Lecture 30 (Nov. 18, 2013)

Learning Goals: ① Derivative of Inverse Trig. functions
② Related rates problem
③ Linear Approximation.

• Derivative of Inverse Trig. functions

Assume $y = f(x)$ and $y' = f'(x)$ are given, can we find $\frac{d}{dx}[f^{-1}(x)]$?

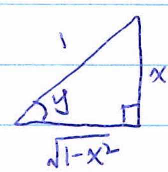
$y = f^{-1}(x) \iff x = f(y)$ ⇐ apply implicit differentiation on both sides w.r.t. x

$$\Rightarrow 1 = \frac{d}{dy}[f(y)] \cdot \frac{dy}{dx} = f'(y) \cdot \frac{dy}{dx} \qquad \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(y)\big|_{y=f^{-1}(x)}}$$

① $y = f(x) = \sin(x)$, $f'(x) = \cos(x)$

$y = f^{-1}(x) = \arcsin(x) \Rightarrow x = \sin(y) \xRightarrow{\text{implicit differentiation w.r.t } x} 1 = \cos(y) \cdot \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$$



$\sin(y) = x = \frac{x}{1}$

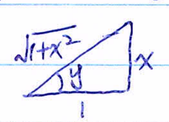
$\Rightarrow \cos(y) = \sqrt{1-x^2}$

② $y = f(x) = \cos(x)$, $f'(x) = -\sin(x)$

$y = f^{-1}(x) = \arccos(x)$, $\frac{d}{dx}[\arccos(x)] = -\frac{1}{\sqrt{1-x^2}}$

③ $y = f(x) = \tan(x)$, $f'(x) = \sec^2(x) = \frac{1}{\cos^2(x)}$

$y = f^{-1}(x) = \arctan(x)$, $\frac{d}{dx}[\arctan(x)] = \frac{1}{\frac{1}{\cos^2(y)}} = \cos^2(\arctan(x)) = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$

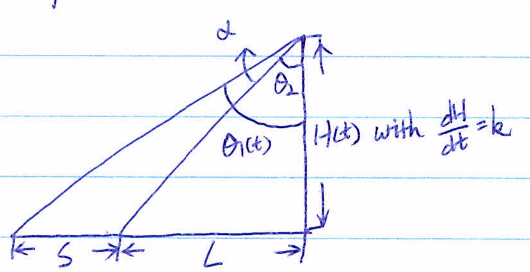


$\tan(y) = x = \frac{x}{1}$

$\Rightarrow \cos(y) = \frac{1}{\sqrt{1+x^2}}$

• Related rates problem

Example 1:



Find $\frac{d\alpha}{dt}$ at $H = h$

Option 1: Law of cosine, $s^2 = L^2+H^2 + (s+L)^2+H^2 - 2\sqrt{L^2+H^2}\sqrt{(s+L)^2+H^2}\cdot\cos\alpha$

Apply implicit differentiation on both sides w.r.t t

Question: what's $\sin(\alpha)$ at $H(t)=h$?

Option 2: $\alpha(t) = \theta_1(t) - \theta_2(t)$

$$= \arctan\left(\frac{s+L}{H}\right) - \arctan\left(\frac{L}{H}\right)$$

Treat $u = \frac{s+L}{H}$ as a function of H and remember $H=H(t)$ is a function of t

$v = \frac{L}{H}$ ... " " "

$$\frac{d\alpha}{dt} = \frac{d\theta_1}{du}\cdot\frac{du}{dH}\cdot\frac{dH}{dt} - \frac{d\theta_2}{dv}\cdot\frac{dv}{dH}\cdot\frac{dH}{dt}$$

$$= \frac{1}{1+\left(\frac{s+L}{H}\right)^2}\cdot\left[-\frac{s+L}{H^2}\right]\cdot\frac{dH}{dt} - \frac{1}{1+\left(\frac{L}{H}\right)^2}\cdot\left[-\frac{L}{H^2}\right]\cdot\frac{dH}{dt}$$

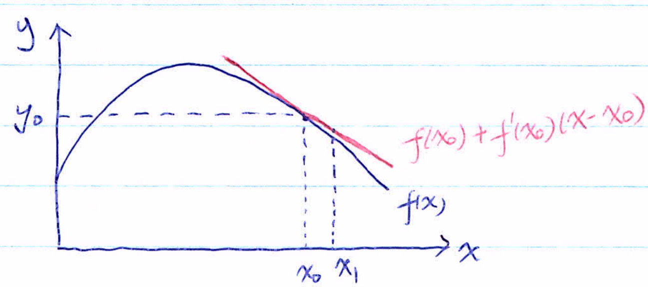
$$\Rightarrow \left.\frac{d\alpha}{dt}\right|_{H=h} = \left(-\frac{s+L}{h^2+(s+L)^2} + \frac{L}{h^2+L^2}\right)\cdot k$$

- Linear Approximation:

Assume $f(x)$ is given and (x_0,y_0) satisfies $y_0=f(x_0)$, we can find the approximation of $f(x)$ near $x=x_0$ by

$$\underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\text{slope of the secant line passing } (x,y)\ \&\ (x_0,y_0)} \approx \underbrace{f'(x_0)}_{\text{slope of the tangent line passing through } (x_0,y_0)}$$

$$\Rightarrow \boxed{f(x) \approx f(x_0) + f'(x_0)(x-x_0)}$$



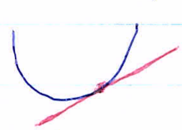
Notice: ① a function of x, x near x_0

② (x_0,y_0), $f'(x_0)$ are necessary

③ $f(x_0)+f'(x_0)(x-x_0)$ is the tangent line.

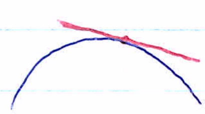
④ underestimate/overestimate: concavity of $f(x)$ at $x=x_0$

concave up



underestimate

concave down



overestimate