Guest lecture on November 11, 2013 on derivatives of trigonometric functions

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1 Hook and Objective

Today I am going to talk about the derivatives of trigonometric functions and trigonometric related rates. If time permits, we'll also look at some examples of second order ordinary differential equations.

Learning goals:

- Derivatives of trigonometric functions
- Related rates
- Introduction to second order ODE (if time permits)

Let's start with a revision.

Question: Anyone knows the limit

$$\lim_{h \to 0} \frac{\sin h}{h}?$$

A. 0

B. 1

C. ∞

How do you know the answer is B?

One way to figure that out is by considering the sector OPQ where $OP = OQ = 1$ and $\angle POQ = h$ radian. Draw $PR \perp OQ$ and $SQ \perp OQ$ such that OPS and ORQ are straight lines. By considering

$$Area(\Delta OPR) < Area(\text{sector } OPQ) < Area(\Delta OQS),$$

we arrive at

$$\cos h < \frac{\sin h}{h} < \frac{1}{\cos h}.$$

As $h \to 0$, $\cos h \to 1$. By Squeeze Theorem,

$$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$

This limit will be crucial.

Another question:

$$\begin{aligned}
\sin(A+B) &= \\
\sin(A-B) &= \\
\cos(A+B) &= \\
\cos(A-B) &=
\end{aligned}$$

Here's the answer.

$$\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin B \\
\sin(A-B) &= \sin A \cos B - \cos A \sin B \\
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
\cos(A-B) &= \cos A \cos B + \sin A \sin B
\end{aligned}$$

And from these you get, in particular,

$$\begin{aligned}
\sin A - \sin B &= \sin\left(\frac{A+B}{2} + \frac{A-B}{2}\right) - \sin\left(\frac{A+B}{2} - \frac{A-B}{2}\right) \\
&= 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \\
\cos A - \cos B &= \cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right) \\
&= -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}
\end{aligned}$$

Soon we'll use these equations with $A = x + h$ and $B = x$ to get

$$\begin{aligned}
\sin(x+h) - \sin x &= 2\cos\left(x + \frac{h}{2}\right)\sin\frac{h}{2} \\
\cos(x+h) - \cos x &= -2\sin\left(x + \frac{h}{2}\right)\sin\frac{h}{2}
\end{aligned}$$

2 Body 1: Derivative of trigonometric functions

Now we have the right tools to find the derivative of sine and cosine. We recall that the derivative of a function $f(x)$ is

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Derivative of sin and cos

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h} \\
&= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}} \\
&= \cos(x+0)\cdot 1 \\
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h} \\
&= \lim_{h\to 0}-\sin\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}} \\
&= -\sin(x+0)\cdot 1 \\
\frac{d}{dx}\cos x &= -\sin x
\end{aligned}$$

Using the quotient rule, we can compute the the derivative of tangent, secant, cotangent and cosecant.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$$

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{g(x)^2}$$

$$
\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) \\
&= -\frac{-\sin x}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} = \sec x \tan x
\end{aligned}
$$

3 Questions

Now, try to work out the derivatives of cotangent and cosecant. You have five minutes. Feel free to ask any question.

For reference:

$$
\begin{aligned}
\frac{d}{dx}\cot x &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) \\
&= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} \\
&= -\frac{1}{\sin^2 x} = -\csc^2 x
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\csc x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) \\
&= -\frac{\cos x}{\sin^2 x} = -\csc x \cot x
\end{aligned}
$$

To summarize,

$$
\frac{d}{dx}\sin x = \cos x
$$

$$
\frac{d}{dx}\cos x = -\sin x
$$

$$
\frac{d}{dx}\tan x = \sec^2 x
$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\csc x = -\csc x \cot x$$

4 Body 2: Trigonometric related rates

Now let's talk about a problem of trigonometric related rates.

Question: Anyone remembers the Law of Cosines? (Draw the triangle with sides a, b and c, and label θ as the angle opposite to c.)

Recall that the Law of Cosines is

Law of cosines
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Here a, b and c be the sides and θ the angle opposite to c.

Problem Suppose that the sides $a = 3$ and $b = 4$ are fixed, and the angle θ increases at a constant rate, $d\theta/dt = k$. If $a = 3$ and $b = 4$, determine the rate of change of c at the instant that $c = 5$.

So, how should we start? The first thing we know is that a and b are constants, while c and θ depends on time t. Therefore, c and θ is a function of t. We can write

$$c(t)^2 = 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cdot \cos\theta(t)$$
$$c(t)^2 = 25 - 24\cos\theta(t)$$

We are asked to find the rate of change of c when $c = 5$, and this means we need to find $\frac{dc}{dt}$ when $c = 5$.

Need to find $\frac{dc}{dt}$ when $c = 5$.
Question: Where does the term $\frac{dc}{dt}$ come from?

Let's differentiate both sides with respect to t.

$$\begin{aligned}
\frac{d}{dt}(c(t)^2) &= -24\frac{d}{dt}(\cos\theta(t)) \\
2c(t)\frac{dc}{dt} &= -24(-\sin\theta(t))\frac{d\theta}{dt} \\
c(t)\frac{dc}{dt} &= 12\sin\theta(t)\frac{d\theta}{dt}
\end{aligned}$$

At the instant $c = 5$, we know that $\theta = \pi/2$ by the Pythagoras' theorem. (Alternatively, you can get this from the law of cosines.)

When $c = 5$, since $3^2 + 4^2 = 5^2$, $\theta = \frac{\pi}{2}$.

Putting everything back in, we arrive at

$$\begin{aligned}
\frac{dc}{dt} &= \frac{12\sin\theta}{c}\frac{d\theta}{dt} \\
&= \frac{12\sin\frac{\pi}{2}}{5}k \\
&= \frac{12k}{5}
\end{aligned}$$

5 Body 3: Second order ODEs

Let's look at the derivatives of $\sin x$ and $\cos x$:

$$\begin{aligned}
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= -\sin x
\end{aligned}$$

If we take the second derivative, we can get

$$\begin{aligned}
\frac{d^2}{dx^2}\sin x &= \frac{d}{dx}\cos x \\
&= -\sin x
\end{aligned}$$

$$\begin{aligned}\frac{d^2}{dx^2}\cos x &= \frac{d}{dx}(-\sin x)\\ &= -\cos x\end{aligned}$$

The second derivatives of both $\sin x$ and $\cos x$ are equal to the negative of themselves. That means, both of these functions satisfies the second order ODE

$\sin x$ and $\cos x$ satisfy

$$y''(x) + y(x) = 0$$

If you put $y(x) = \sin x$ the equation holds, and similarly for $y(x) = -\cos x$. In fact,

$\sin(kx)$ and $\cos(kx)$ satisfy

$$y''(x) + k^2 y(x) = 0$$

Let's verify that the method of superposition is valid for this linear second order ODE, which means that

For any numbers A and B, $A\sin(kx) + B\cos(kx)$ satisfy

$$y''(x) + k^2 y(x) = 0$$

<u>Verification</u>

$$\begin{aligned}\frac{d^2}{dx^2}(A\sin(kx) + B\cos(kx)) &= \frac{d}{dx}(A\cos(kx)\cdot k - B\sin(kx)\cdot k)\\ &= k\frac{d}{dx}(A\cos(kx) - B\sin(kx))\\ &= k(-A\sin(kx)\cdot k - B\cos(kx)\cdot k)\\ &= -k^2(A\sin(kx) + B\cos(kx))\end{aligned}$$

Therefore the previous result is true, by taking $A = 1$ and $B = 0$ or $A = 0$ and $B = 1$.

Finally, we remark that the converse is also true.

If $y(x)$ satisfies

$$y''(x) + k^2 y(x) = 0,$$

then $y(x) = A\sin(kx) + B\cos(kx)$ for some numbers A and B.

6 Summary

To summarize, we have calculated the derivatives of trigonometric functions, used them to solve a rate-of-change problem, and discussed an important class of second order ordinary differential equations.