# Exercises for Math 102

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October 10, 2011

## 4. The derivative; What the derivative tells us about a function

**Exercise 1:** Let $f(x) = \frac{1}{1+x^2}$ for $-\infty < x < \infty$.

(a) Find the first-order derivative and the second-order derivative of $f(x)$.

(b) Find the zeros, the critical points, and the inflection points of $f(x)$, if any.

(c) Draw the graph of $f(x)$ with explanations. (Cf. the example in Section 5.6 in the course notes.)

**Exercise 2:** You learn in the notes on how to find the derivative of the power function $f(x) = x^3$ from its definition as the limit $\lim_{d\to 0} \dfrac{f(x+d) - f(x)}{d}$. It is given by $f'(x) = 3x^2$.

(a) Write $g(x) = x^3 \cdot x = f(x) \cdot x$. Use the *product rule* and the fact that $f'(x) = 3x^2$ to show that the derivative of the power function $g(x) = x^4$ is $g'(x) = 4x^3$.

(b) Find the derivative of $h(x) = \frac{1}{x}$ from its definition as the limit $\lim_{d\to 0} \dfrac{h(x+d) - h(x)}{d}$.

(c) You can also write

$$g(x) = \frac{f(x)}{1/x},$$

whenever $x \neq 0$. Show that the derivative $g'(x)$ for $x \neq 0$ is equal to $4x^3$ by using the *quotient rule*, your answer for (b), and the foregoing equality.

**Exercise 3:**

(a) Use antiderivatives to show that if the second-order derivative of a function $f''$ is identically zero, then $f$ is a polynomial of degree at most 1.

(b) Show that the converse is also true. That is, every polynomial $f$ of degree at most 1 has zero second-order derivative.

**Exercise 4:** Find the global maximum and global minimum values of $f(x)$ on the given interval.

(a) $f(x) = \frac{x}{x^2+1}$ on $[0, 2]$

(b) $f(x) = x^2 + \frac{2}{x}$ on $[\frac{1}{2}, 2]$

**Exercise 5:** Between $0\,^\circ$C and $30\,^\circ$C the volume $V$ (in cubic centimeters) of 1 kg of water at a temperature $T$ is given approximately by

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

Find the temperature at which water has its maximum density and show that this is indeed a maximum. *Hint:* density is defined as mass/volume.
(Note: here calculators are perfectly acceptable).

**Exercise 6:** Find the global maximum and minimum values achieved by $f(x) = \frac{1}{1+x^2}$ on the interval $-1 \leq x \leq 2$.

**Exercise 7:** Some values of the function $f$ are given in the table below. Assume the values in the table are representative of the function.

| $x$ | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 3 | 4 | 6.5 | 8.2 | 9.8 | 10.8 | 11.2 |

(a) For which $x$-values should we believe that $f(x)$ is concave up? Concave down?

(b) Estimate $f'(3)$ and $f''(3)$.

(c) Find an equation for the tangent line to $f(x)$ at $x = 3$.

**Exercise 8:** Compute

$$\lim_{x \to 1} \frac{3x^5 - x^3 + x^2 - x - 2}{x - 1}.$$

*Hint:* There is an easy way to use the function $f(x) = 3x^5 - x^3 + x^2 - x - 2$, the fact $f(1) = 0$, and the notion of the derivative.

**Exercise 9:** Sketch $y = f(x)$ that has the following properties:

- $f'(x) > 0$ for $x < 1$;
- $f'(x) < 0$ for $x > 1$;
- $f''(x) > 0$ for $x < -2$ and $x > 2$;
- $f''(x) < 0$ for $-2 < x < 2$;
- $\lim_{x \to -\infty} f(x) = -2$ and $\lim_{x \to \infty} f(x) = 0$;
- $-2 < f(-2) < 0$;
- $f(0) > 0$ ;
- $f(2) > 0$.

**Exercise 10:** Consider $y = f(x) = x^4 - 4x^3$.

(a) Find all the critical points, local maximum, local minimum, and inflection points.

(b) Find the global maximum of the function $f(x)$ *on the interval* $1 \leq x \leq 2$.