# Exercises for Math 102

University of British Columbia
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## 7. Exponential Functions

**Exercise 1:**

(a) Consider the limit of $(x^2 - e^x)$ as $x \to \infty$. Determine whether it converges to a finite number. If not, indicate whether it tends to $+\infty$, $-\infty$ or neither.

(b) Find $\lim_{x\to\infty} x^2 e^{-x}$.

**Exercise 2:** Let
$$f(x) = x^2 \ln x.$$

(a) Determine the domain and range of $f(x)$.

(b) Find the critical points of $f(x)$.

(c) Find the intervals where $f(x)$ is increasing, decreasing, concave up or concave down.

(d) Compute the limit of $f(x)$ as $x \to 0$ and $x > 0$.

(e) Find the coordinates of the global minimum of $f(x)$, i.e. the global minimum value and where it occurs.

**Exercise 3:** Consider the *double exponential* function:
$$g(x) = e^{-|x|}, \quad x \in (-\infty, \infty).$$

(a) Find the critical points, zeros and the inflection points, if any.

(b) Draw the graph.

(c) Find the local maxima and minima, if any. Is each of them a critical point? (Recall that by definition a critical point is a point at which the derivative of the function is defined and is zero.)

(d) Find the global maximum and minimum.

**Exercise 4:** Show that $e^x \geq 1 + x$ for any $x \geq 0$.

**Exercise 5:** **(Change of base).** Instead of using the base $e$, we can also define the logarithm $y = \log_a x$ for any positive number $a$. The logarithm $y = \log_a x$ is again defined to be the inverse function of $y = a^x$. Show the change-of-base formula
$$\log_a x = \frac{\ln x}{\ln a}.$$
(*Hint:* It is enough to obtain $\ln a \cdot \log_a x = \ln x$. Compare $e^{\ln a \cdot \log_a x}$ and $e^{\ln x}$. See also Section 8.8.1.)

**Exercise 6:** Use the formulas in the previous exercise to find the derivative of the function where the base of the logarithm is itself a function of $x$:

$$G(x) = \ln_{x^2+1}(x^4+1).$$

**Exercise 7:** Find the derivative of

$$f(x) = x^x.$$

*Hint:* use exponential functions!

**Exercise 8:**

(a) Write an equation that defines the exponential function with base $a > 0$.

(b) What is the domain of this function?

(c) If $a$ is not equal to 1, what is the range of this function?

(d) Sketch the general shape of the graph of the exponential function for each of the following cases: $a > 1$, $a = 1$, $0 < a < 1$.

**Exercise 9:** Under ideal conditions a certain bacteria population is known to double every three hours. Suppose there are initially 100 bacteria.

(a) What is the population after 15 hours?

(b) What is the size of the population after $t$ hours?

(c) Estimate the size of the population after 20 hours.

**Exercise 10:** Find the domain and range of each of the following functions:

(a) $f(x) = \frac{1}{1+e^x}$

(b) $f(x) = \frac{1}{1-e^x}$

(c) $g(t) = \sqrt{1-2^t}$

**Exercise 11:** Let $f(x) = \ln(x^2)$ and $g(x) = (\ln x)^2$. Are there any values of $x$ where the tangent lines of these two functions have the same slope?

**Exercise 12:** Put these in order of size: $(10^9)^{(10^3)}$, $(10^6)^{(10^6)}$ and $(10^3)^{(10^9)}$.