Today

- Bacterial growth example
- Doubling time, half life, characteristic time
- Exponential behaviour as solution to DE
- Linear DE ($y' = ky$)
- Nonlinear DE (e.g. $y' = y\,(1-y)$)
- Qualitative analysis (phase line)

A single cell is placed in a dish containing a growth medium. The dish can support 100,000 cells. If each cell divides once every 24 hours, how long before the dish is full?

Rough estimate (no calcs, just intuition):

(A) ~1 week.

(B) ~2 weeks.

(C) ~1 month.

(D) ~1 year.

(E) ~10^4 days ≈ 27 years.

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(A) $p(t) = e^{t/24}$.

(B) $p(t) = 100{,}000\ 2^{t/24}$.

(C) $p(t) = e^{\ln(2)t}$.

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(C) $p(t) = e^{\ln(2)t}$. $= 2^t$ <-- t measured in days.

(D) $p(t) = 2^{-t/24}$.

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(A) $t=\ln(10^5)/\ln(2)$

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$$p(t) = 2^t$$

$$10^5 = 2^t$$

$$\ln(10^5) = \ln(2^t)$$

$$\ln(10^5) = t \ln(2)$$

$$t \approx 16.6 \text{ days}$$

Doubling time

- Let $c(t) = c_0 e^{kt}$. At $t=0$, $c(0) = c_0 e^0 = c_0$.
- If $k>0$, $c(t)$ is increasing and doubles when $c_0 e^{kt} = 2c_0$.
- That is when $t=\ln(2)/k$.
- This is called the doubling time.

Half-life

- Let $c(t) = c_0e^{kt}$. At $t=0$, $c(0) = c_0e^0 = c_0$.
- If $k<0$, $c(t)$ is decreasing and halves when $c_0e^{kt} = c_0/2$.
- That is when $t=-\ln(2)/k$.
- If written $c(t) = c_0e^{-kt}$ with $k>0$ then $t=\ln(2)/k$ (same as doubling time).
- This is called the half-life.

Characteristic time / mean life

- Let $c(t) = c_0e^{-kt}$. At $t=0$, $c(0) = c_0e^0 = c_0$.
- If $k>0$, $c(t)$ is decreasing and reaches 1/e its original value when $c_0e^{-kt} = c_0/e$.
- That is when $t=1/k$.
- This is called the characteristic time or mean life. Just like half-life but replace 2 with e (could be called 1/e-life).

Differential equations (DE)

Carbon dating: The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write down a DE for the amount of C-14 at time t.

(A) $C'(t) = k\, C(t)$ where $k>0$.

(B) $C'(t) = k\, C(t)$ where $k<0$.

(C) $C(t) = C_0 e^{kt}$.

(D) $C'(t) = C_0 e^{-kt}$.

Differential equations (DE)

Carbon dating: The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write down a DE for the amount of C-14 at time t.

(A) $C'(t) = k\ C(t)$ where $k>0$. <---solution grows!

(B) $C'(t) = k\ C(t)$ where $k<0$.

(C) $C(t) = C_0e^{kt}$. <---if $k<0$, this might be the solution but it's not a DE.

(D) $C'(t) = C_0e^{-kt}$. <---this is not a DE either.

Differential equations (DE)

Carbon dating: The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write down a DE for the amount of C-14 at time t.

$$C'(t) = -r\, C(t) \quad \text{where } r > 0.$$

Solution:

(A) $C(t) = e^{-rC}$

(B) $C(t) = 17e^{-rt}$

(C) $C(t) = -rC^2/2$

(D) $C(t) = 5e^{rt}$

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- In fact, $C(t) = C_0e^{-rt}$ is a solution for all values of C_0 - show on board.
- DEs are often given with an initial condition (IC) e.g. $C(0)=17$ which can be used to determine C_0.
- DE + IC is called an Initial Value Problem (IVP)

Summary of what you should be able to do

- Take a word problem of the form "Quantity blah changes at a rate proportional to how much blah there is" and write down the DE:
  - $Q'(t) = k\, Q(t)$
- Write down the solution to this equation:
  - $Q(t) = Q_0 e^{kt}$
- Determine k and Q_0 from given values or %ages of Q at two different times (i.e. data).
- Determine half-life/doubling time from data or k.