4. THE DERIVATIVE; WHAT THE DERIVATIVE TELLS US ABOUT A FUNCTION

Exercise 1: Let \( f(x) = \frac{1}{1+x^2} \) for \(-\infty < x < \infty\).

(a) Find the first-order derivative and the second-order derivative of \( f(x) \).

(b) Find the zeros, the critical points, and the inflection points of \( f(x) \), if any.

(c) Draw the graph of \( f(x) \) with explanations. (Cf. the example in Section 5.6 in the course notes.)

Exercise 2: You learn in the notes on how to find the derivative of the power function \( f(x) = x^3 \) from its definition as the limit \( \lim_{d \to 0} \frac{f(x+d) - f(x)}{d} \). It is given by \( f'(x) = 3x^2 \).

(a) Write \( g(x) = x^3 \cdot x = f(x) \cdot x \). Use the product rule and the fact that \( f'(x) = 3x^2 \) to show that the derivative of the power function \( g(x) = x^4 \) is \( g'(x) = 4x^3 \).

(b) Find the derivative of \( h(x) = \frac{1}{x} \) from its definition as the limit \( \lim_{d \to 0} \frac{h(x+d) - h(x)}{d} \).

(c) You can also write
\[
g(x) = \frac{f(x)}{1/x},
\]
whenever \( x \neq 0 \). Show that the derivative \( g'(x) \) for \( x \neq 0 \) is equal to \( 4x^3 \) by using the quotient rule, your answer for (b), and the foregoing equality.

Exercise 3:

(a) Use antiderivatives to show that if the second-order derivative of a function \( f'' \) is identically zero, then \( f \) is a polynomial of degree at most 1.

(b) Show that the converse is also true. That is, every polynomial \( f \) of degree at most 1 has zero second-order derivative.

Exercise 4: Find the global maximum and global minimum values of \( f(x) \) on the given interval.

(a) \( f(x) = \frac{x}{x^2+1} \) on \([0, 2]\)

(b) \( f(x) = x^2 + \frac{2}{x} \) on \([\frac{1}{2}, 2]\)

Exercise 5: Between 0°C and 30°C the volume \( V \) (in cubic centimeters) of 1 kg of water at a temperature \( T \) is given approximately by
\[
V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3
\]

Find the temperature at which water has its maximum density and show that this is indeed a maximum. \textit{Hint:} density is defined as mass/volume.
(Note: here calculators are perfectly acceptable).
Exercise 6: Find the global maximum and minimum values achieved by $f(x) = \frac{1}{1+x^2}$ on the interval $-1 \leq x \leq 2$.

Exercise 7: Some values of the function $f$ are given in the table below. Assume the values in the table are representative of the function.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>3</td>
<td>4</td>
<td>6.5</td>
<td>8.2</td>
<td>9.8</td>
<td>10.8</td>
<td>11.2</td>
</tr>
</tbody>
</table>

(a) For which $x$-values should we believe that $f(x)$ is concave up? Concave down?

(b) Estimate $f'(3)$ and $f''(3)$.

(c) Find an equation for the tangent line to $f(x)$ at $x = 3$.

Exercise 8: Compute

$$\lim_{x \to 1} \frac{3x^5 - x^3 + x^2 - x - 2}{x - 1}.$$ 

Hint: There is an easy way to use the function $f(x) = 3x^5 - x^3 + x^2 - x - 2$, the fact $f(1) = 0$, and the notion of the derivative.

Exercise 9: Sketch $y = f(x)$ that has the following properties:

- $f'(x) > 0$ for $x < 1$;
- $f'(x) < 0$ for $x > 1$;
- $f''(x) > 0$ for $x < -2$ and $x > 2$;
- $f''(x) < 0$ for $-2 < x < 2$;
- $\lim_{x \to -\infty} f(x) = -2$ and $\lim_{x \to \infty} f(x) = 0$;
- $-2 < f(-2) < 0$;
- $f(0) > 0$;
- $f(2) > 0$.

Exercise 10: Consider $y = f(x) = x^4 - 4x^3$.

(a) Find all the critical points, local maximum, local minimum, and inflection points.

(b) Find the global maximum of the function $f(x)$ on the interval $1 \leq x \leq 2$. 
