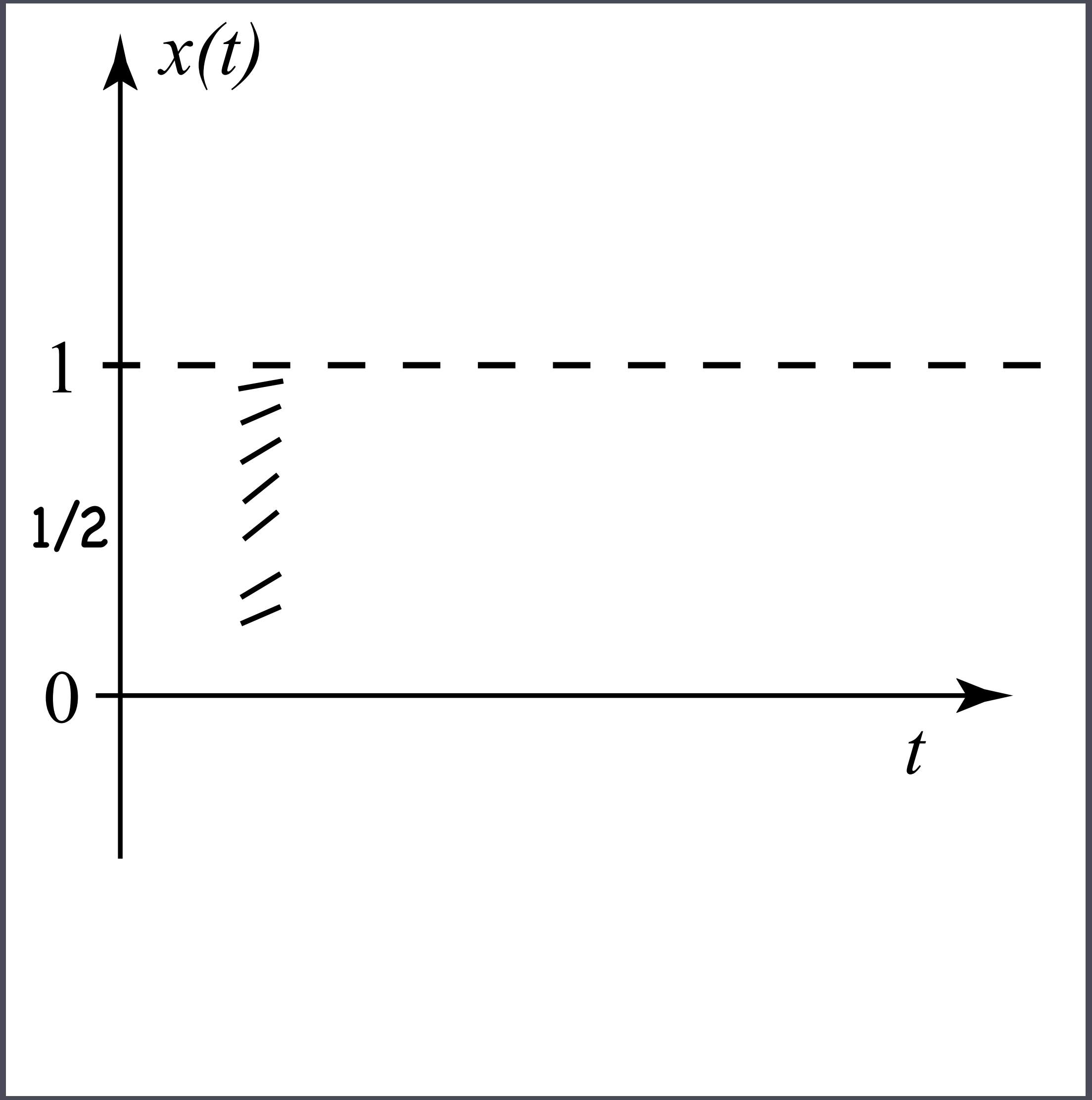

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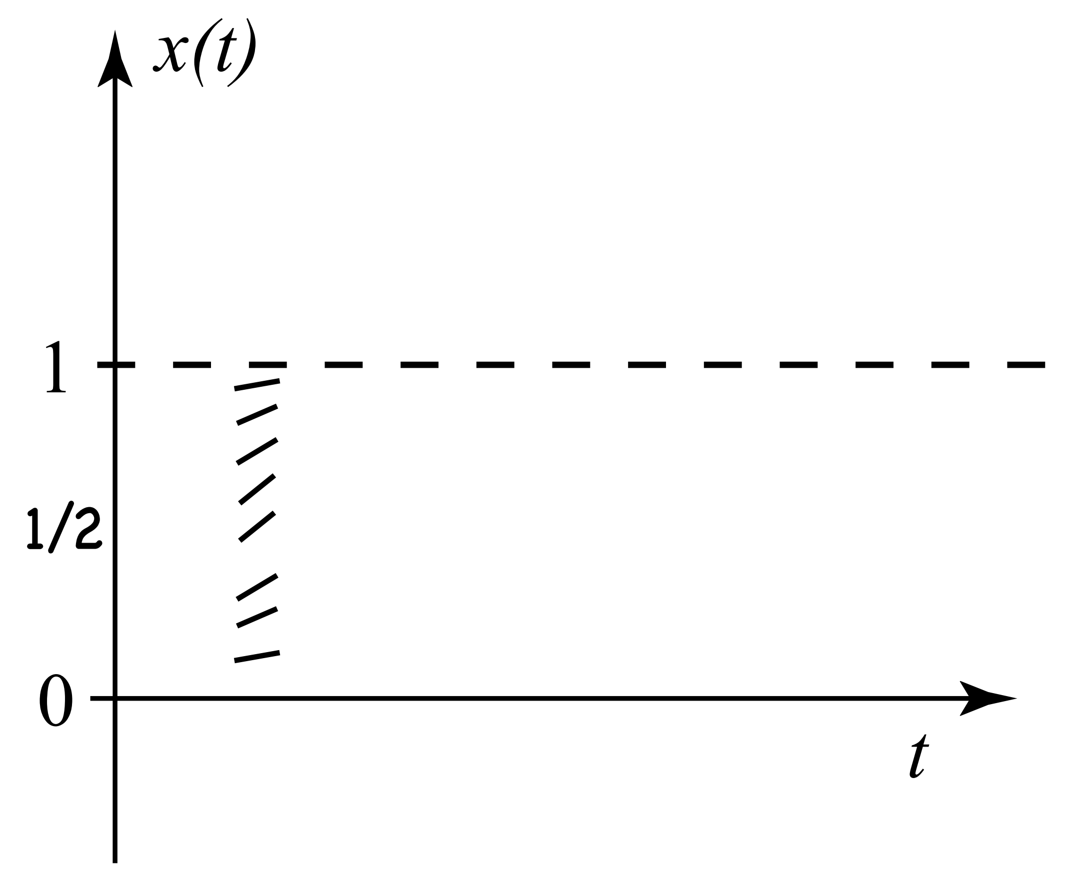

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Slope field

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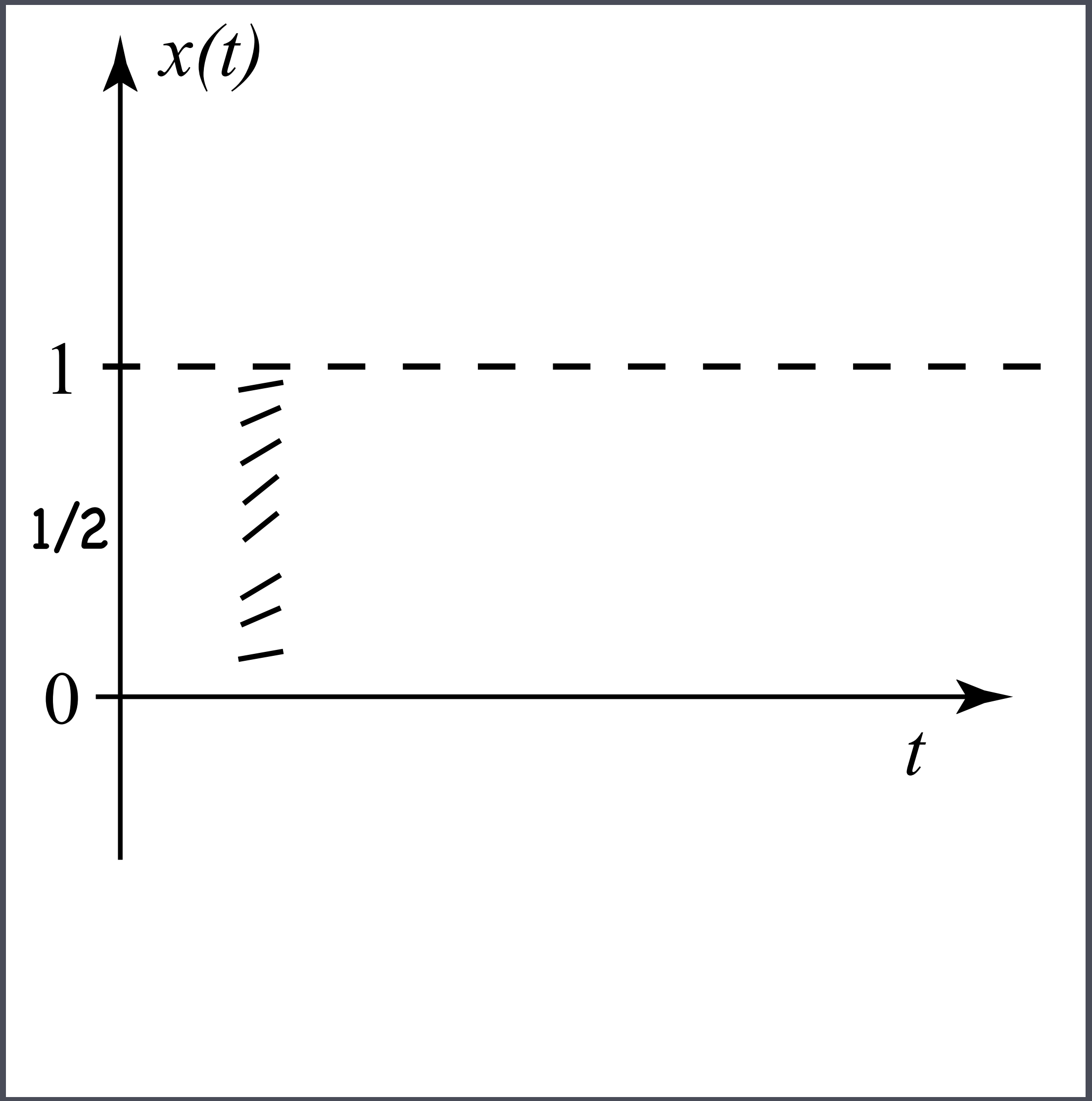

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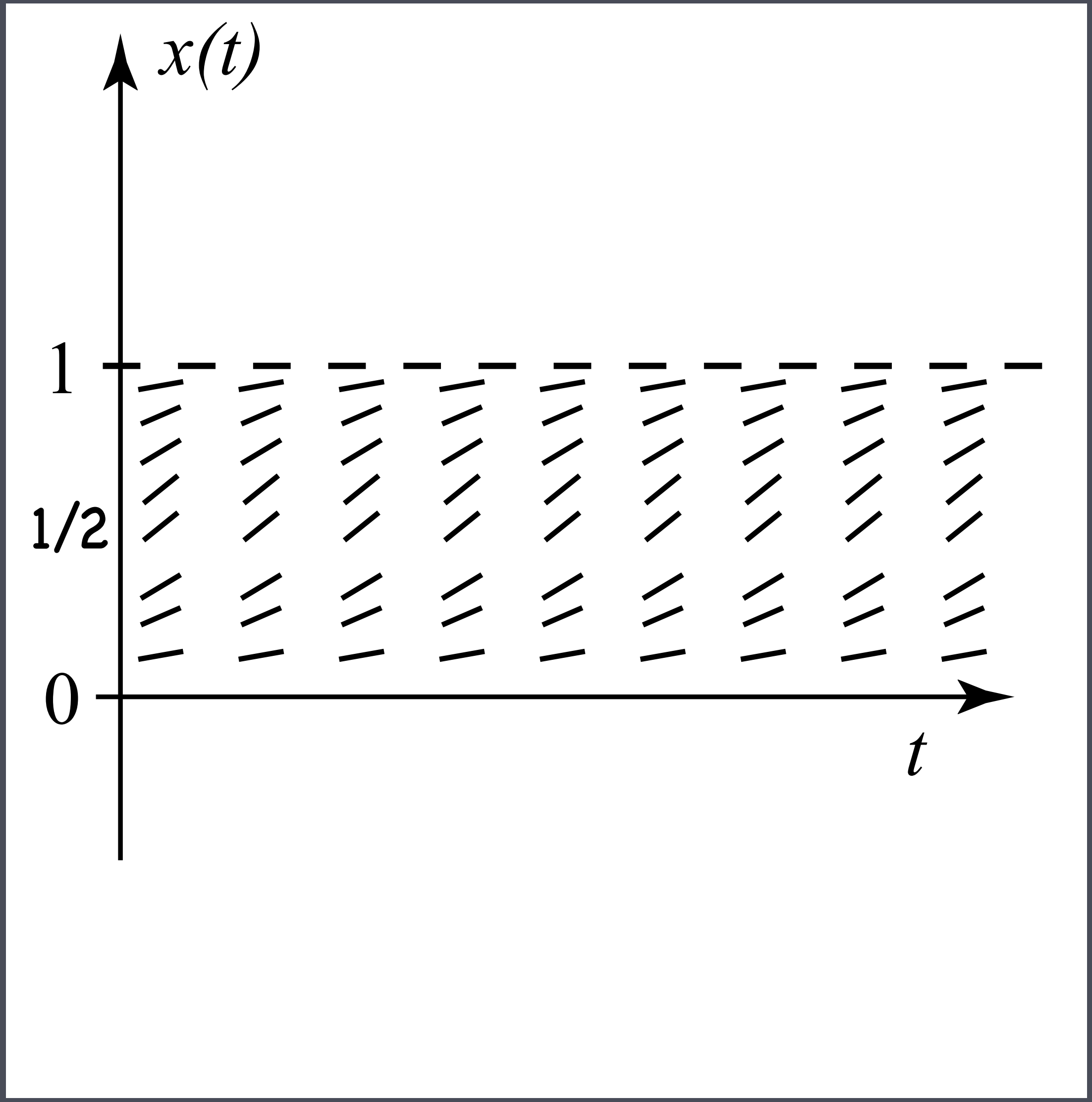

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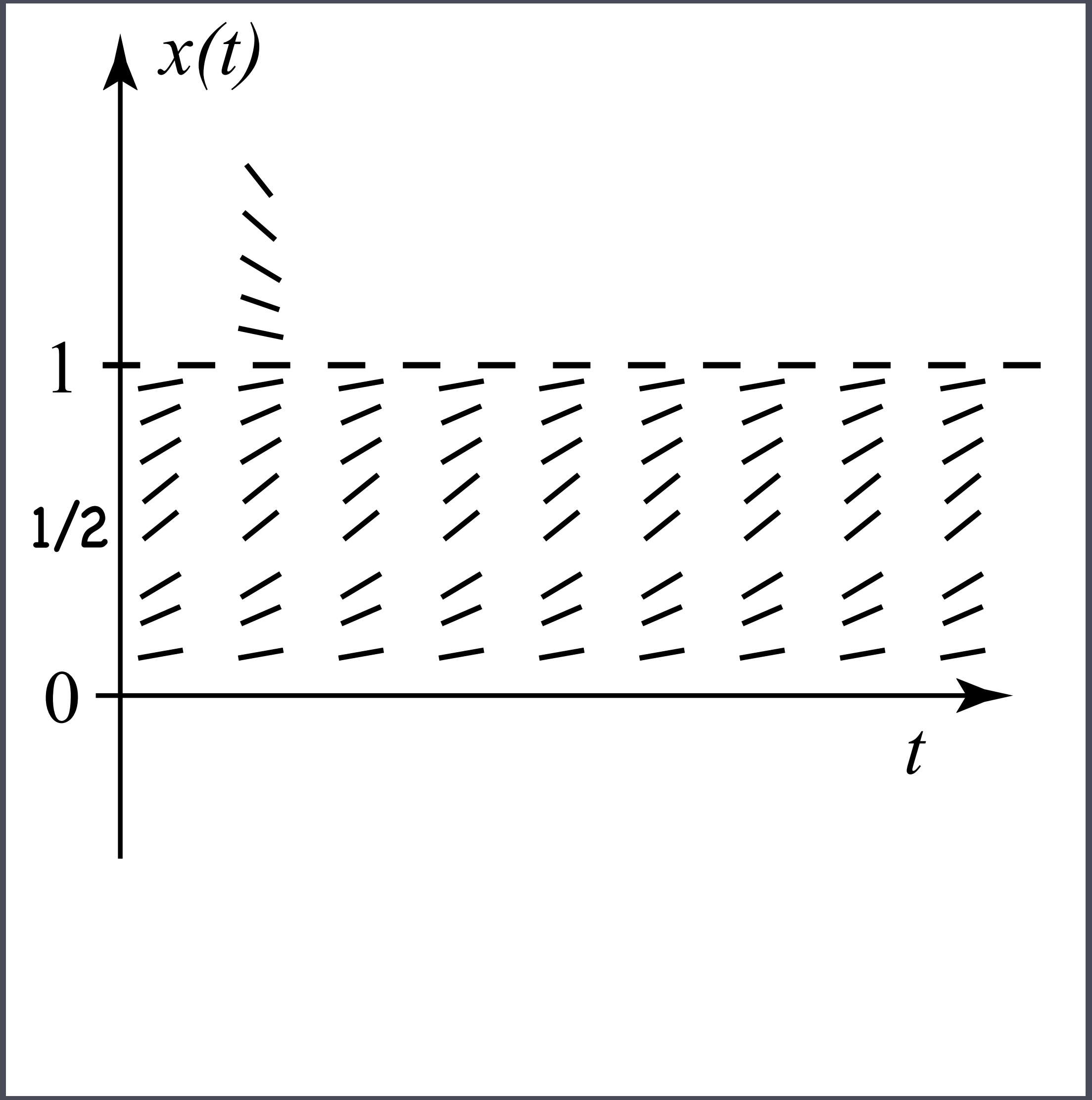

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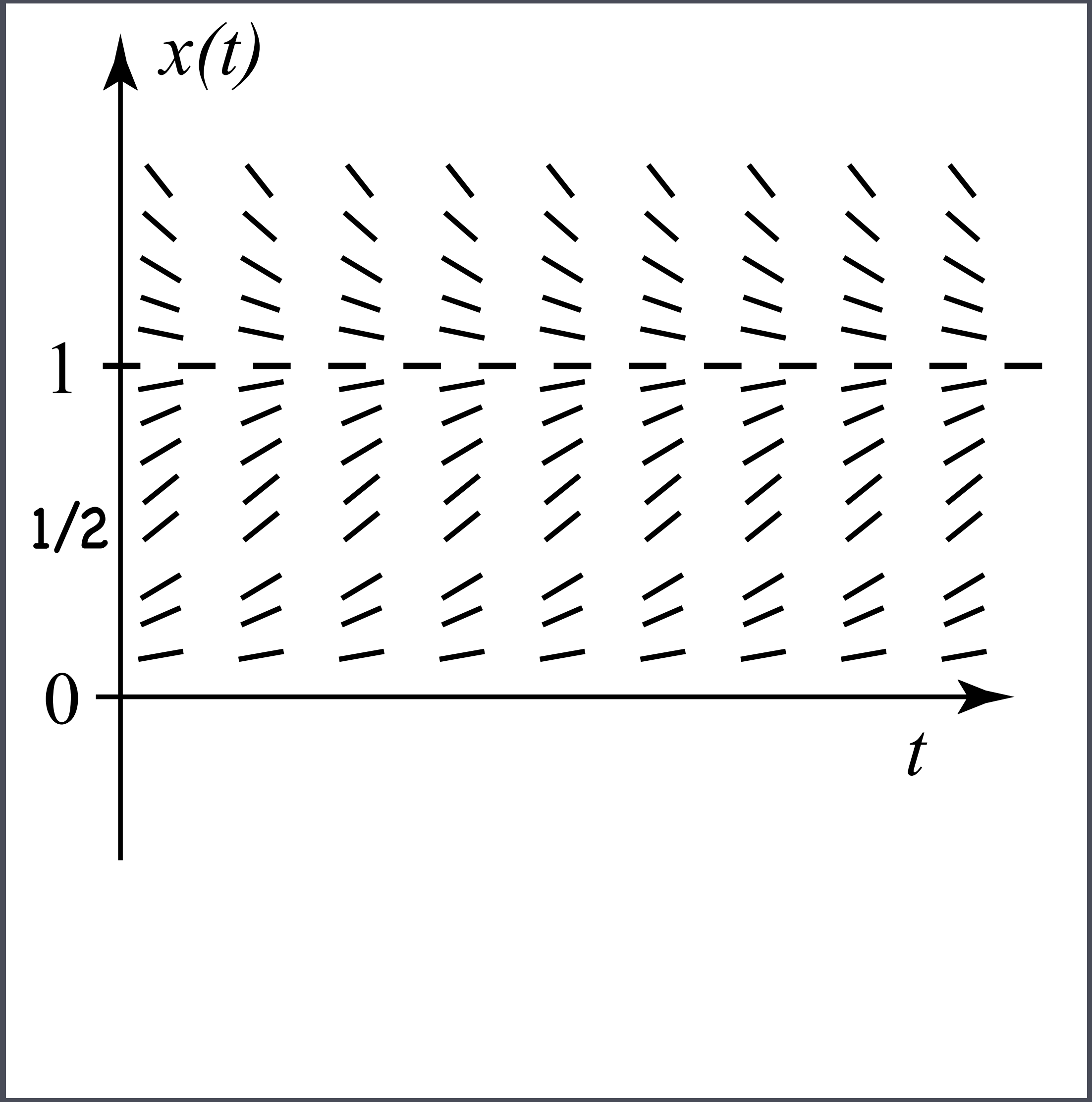

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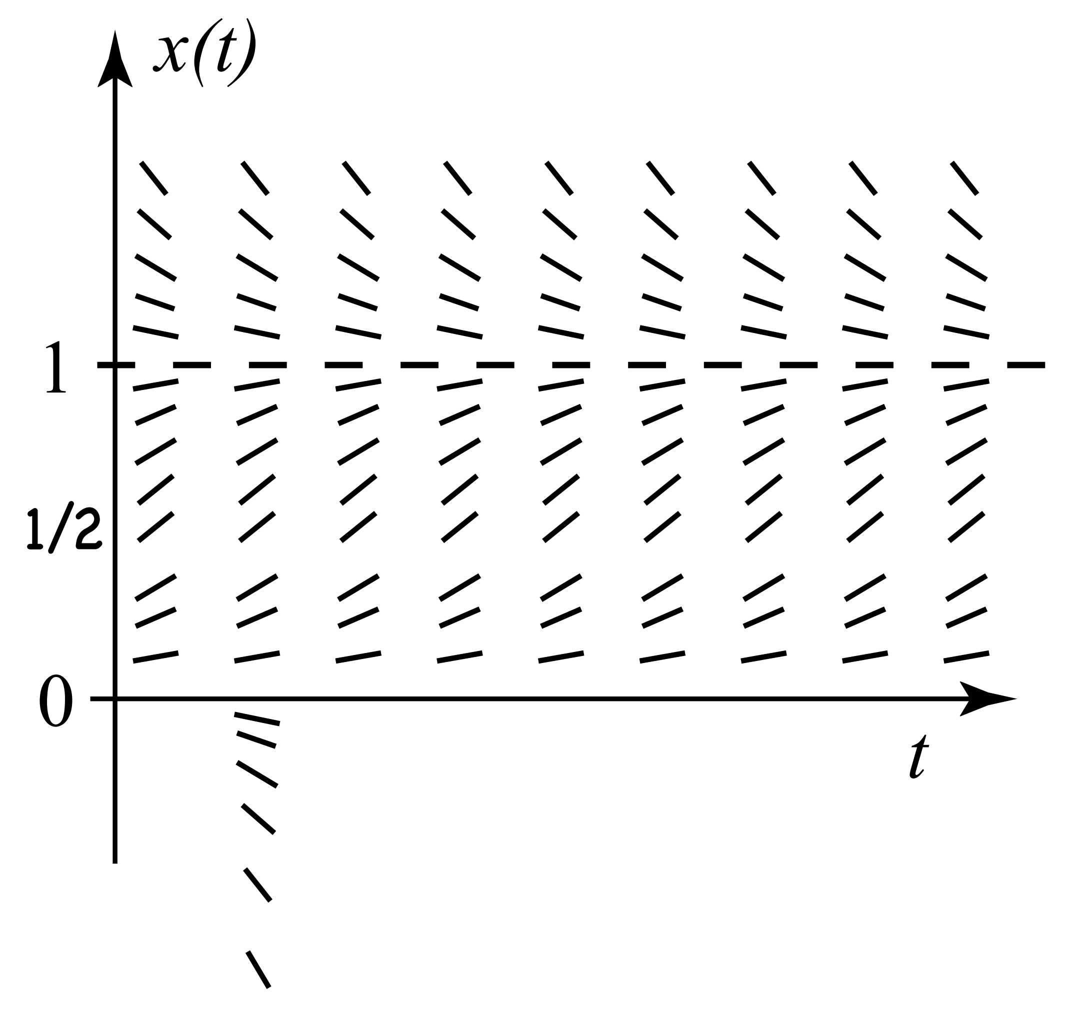

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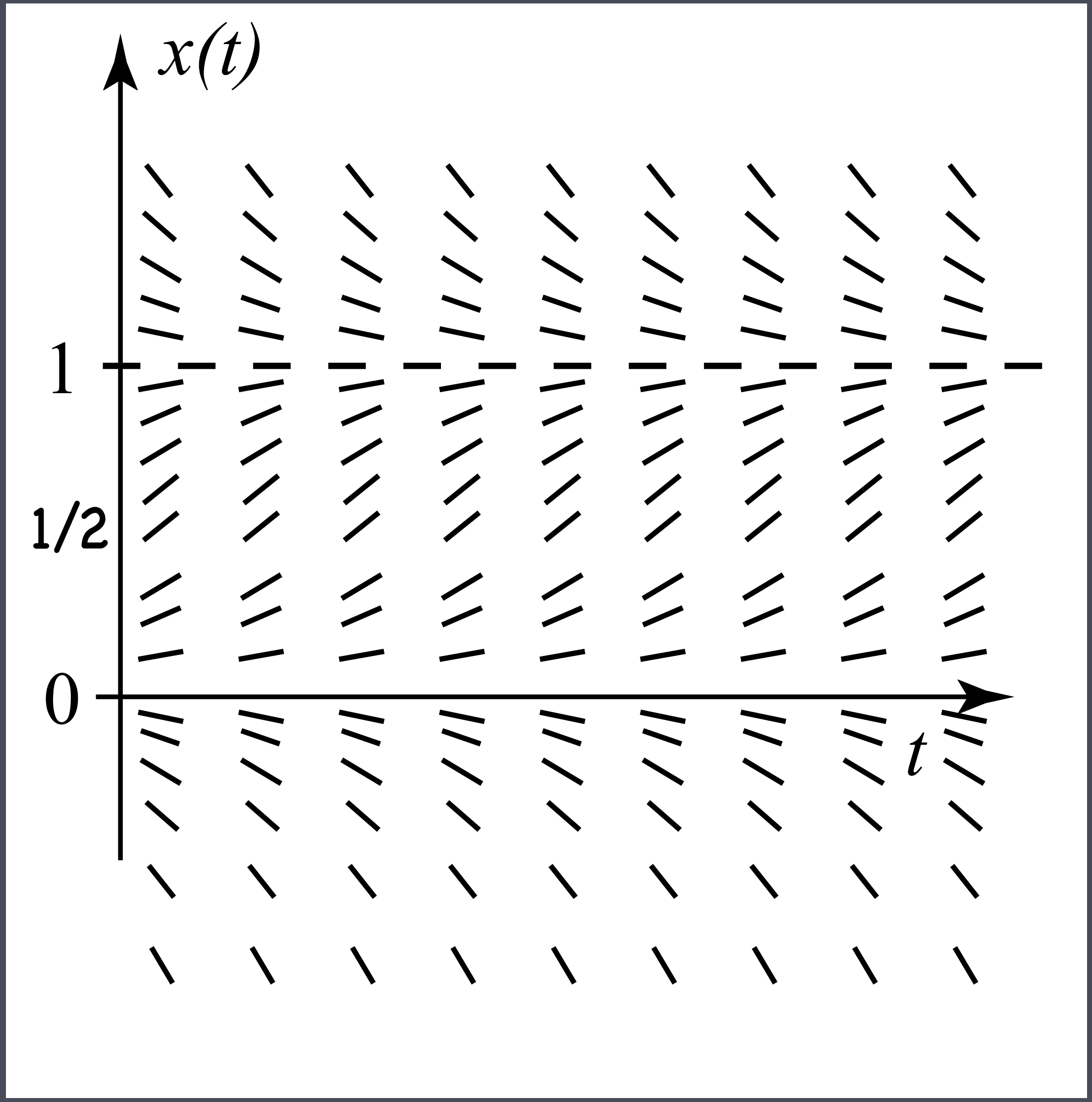

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velocity

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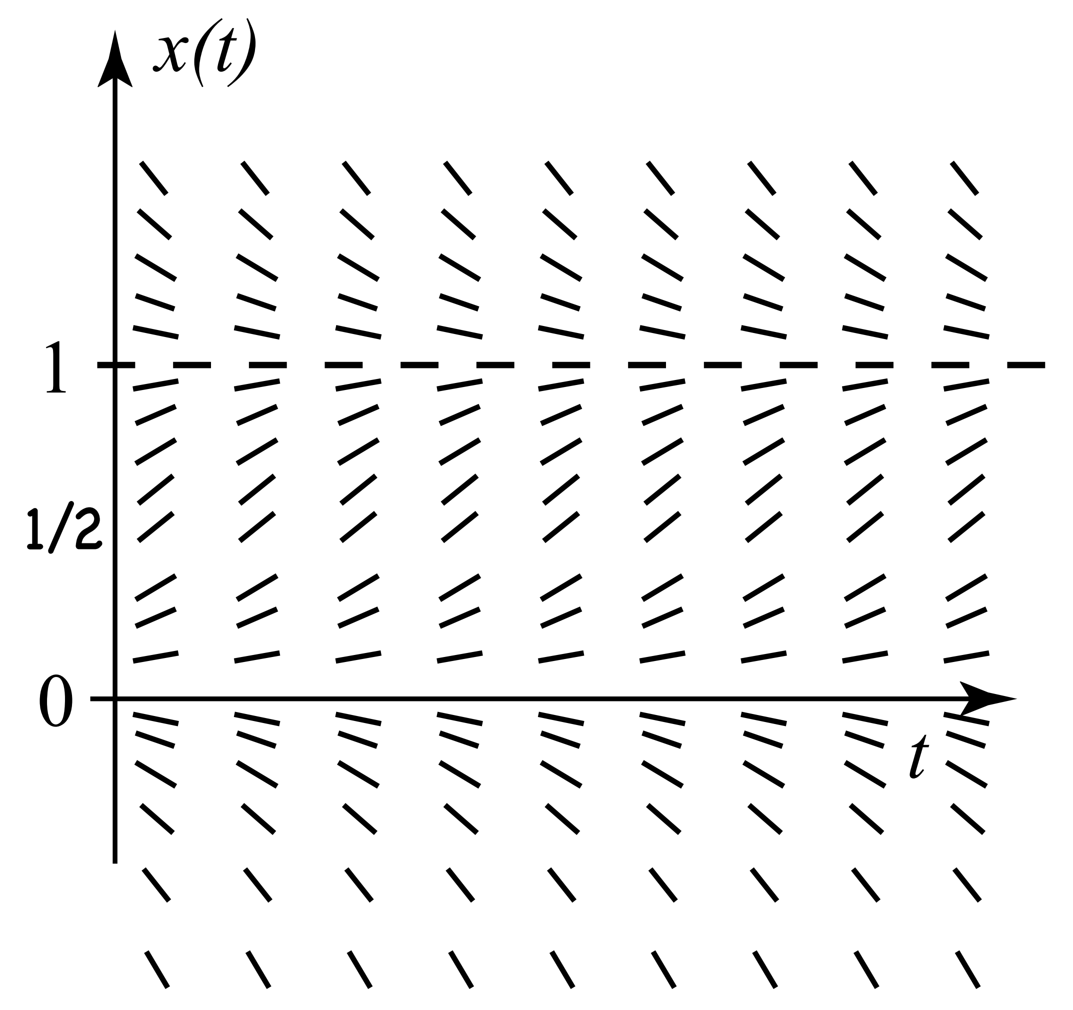

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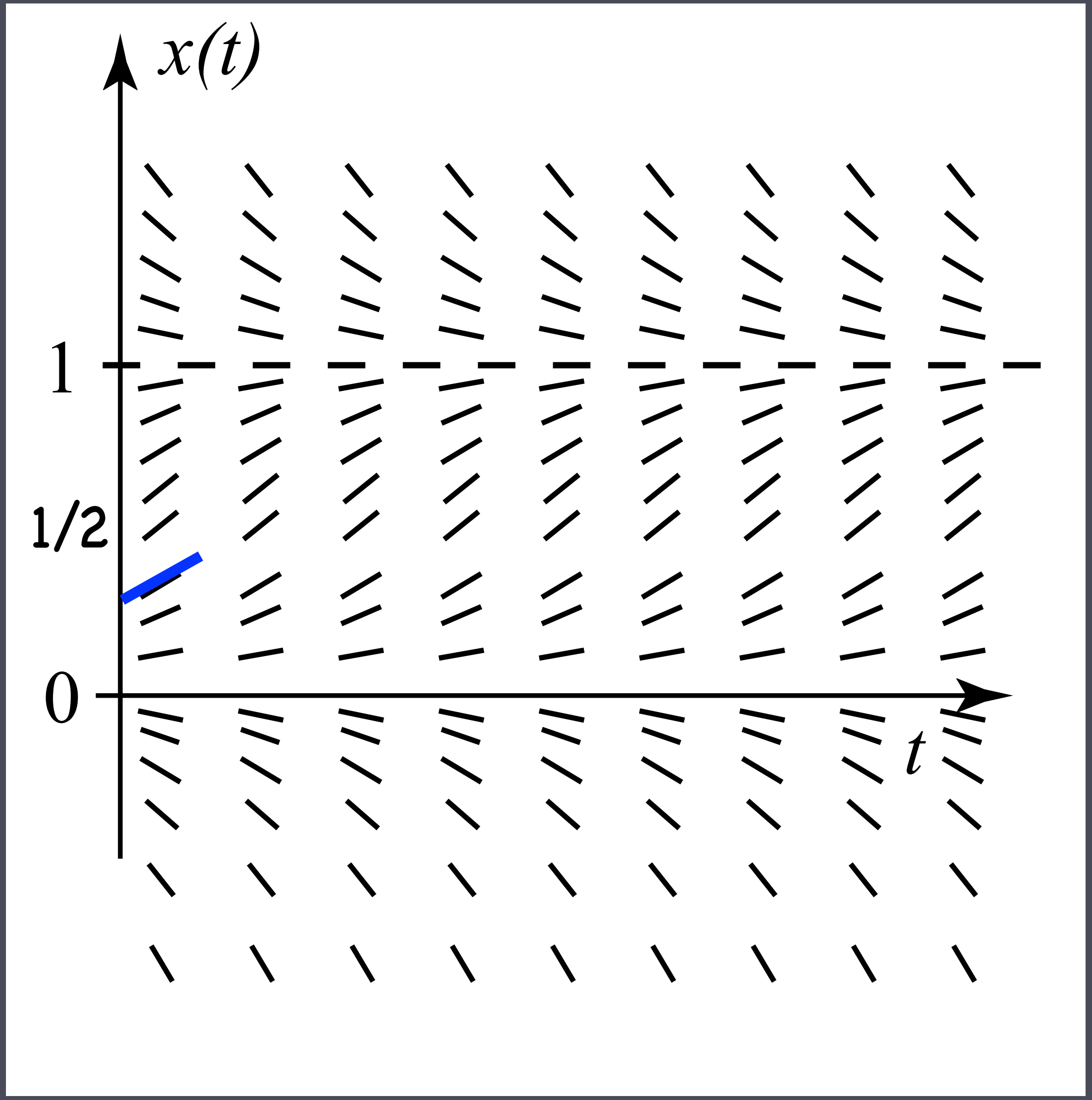
↑ velocity ↑ position

Slope field



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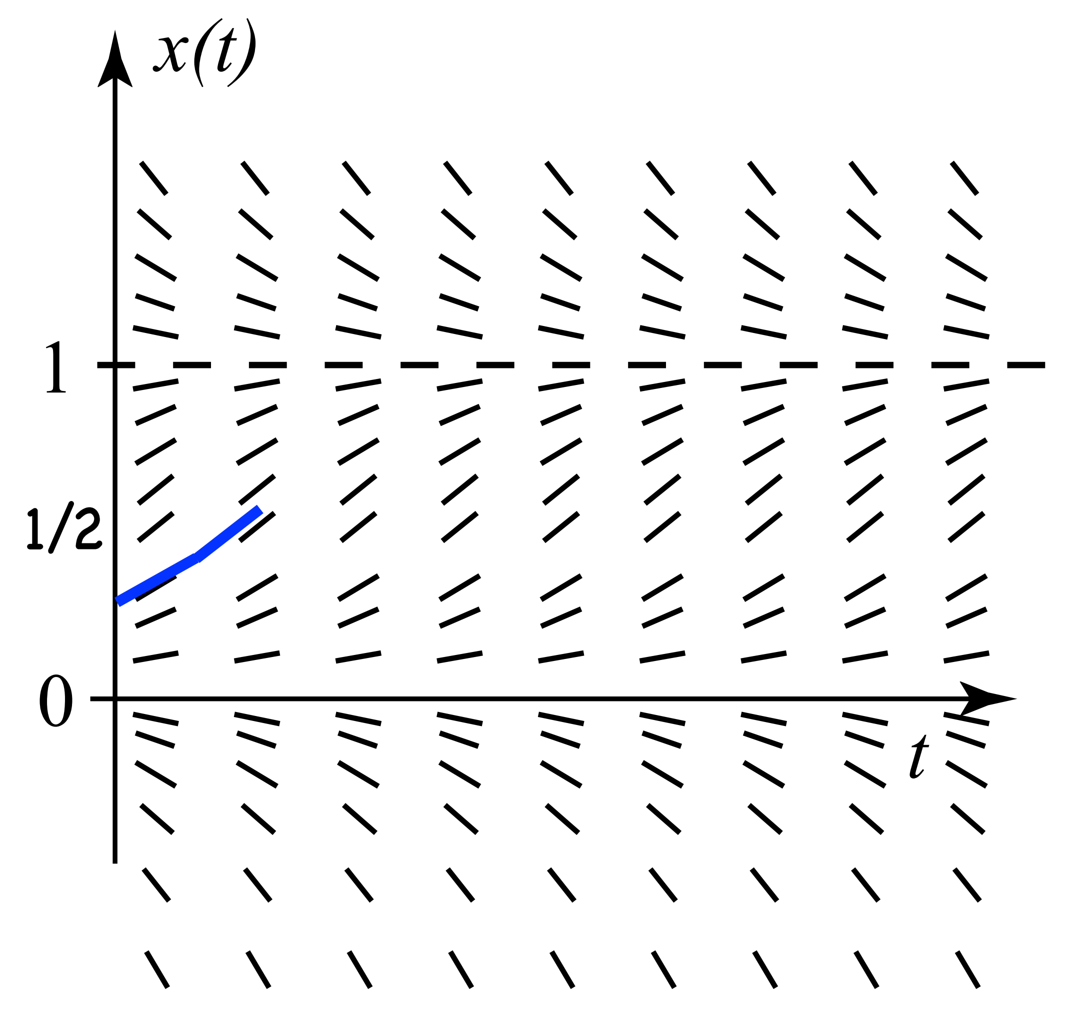

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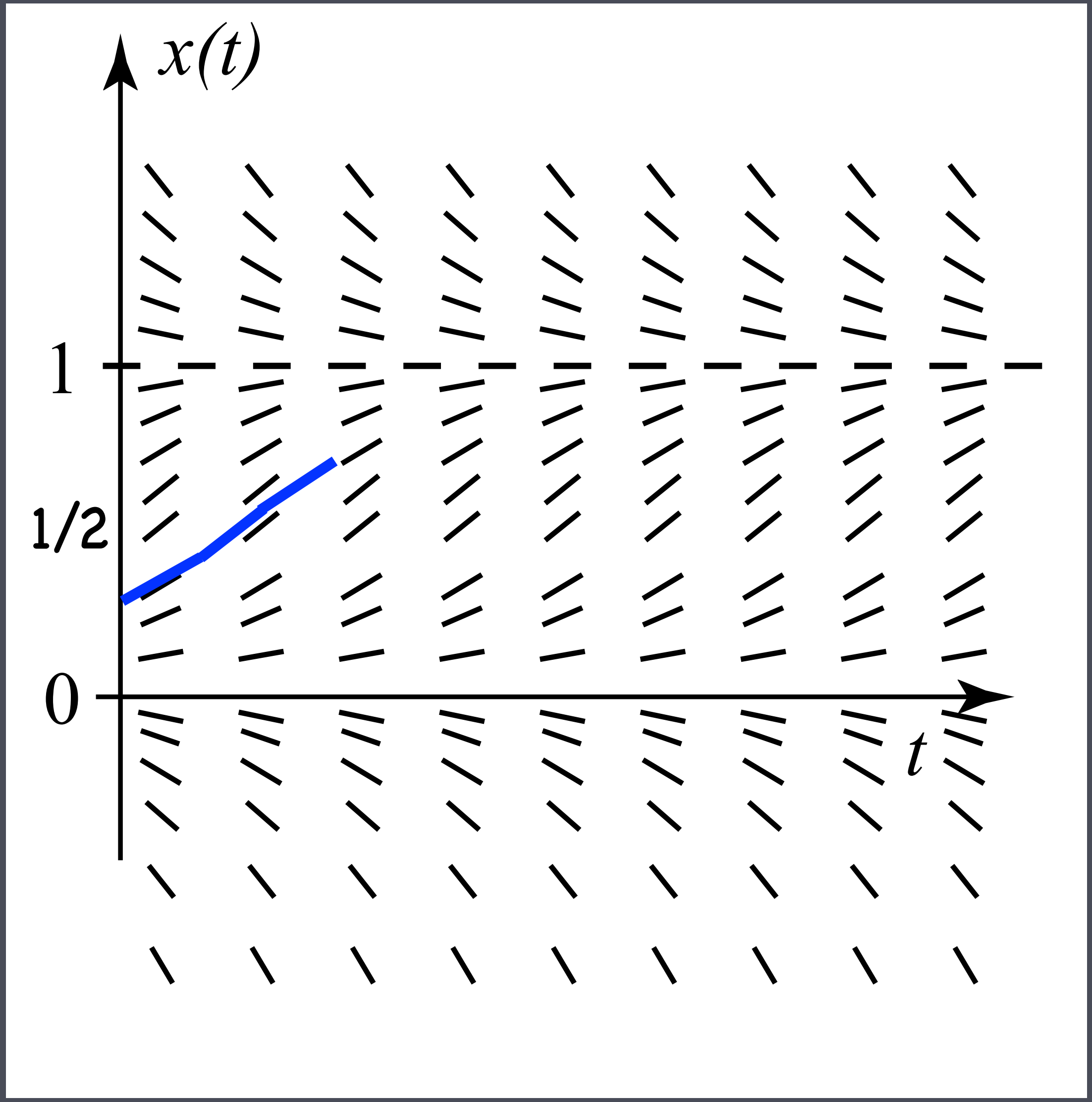

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position

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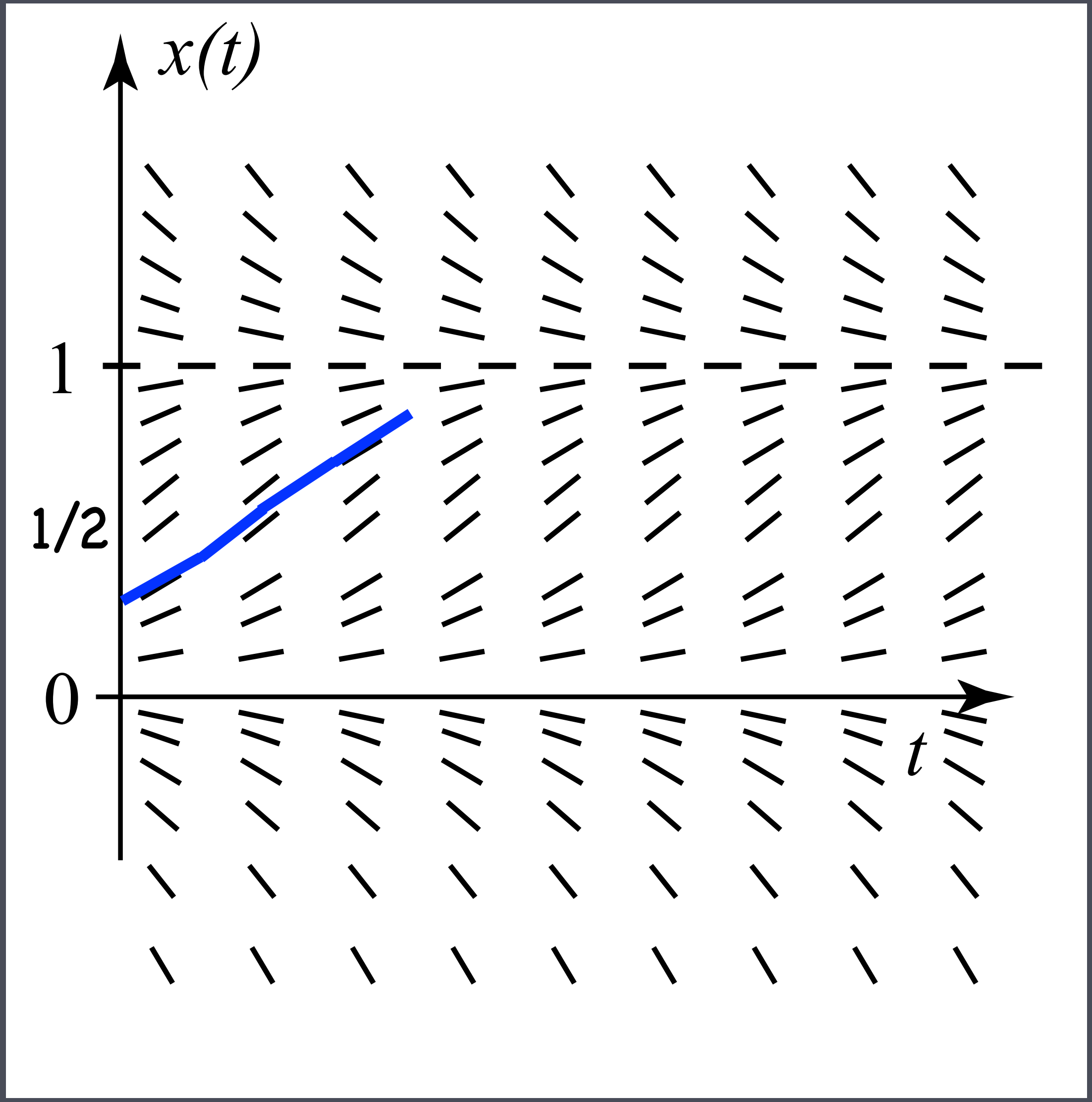

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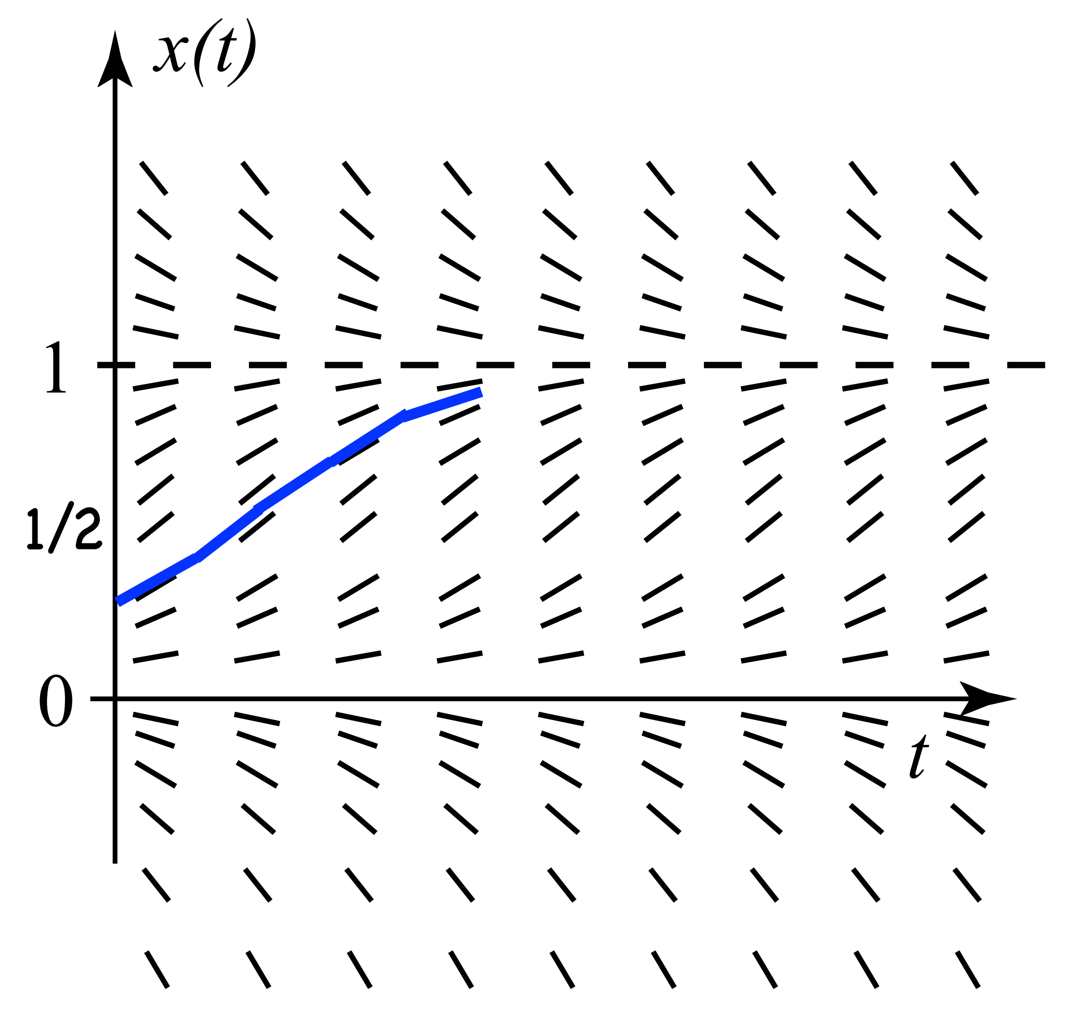

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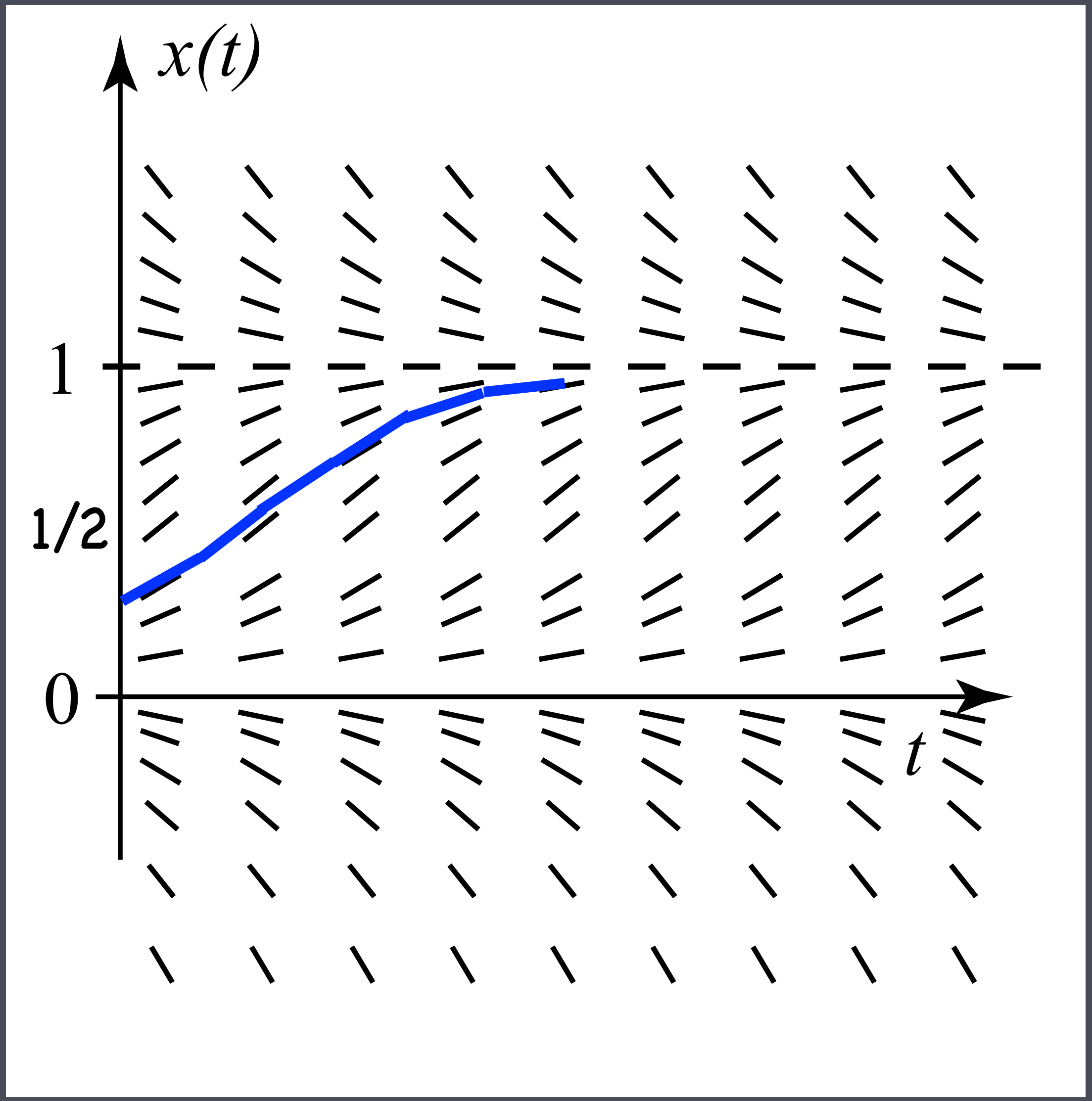

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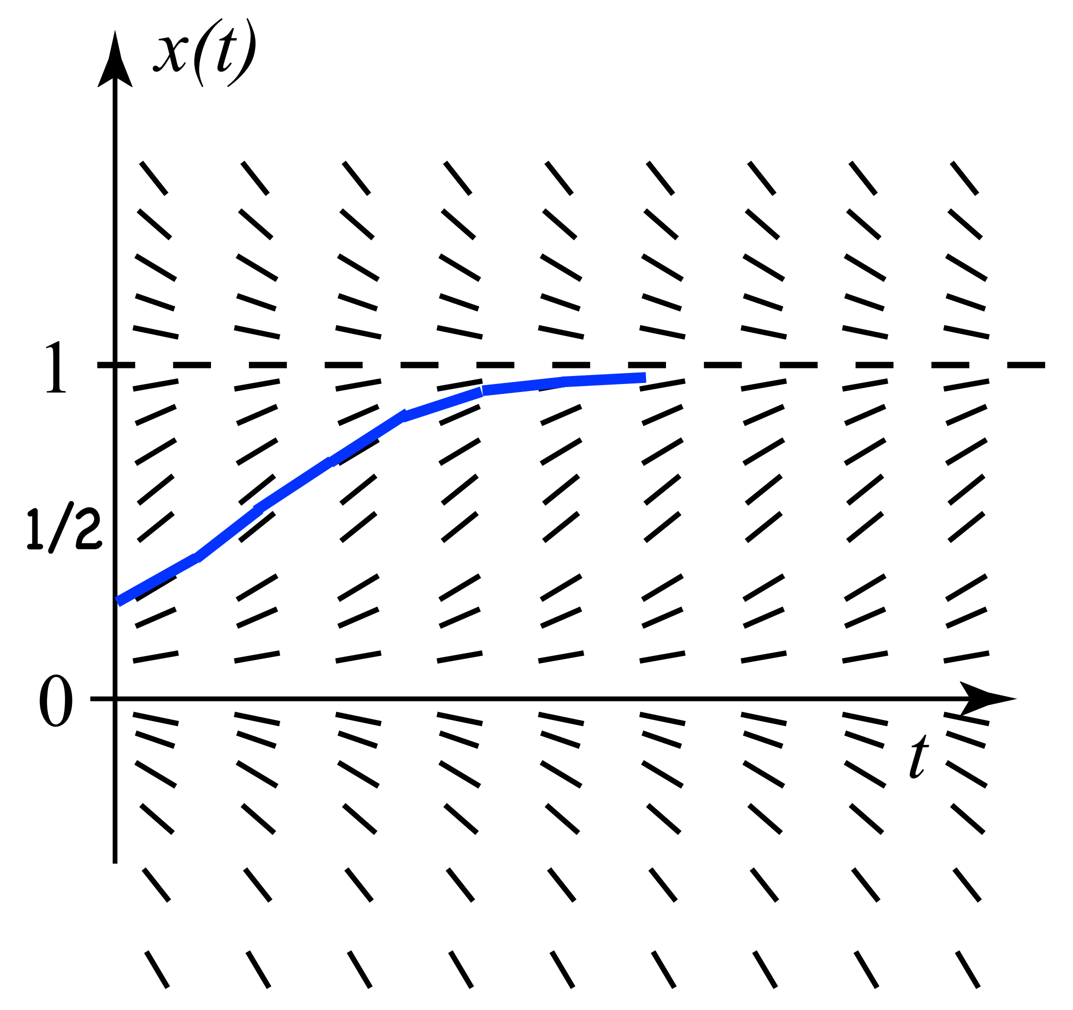

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$$\underset{\substack{\uparrow\\ \text{velocity}}}{x'} = x(1-\underset{\substack{\uparrow\\ \text{position}}}{x})$$

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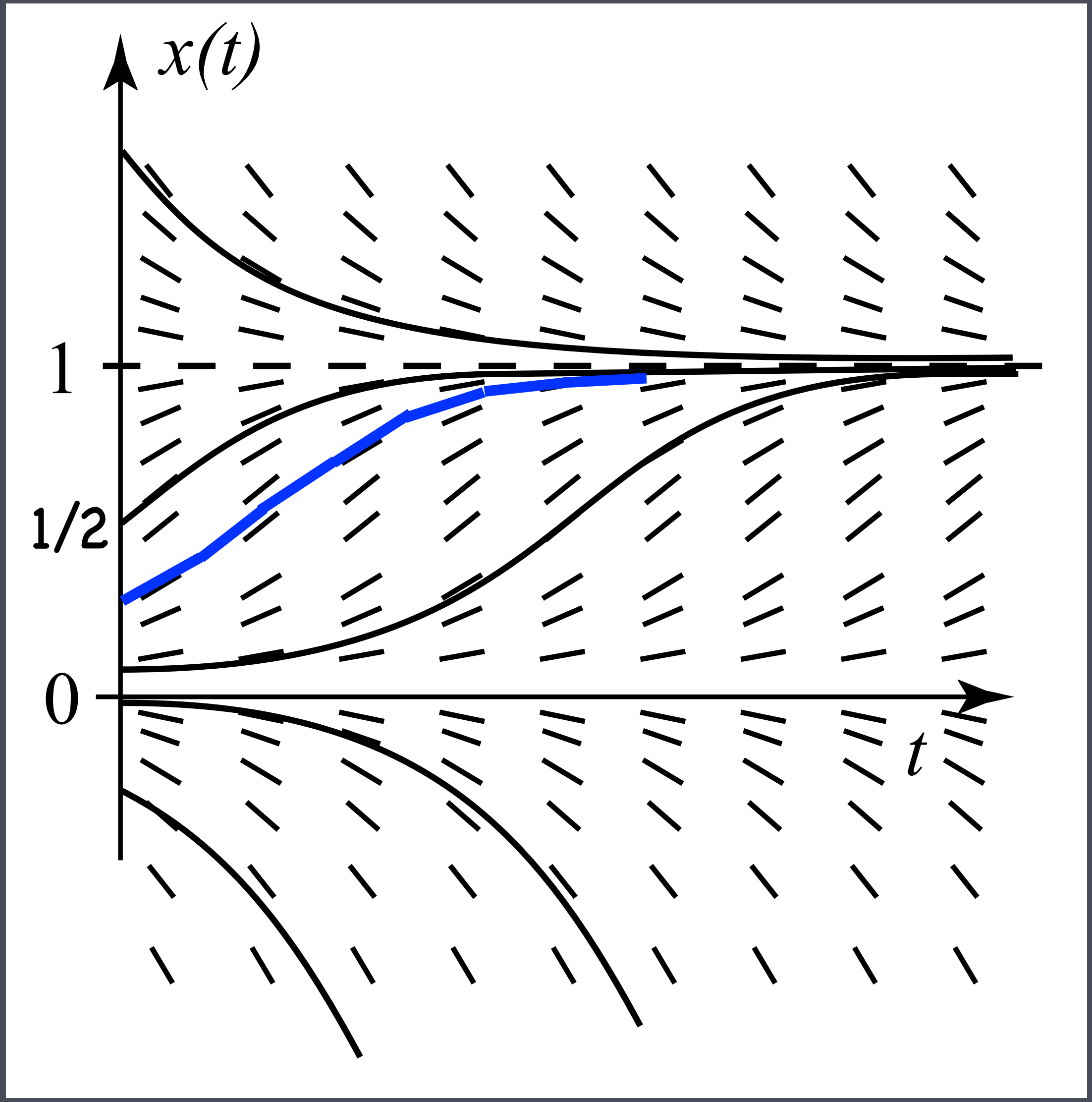
- Solution curves must be tangent to slope field everywhere.

Slope field



$y' = -y(y-1)(y+1)$

- What are the steady states of this equation?
- Draw the slope field for this equation.
    - Include the steepest slope element in each interval between steady states and two others (roughly).