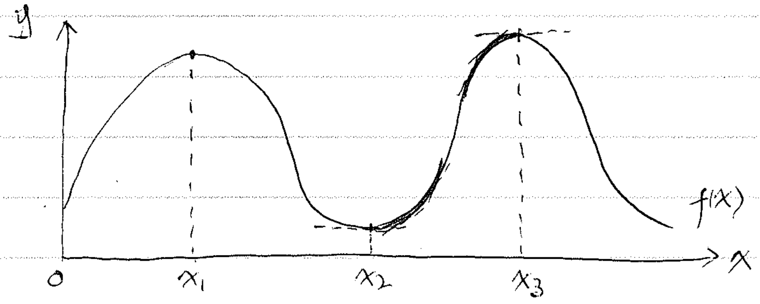
Lecture 9 (Sept 25, 2013)

Learning Goals: ① understand the critical points

② understand the inflection points

③ local extrema (derivative test)



- Trend of functions: $(0, x_1)$ $f(x)$ is increasing $\Leftrightarrow$ $f'(x) > 0$

  (x_1, x_2) $f(x)$ is decreasing $\Leftrightarrow$ $f'(x) < 0$

  Q: $f'(x) = 0$ indicates what?

- Critical Point: A point a in the domain of f(x) at which $f'(a) = 0$ or $f'(a)$ does not exist (DNE)

Example 1: (T/F) (1) $x=0$ is a critical point of $f(x) = x^3$

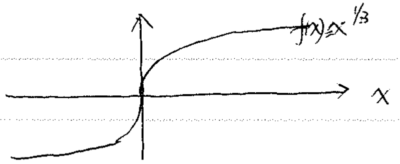
(2) $x=0$ is a critical point of $f(x) = x^{-2}$

(3) $x=0$ is a critical point of $f(x) = x^{\frac{1}{3}}$, given $f'(x) = \frac{1}{3} \cdot x^{-\frac{2}{3}}$

(1) T. $f'(x) = 3x^2 = 0 \Rightarrow x = 0$

(2) F. At $x=0$, $f(x)$ DNE. $x=0$ isn't in the domain of $f(x)$

(3) T. $f(0) = 0$, $f'(0)$ DNE



at $x=0$, tangent line is y-axis

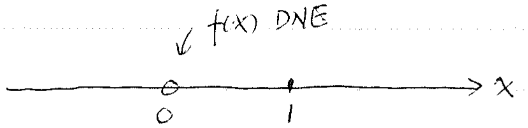
Example 2: In which open interval, function $f(x) = x^2 + \frac{2}{x}$ is decreasing

① take derivative $f'(x) = 2x - \frac{2}{x^2}$

② find critical points. set $f'(x) = 0$ $\Rightarrow$ $\frac{2}{x^2}(x^3-1) = 0$ $\Rightarrow$ $x = 1$

* the critical points separate x-axis into many intervals that we need to check the sign of $f'(x)$

③ choose one point in each interval, plug into $f'(x)$



$(-\infty, 0)$ $f'(-1) = \frac{2}{1} \cdot (-1-1) < 0$ → decreasing

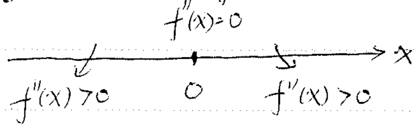
$(0, 1)$ $f'(0.5) < 0$ → decreasing

$(1, +\infty)$ $f'(2) > 0$ → increasing

$f(x)$ is decreasing in $(-\infty, 0) \cup (0, 1)$

- Concavity: Notice in the interval (x_2, x_3), near x_2 and x_3, the shape of the curve is different.
  $f(x)$ is concave up on an open interval $\Leftrightarrow$ $f'(x)$ is increasing $\Leftrightarrow$ $f''(x) > 0$
  $f(x)$ is concave down on an open interval $\Leftrightarrow$ $f(x)$ is decreasing $\Leftrightarrow$ $f''(x) < 0$

- Inflection Point: A point where the concavity changes (in the domain)
  Method: At the inflection point $x = c$, either $f''(c) = 0$ or $f''(c) = \text{DNE}$
  provide possible positions of the inflection point, but we still need to check if concavity changes
  Example 3: (T/F) $x = 0$ is an IP of $f(x) = x^4$
  F. $f'(x) = 4x^3$, $f''(x) = 12x^2 = 0 \Rightarrow x = 0$



$f''(x) > 0$ for all the x except $x = 0$
concavity doesn't change

Notice: $f''(c) = 0$ isn't enough to conclude $x = c$ is an inflection point

- Local extrema: e.g. x_1, x_2, x_3 in the first figure
  $f(x_1)/f(x_2)$ is called local maxima/minima, if in the neighbourhood of x_1/x_2, there's no value of the function $f(x)$ is greater/smaller than $f(x_1)/f(x_2)$

- First Derivative Test: Suppose $x = c$ is a critical point of a differentiable function
  * if $f'(x) > 0$ for $x > c$ and $f'(x) < 0$ for $x < c$, $x = c$ is a local minima
  * if $f'(x) < 0$ for $x > c$ and $f'(x) > 0$ for $x < c$, $x = c$ is a local maxima

- Second Derivative Test: * if $f'(c) = 0$ and $f''(c) > 0$, $f(c)$ is a local minima
  * if $f'(c) = 0$ and $f''(c) < 0$, $f(c)$ is a local maxima
  * if $f'(c) = 0$ and $f''(c) = 0$, it's inconclusive