# Today

- Absolute extrema
- Optimization examples (goat, Kepler).

# Absolute extrema

- An absolute maximum (minimum) is the highest (lowest) value that a function takes on. Important - it must reach the value.
- A continuous function on a closed interval [a,b] takes on its highest (lowest) value either at a local maximum (minimum) or at an end point (x=a or x=b).
- When looking for absolute extrema, check critical points AND end points!

Where does $f(x)=x^3-x^2$ take on its absolute minimum on the interval $[-1,2]$?

(A) $x=-1$

(B) $x=0$

(C) $x=2/3$

(D) $x=2$

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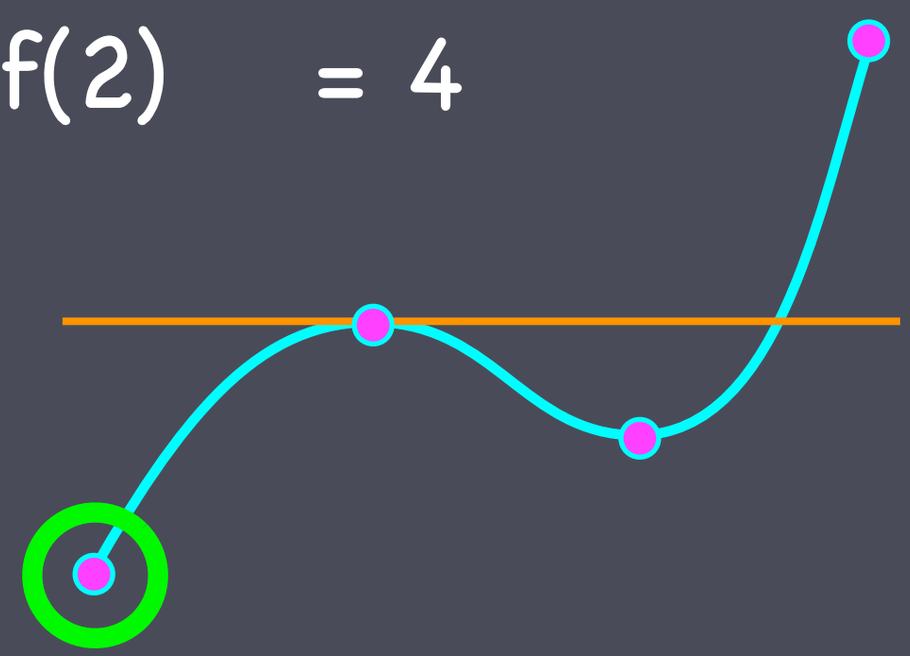
$f(-1) = -2$

$f(0) = 0$

$f(2/3) = -4/27$

$f(2) = 4$



# Finding absolute min/max of f(x) on interval [a,b]:

1. Find all critical points, $x_1$, $x_2$, $x_3$,...
2. Evaluate $f(x_i)$, f(a) and f(b).
3. Max is biggest, min is smallest.
4. If interval is (a,b) or (a,b] or 
   an abs min/max might not exist.
5. No point doing FDT/SDT for each $x_i$.

# Optimization

Given a scenario involving a choice of some number, use calculus to find the best value.

1. Establish an expectation for a reality check later.
2. Translate scenario into a mathematical problem.
3. Solve the problem.
4. Translate back and compare to expectation.

I have 10 meters of fence. I want the enclosure to be as small as possible but it can't be narrower than my goat (1/2 meter).

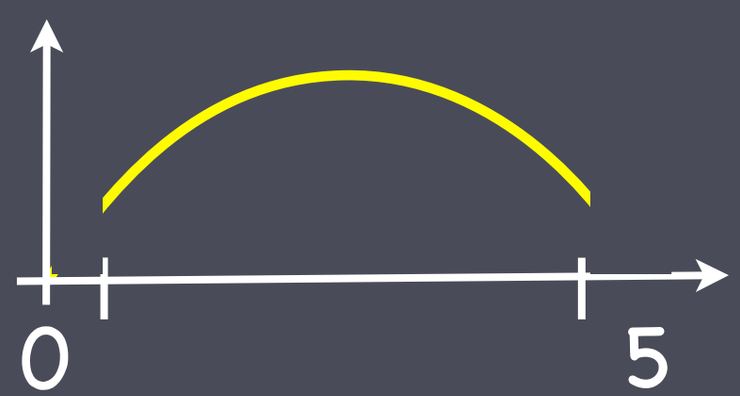
How long and how wide should I make the enclosure?

(A) l = 5/2 m, w = 5/2 m.

(B) l = 0 m, w = 5 m

(C) l = 1/2 m, w = 9/2 m

(D) l = 1/2 m, w = 19/2 m




Find absolute min of $A(w)=w(5-w)$ on $[1/2, 9/2]$.

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# General structure of these problems

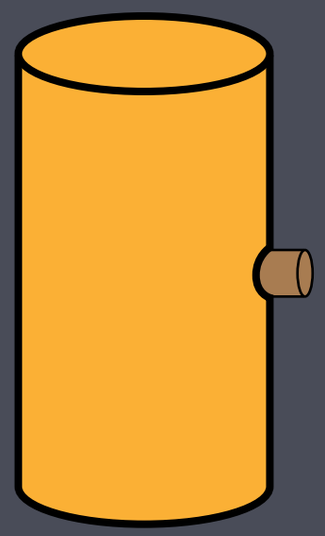
- There's an "objective function" (OF) that you want to maximize/minimize.
- The OF depends on more than one variable.
- There's a constraint relating the two variables.
- The constraint simplifies the OF to one variable.
- The domain is restricted by "physical" considerations.

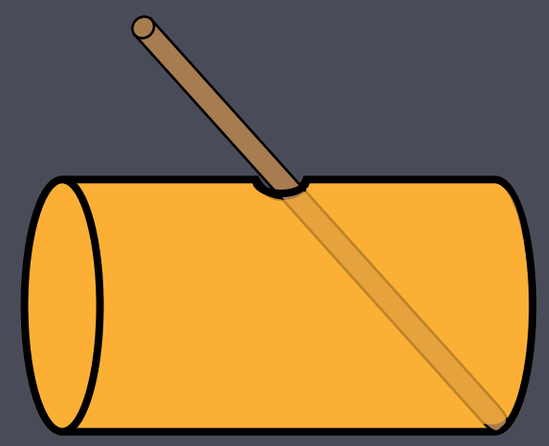
$$A(l,w) = l \times w$$

$$2l + 2w = 10 \longrightarrow l = 5 - w$$

$$A(w)=(5-w)w$$

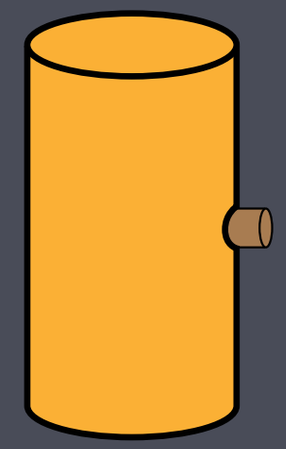
# Wine for Kepler's wedding



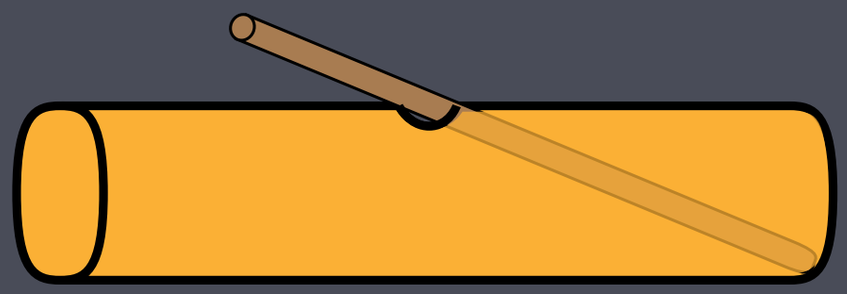


- Wine was sold by "the length of the submerged part of the rod"
- Same length of wet rod = same volume of wine?

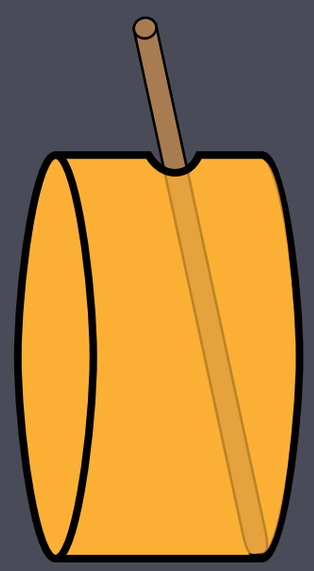
# Which barrel would you buy?



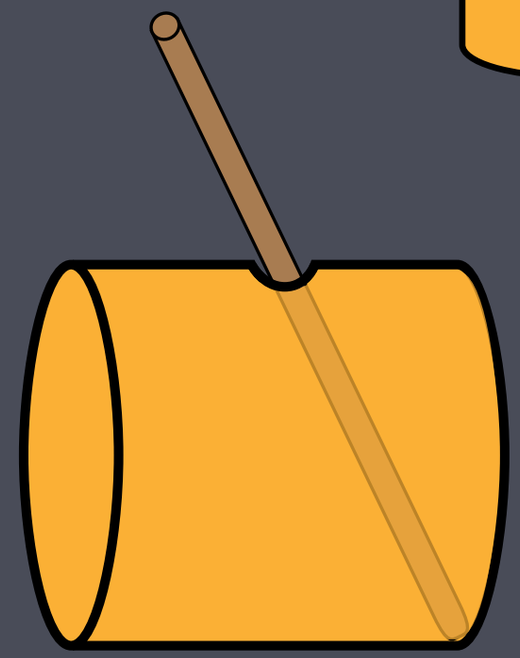
(A)



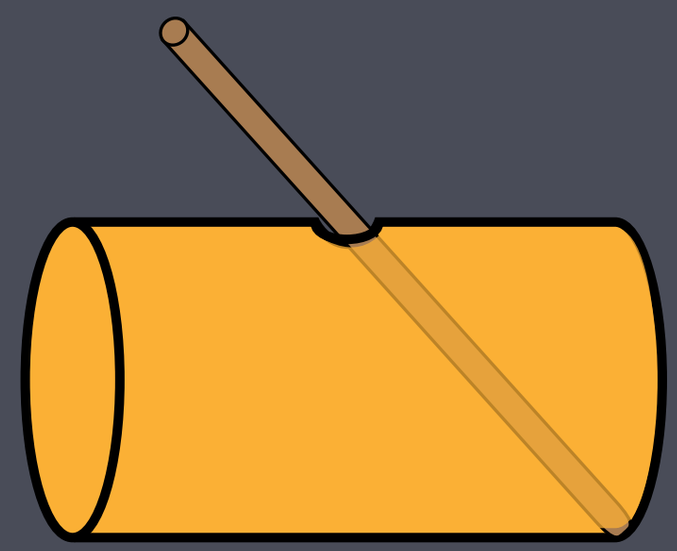
(B)



(C)



(D)



# Kepler should try to

(A) Minimize the length of the rod.

(B) Maximize the volume of the barrel

(C) Maximize the volume while minimizing the length of the rod.

(D) Maximize the volume of the barrel for a fixed rod length.

(E) Minimize the rod length for a fixed volume of the barrel.

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Kepler had enough $ for a rod of length $L_0$. How much wine can he get?

What do you expect to be the best option?

(A) Shortest possible barrel (h=0).

(B) Tallest possible barrel (h = max h).

(C) Somewhere in between.

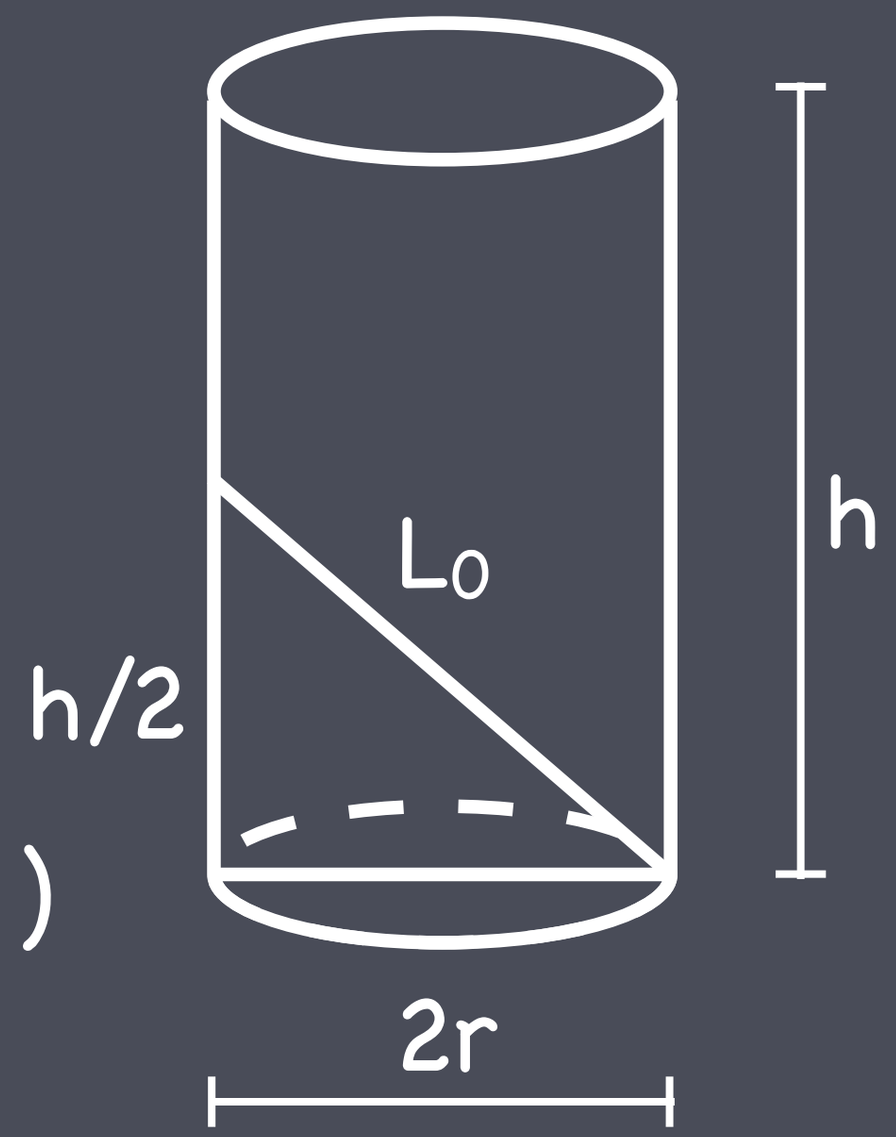
# Objective function? (to be maximized)

(A) $V = 2\pi r h$

(B) $r^2 = L_0^2/4 - h^2/16$

(C) $V = \pi r^2 h$

(D) $L_0 = \mathrm{sqrt}(\ (2r)^2 + (h/2)^2\ )$



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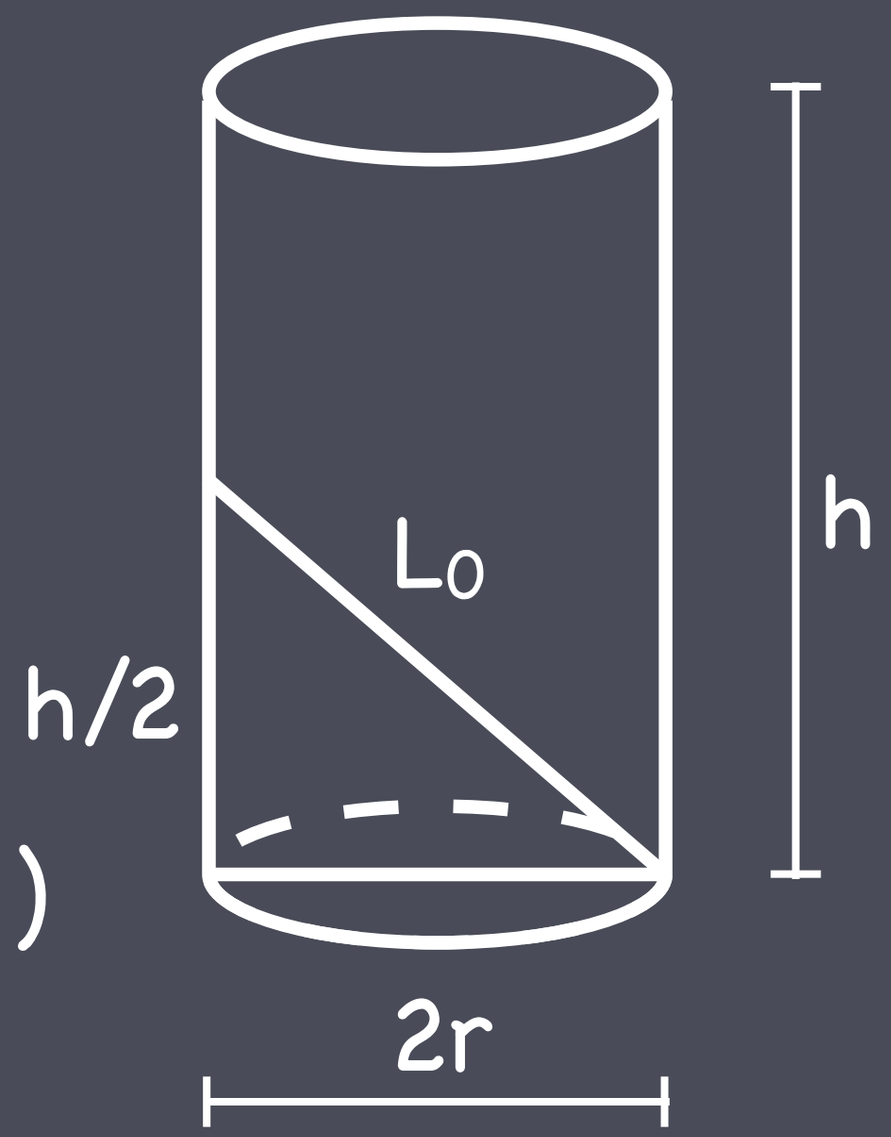


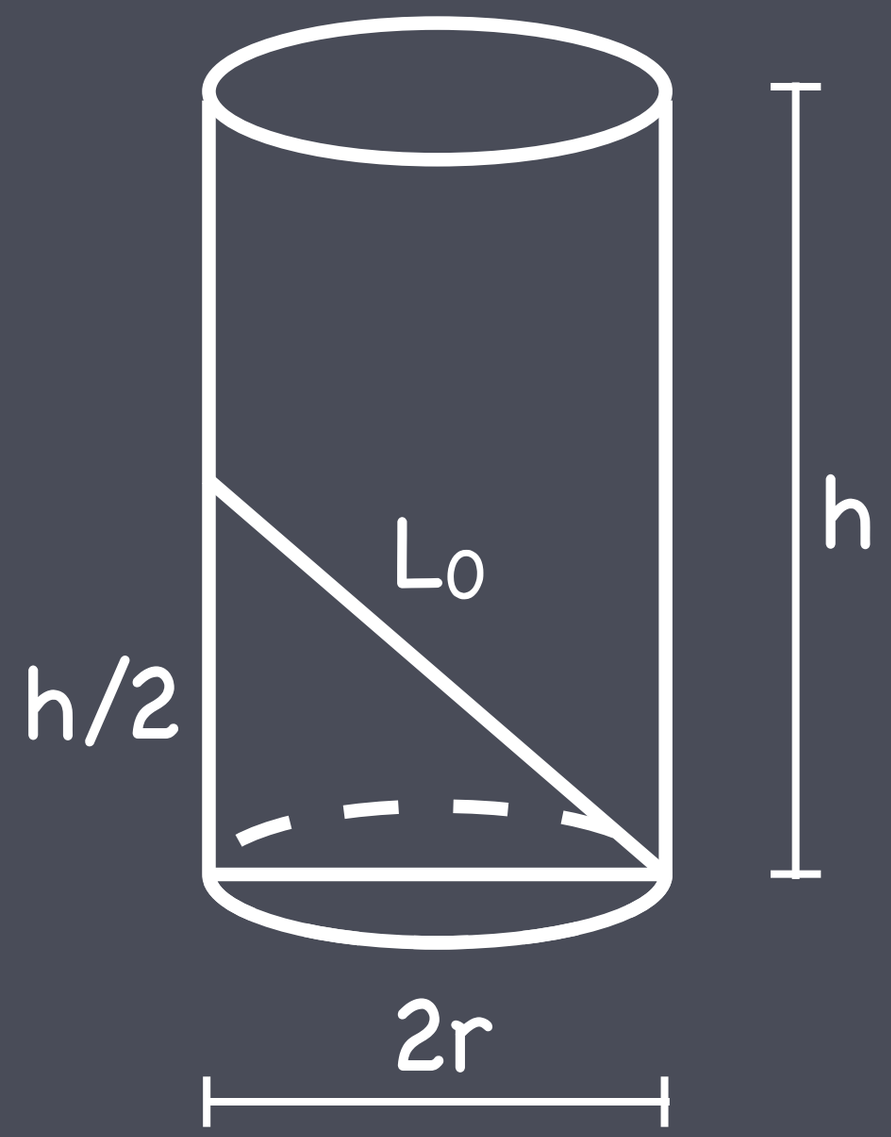
# Constraint?
# (used to simplify OF)

(A) $L_0^2 = (2r)^2 + (h/2)^2$

(B) $L_0^2 = (2r)^2 + h^2$

(C) $V = 2\pi rh$

(D) $L_0 = \tan(h/4r)$



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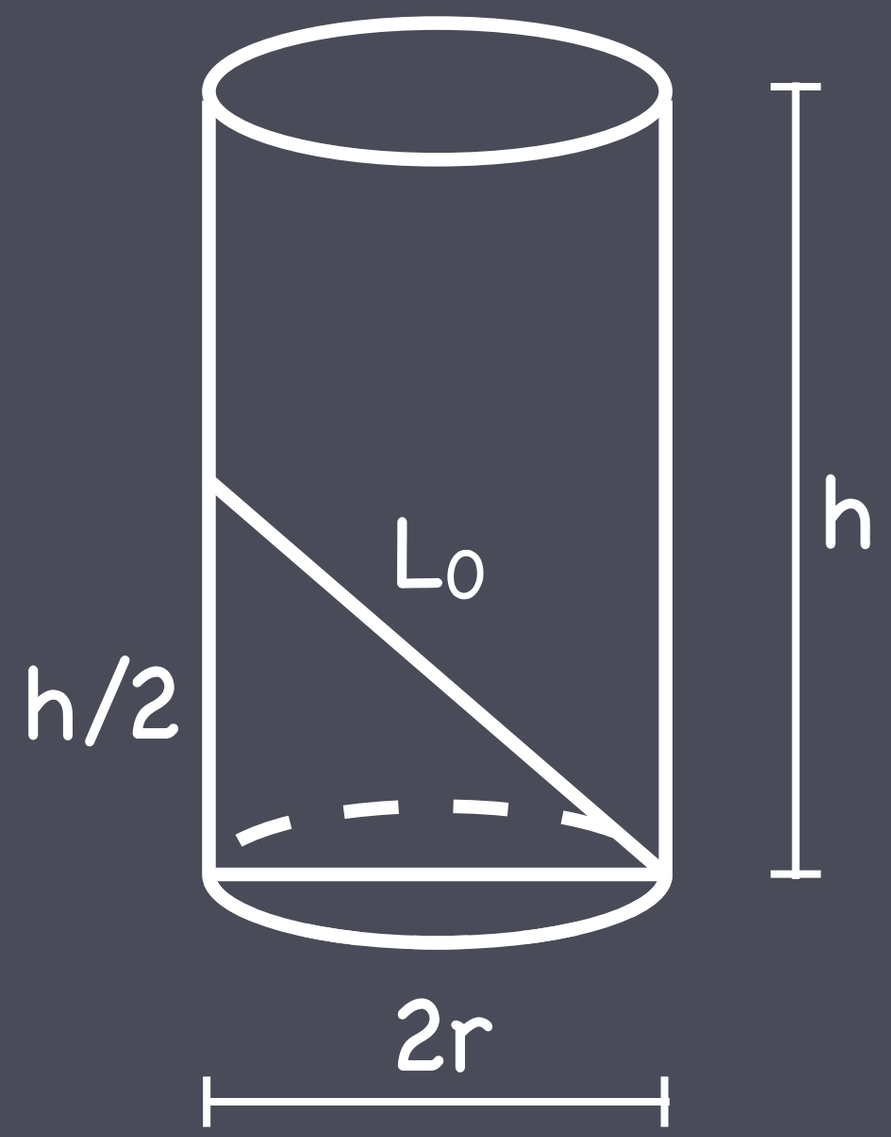


Objective functions: $V = \pi r^2 h$.

Constraint: $L_0^2 = (2r)^2 + (h/2)^2$ .

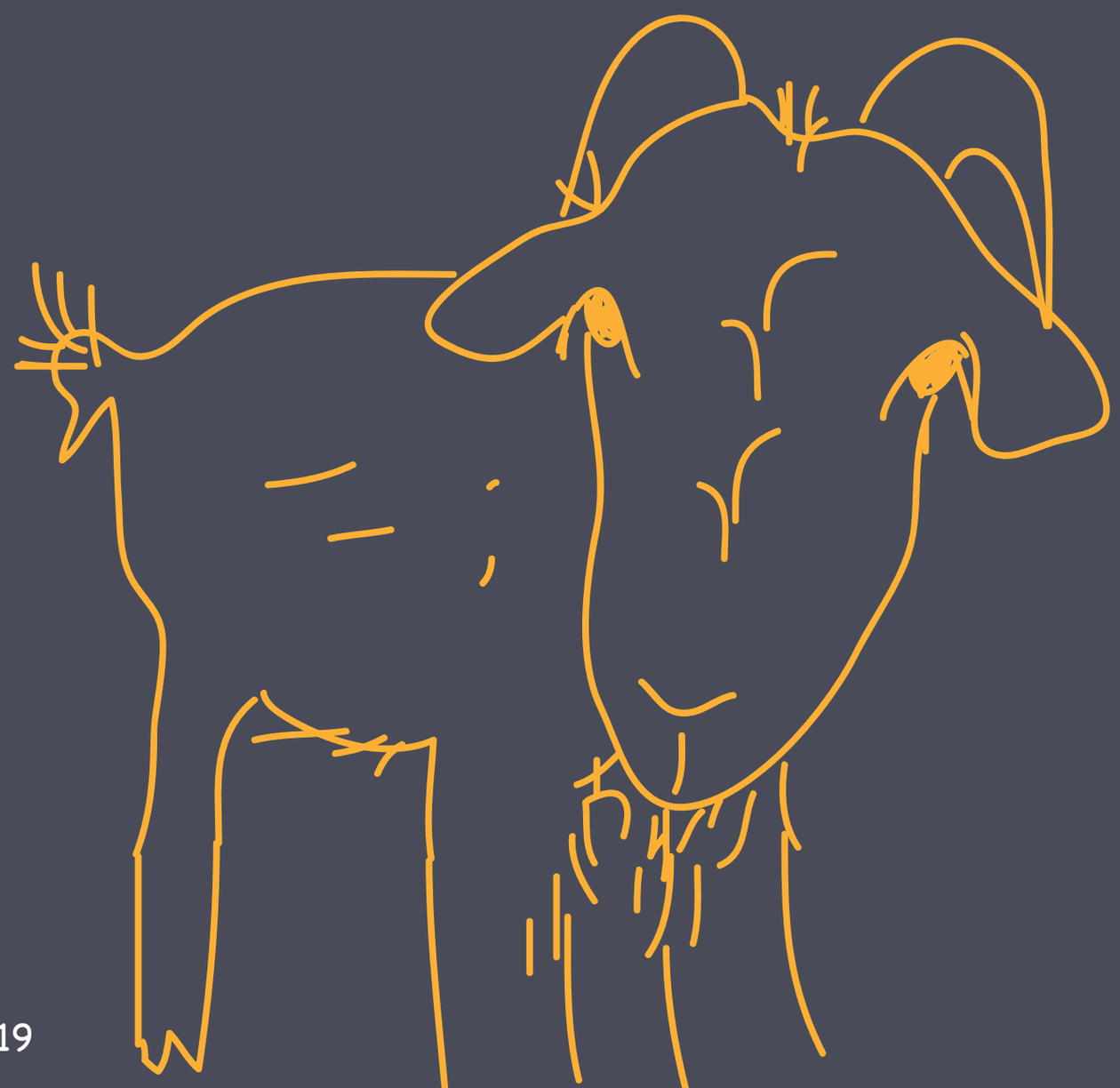
Solve for:

(A) $r$

(B) $r^2$

(C) $h$

(D) $h^2$



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Constraint: $L_0^2 = (2r)^2 + (h/2)^2$ .

Solve for:

(A) r

(B) $r^2$

(C) h

(D) $h^2$

$r^2 = (L_0^2 - h^2/4)\ /4$

Draw a few barrels (h≈0, r≈0)

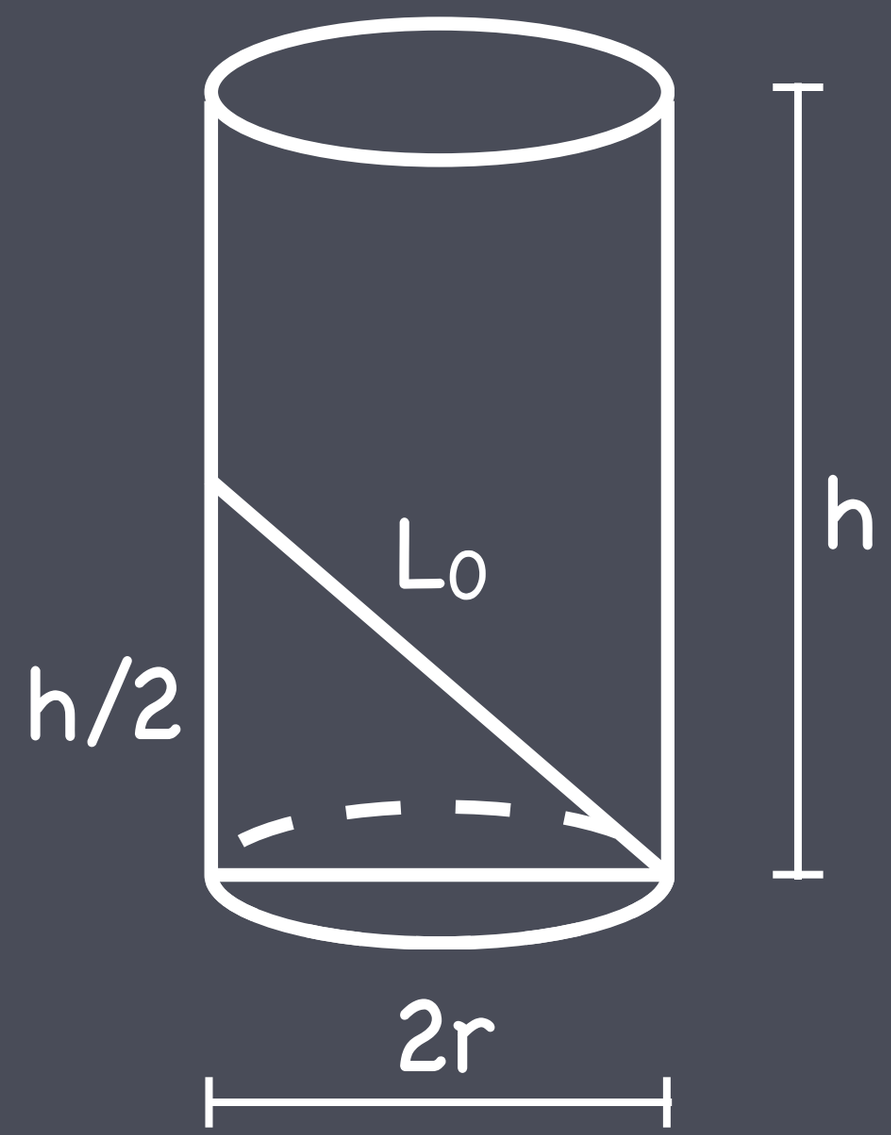
# What is the physical domain for V(h)?

(A) $h > 0$

(B) $0 < h < L_0$

(C) $0 < h < 2L_0$

(D) $0 < h < 2r$



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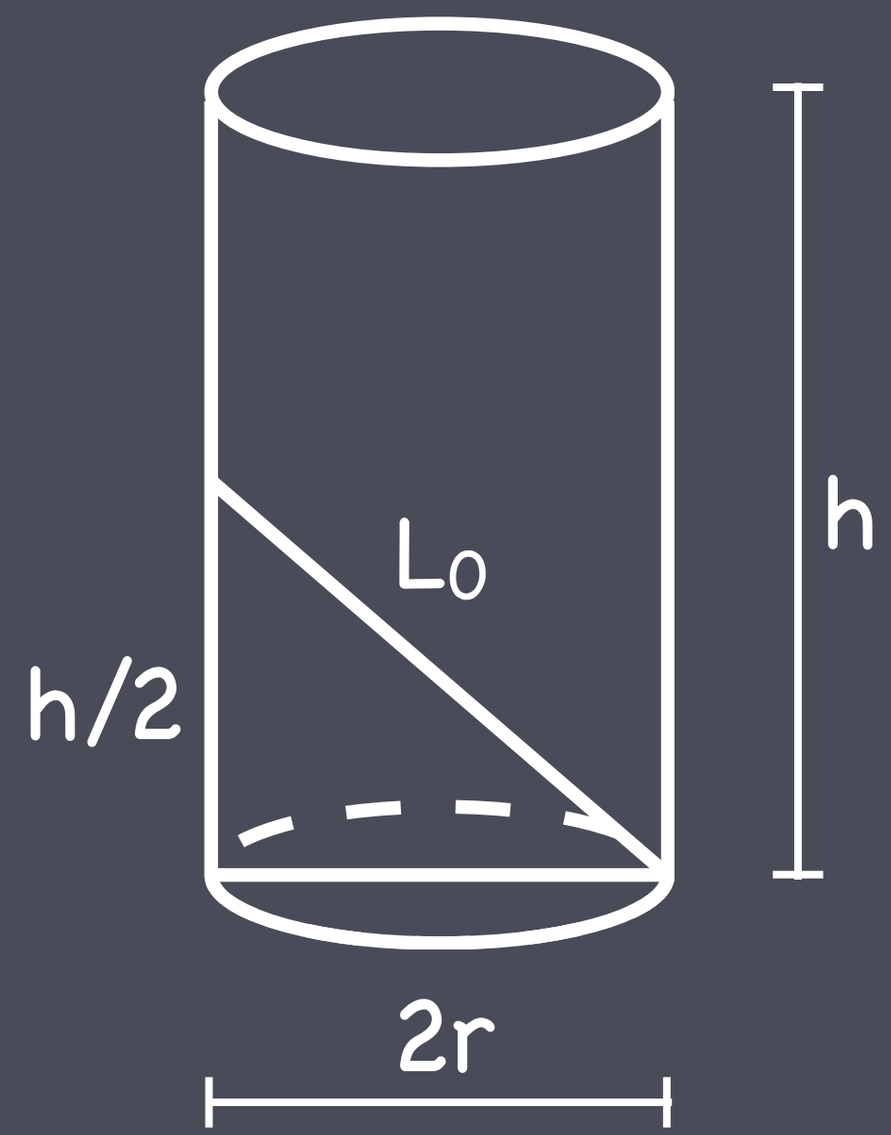


# $V(h) = \pi h(4L_0^2 - h^2)/16$
# What is the best h?

(A) $h = 0$

(B) $h = 2L_0$

(C) $h = \sqrt{3}\ L_0$

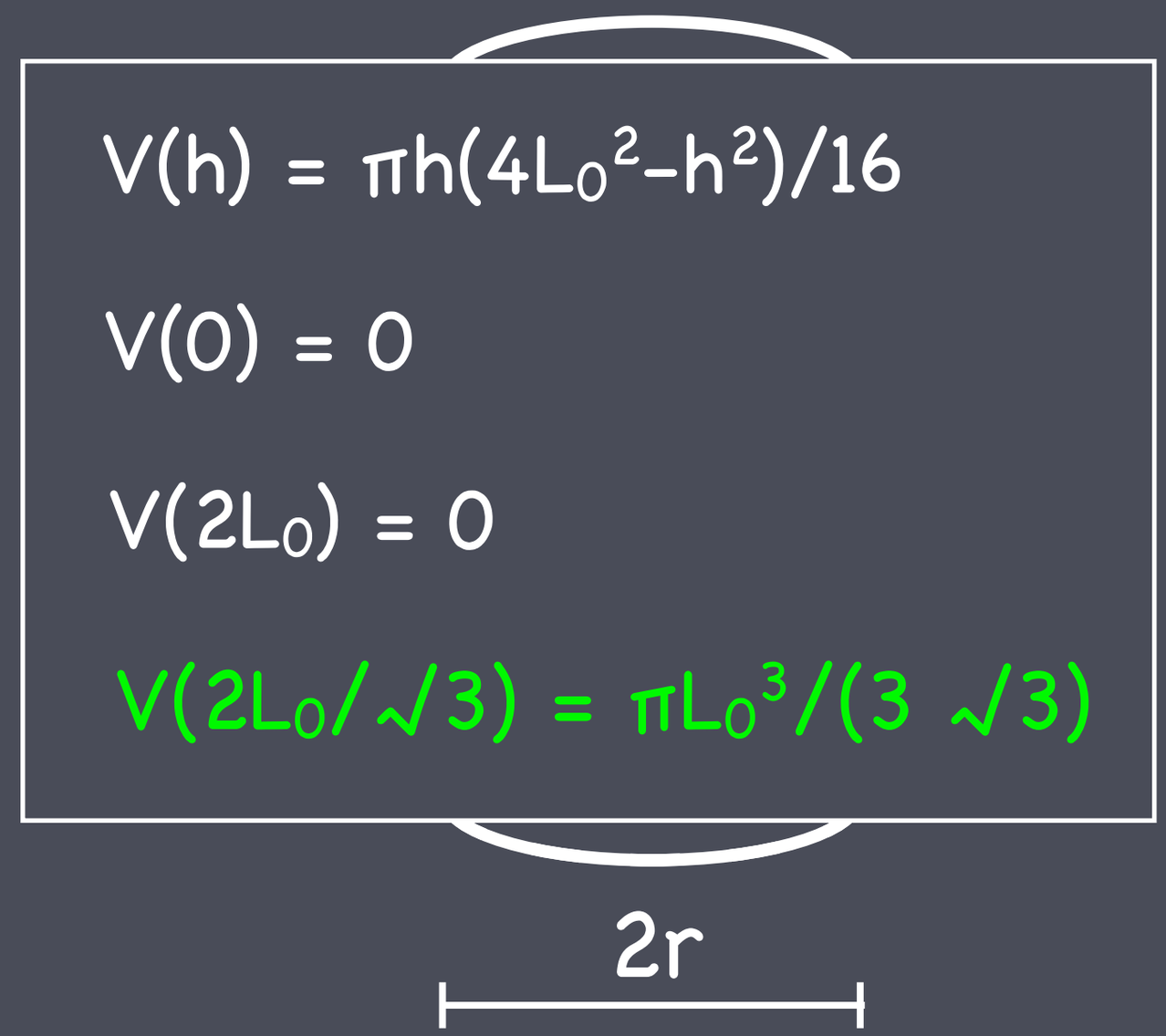
(D) $h = 2L_0/\sqrt{3}$

$V(h) = \pi h(4L_0^2 - h^2)/16$

$V(0) = 0$

$V(2L_0) = 0$

$V(2L_0/\sqrt{3}) = \pi L_0^3/(3\sqrt{3})$



http://www.matematicasvisuales.com/english/html/history/kepler/doliometry.html

# Overall procedure

1. Draw some sketches, establish an expectation.
2. Determine the objective function.
3. Determine the constraint (if necessary).
4. More sketching, check/update expectation.
5. Use constraint --> one variable, make life easy.
6. Find end points and all crit pts.
7. Substitute them into the objective function.
8. Biggest value is the absolute extremum.