Lecture 33 (Nov. 25, 2013)

Learning Goals: ① Qualitative methods of DE

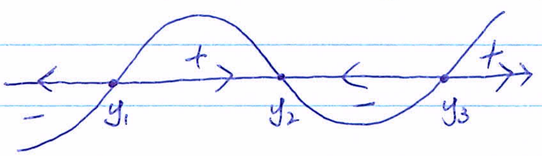
② Applications of nonlinear DE

- (Continue) Qualitative methods for DE

(i) Sketch $\frac{dy}{dt} = f(y)$ as a function of y

For $\frac{dy}{dt} = f(y) > 0$ (above y-axis), $y(t)$ increases

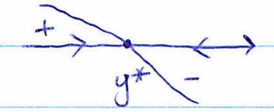
For $\frac{dy}{dt} = f(y) < 0$ (below y-axis), $y(t)$ decreases



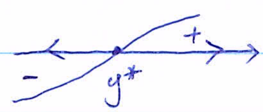
+/− indicates y increases/decreases

→/← on y-axis

(ii) Steady States : y^* satisfies $\frac{dy}{dt} = f(y^*) = 0$



y^* is stable



y^* is unstable

(iii) Slope field: how $y(t)$ changes over time

recall that linear approximation is to use the values on the tangent line estimate the function values.

At any time t_0, we can sketch the segment of tangent line for any value of y

at $t = t_0$, (vertical direction), different y provide different slope.

at $y = y_0$, (horizontal direction), for any t, the slope is the same. as $\frac{dy}{dt} = f(y_0)$

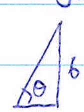
Example 1 Slope field of $\frac{dy}{dt} = y - y^3$

set $\frac{dy}{dt} = 0$, we have $y^* = 0$ or $y^* = \pm 1$

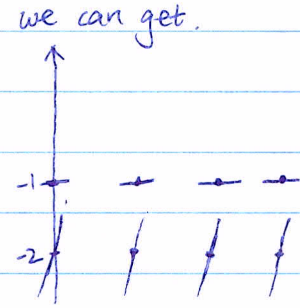
To sketch the slope field, we choose y in $[-2, 2]$ and t in $[0, 20]$

more points we sketch, more information we can get.

Start with $t_0 = 0$, $y_0 = -2$. $\frac{dy}{dt} = 6$



$\Leftrightarrow \tan\theta = \frac{dy}{dt} = 6$



for $y = -1$. $\frac{dy}{dt} = 0$ horizontal

Examples are at the bottom of today's notes

- Application 1: Logistic growth $G(N) = rN\left(\frac{K-N}{K}\right)$ (Oct. 04's lecture)

As $G(N)$ represents the change rate of the density of population N, it can be ~~p~~ replaced by $\frac{dN}{dt}$

$$\frac{dN}{dt} = rN\left(\frac{K-N}{K}\right),$$

r - intrinsic growth rate
K - carrying capacity of N } constants

Compare the model with simple exponential growth model $\frac{dN}{dt} = r \cdot N$

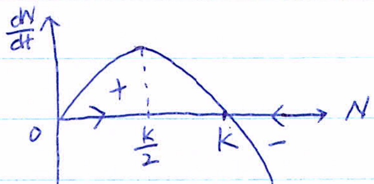
rewrite the logistic growth model as $\frac{dN}{dt} = R(N) \cdot N = \underbrace{r\left(\frac{K-N}{K}\right)}_{\text{growth rate changing with } N} \cdot N$

Compare the model with the exponential growth model including mortality rate $\frac{dN}{dt} = r \cdot N - m \cdot N$

rewrite the logistic growth model as $\frac{dN}{dt} = r \cdot N - \underbrace{r \cdot \frac{N}{K}}_{\text{mortality rate changing with } N} \cdot N$

- Steady States: $\frac{dN}{dt} = rN\left(\frac{K-N}{K}\right) = 0 \Rightarrow N^* = 0$ or $N^* = K$

Sketch $\frac{dN}{dt}$ vs. N

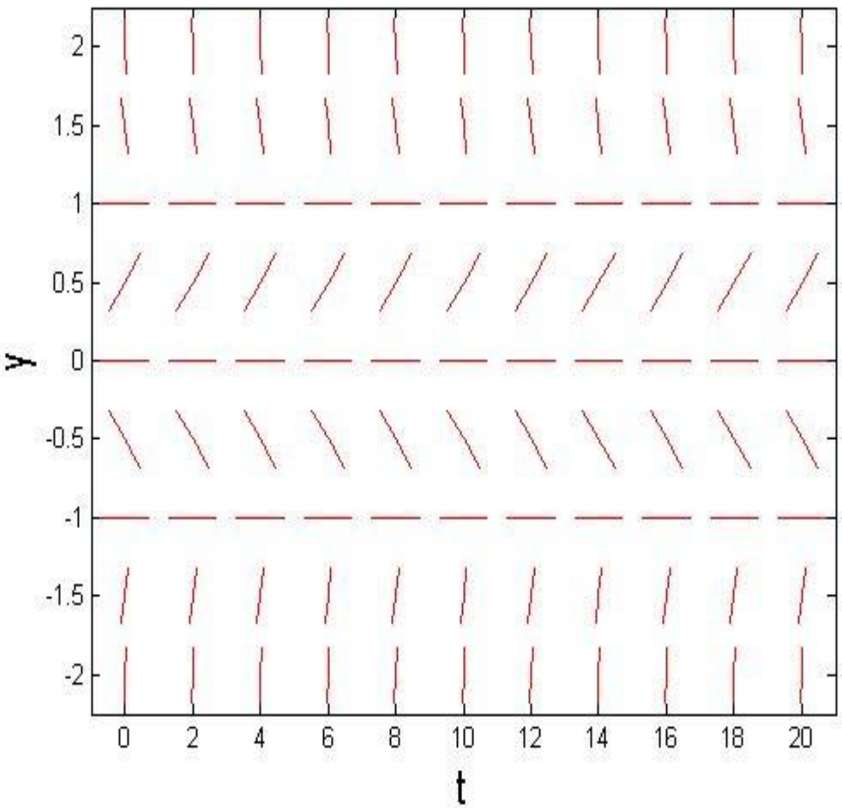


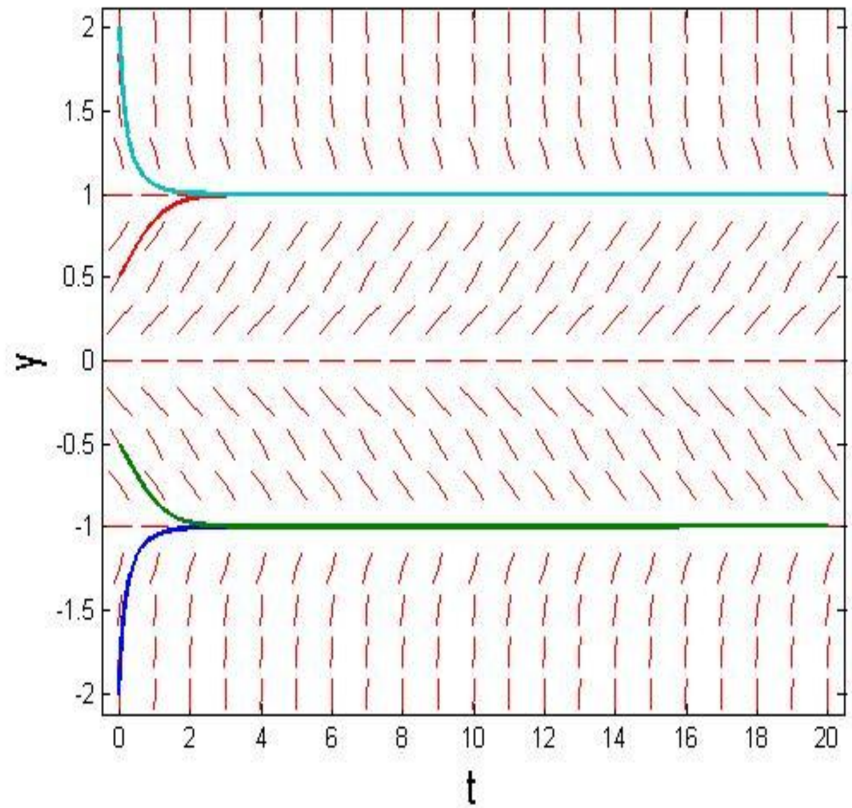
$\Rightarrow N^* = 0$ is unstable
$N^* = K$ is stable

Longterm behaviour: For $N > 0$, no matter $N > K$ or $N < K$, it approaches to K overtime.

▪Slope Field of $\frac{dy}{dt} = y - y^3$

y	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	
y'	6	1.875	0	-0.375	0	0.375	0	-1.875	-6	





■Slope Field of $\dfrac{dy}{dt} = y^4 - y^2$

y	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	
y'	12	2.8125	0	-0.1875	0	-0.1875	0	2.8125	12