Today

- Introduction to Differential Equations
- Linear DE (\( y' = ky \) )
- Nonlinear DE (e.g. \( y' = y (1-y) \) )
- Qualitative analysis (phase line)
Carbon dating: The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write down a DE for the amount of C-14 at time \( t \).

(A) \( \frac{dC}{dt} = k \cdot C(t) \) where \( k > 0 \).

(B) \( \frac{dC}{dt} = k \cdot C(t) \) where \( k < 0 \).

(C) \( C(t) = C_0 e^{kt} \).

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$$C'(t) = -r C(t) \quad \text{where} \quad r > 0.$$ 

Solution:

(A) $C(t) = e^{-rc}$
(B) $C(t) = 17e^{-rt}$
(C) $C(t) = -rC^2/2$
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DE + IC is called an Initial Value Problem (IVP)
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Write down the solution to this equation:

$$Q(t) = Q_0 e^{kt}$$
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- Determine \( k \) and \( Q_0 \) from given values or %ages of \( Q \) at two different times (i.e. data).
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- Determine half-life/doubling time from data or k.
DEs – a broad view
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  - \( v' = g - v^2 \), \( y' = -\sin(y) \), \( (h')^2 = bh \).
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- Object falling through air
- Pendulum under water
- Water draining from a vessel
DEs – a broad view
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Population growth:

\[ N' = bN - dN = kN \] (linear)

where \( b \) is per-capita birth rate, \( d \) is per-capita death rate and \( k = b - d \).
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\[ N' = bN - (cN)N = bN - cN^2 \]
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This is called the logistic equation, usually written as

\[ \frac{dN}{dt} = rN \left( 1 - \frac{N}{K} \right) \]
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where \( r=b \) and \( K=1/c \). This is a nonlinear DE because of the \( N^2 \).
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Qualitative analysis - extract information about the general solution without solving.

- Steady states
- Slope fields
- Stability of steady states
- Plotting y’ versus y (state space/phase line)
Steady state. Where can you stand so that the DE tells you not to move?

(A) $x=-1$

(B) $x=0$

(C) $x=1/2$

(D) $x=1$

This is the logistic eq with $r=1$, $K=1$. 

\[ x' = x(1 - x) \]
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**Steady state.** Where can you stand so that the DE tells you not to move?

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A **steady state** is a constant solution.
$y' = -y(y-1)(y+1)$

What are the steady states of this equation?
\[ x' = x(1 - x) \]

velocity

position

Slope field
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**Slope field.**

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When \( x(t) = 1/2 \) what is \( x' \)?

- (A) 0
- (B) 1/4
- (C) 1/2
- (D) 1
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- **Velocity**
- **Position**

- **Slope field.**

- At any \( t \), don’t know \( x \) yet so plot all possible \( x' \) values

- Now draw them for all \( t \).
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\[ \begin{align*}
\text{velocity} & \quad \uparrow \\
\text{position} & \quad \uparrow
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\( y' = -y(y-1)(y+1) \)

What are the steady states of this equation?

Draw the slope field for this equation.

Include the steepest slope element in each interval between steady states and two others (roughly).