Exercises for Math 102

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5. Optimization

Exercise 1: The average number of new virus particles (virions) produced by a virally-infected cell during its lifetime is called the *burst size*. We will call the burst size N. Under reasonable assumptions, N can be approximated by $N = p/\mu(p)$ where p is the rate of production of new viruses and $\mu(p)$ is the cell death rate. Over many generations, the virus population can evolve so that its production rate p lies anywhere on the range $0 < p < p_{\max}$. The upper bound $p_{\max}$ is set by biological constraints (e.g. nutrient supply to the cells). We expect the virus to evolve to maximize N.

(a) Suppose that the cell death rate is a constant, $\mu(p) = \mu_0 > 0$. What production rate would you expect the virus to have after a long period of evolutionary change?

(b) Suppose the cell death rate is a linear function of the production rate, $\mu(p) = ap + \mu_0$, where a and μ_0 are positive constants. What production rate would you expect the virus to have after a long period of evolutionary change? Does this depend on a? If so, how?

(c) Finally, suppose the cell death rate is a quadratic function of the production rate, $\mu(p) = bp^2 + \mu_0$, where b and μ_0 are positive constants. What production rate would you expect the virus to have after a long period of evolutionary change? Does this depend on b? If so, how?

Exercise 2: Consider the two-variable functions $f(x, y) = 2x + y$ and $g(x, y) = xy$.

(a) Find (x_0, y_0) such that $f(x_0, y_0)$ is the minimum of $f(x, y)$ subject to the constraints $g(x, y) = 2$ and $x, y > 0$.

(b) The constraint $g(x, y) = 2$ determines a y as a function of x, namely, $y = 2/x$. On the other hand, for any constant C, $f(x, y) = C$ determines a line $L_C : 2x + y = C$ on the x-y plane with y-intercept C. For different C's, you get different lines L_C.

Now, on the same coordinate plane, draw the graph of $y = 2/x$ and three different lines L_{C_0}, L_{C_1}, and L_{C_2}. Here, $C_0 = 2x_0 + y_0$ with (x_0, y_0) found in (1), and you can choose arbitrary C_1 and C_2 such that $C_1 > C_0 > C_2$.

You may observe that L_{C_0} has the smallest value of C among all lines L_C intersecting the graph of $y = 2/x$, and (x_0, y_0) is exactly where the optimal line L_{C_0} touches the curve $y = 2/x$, or $g(x, y) = 2$

Exercise 3: Consider an ellipse

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad 0 < b < a.$$

Let $P = (c, 0)$ be a given point on the major axis of this ellipse, for $-\infty < c < \infty$. Find the *minimum distance* between P and points of E, that is, find the minimum value of

$$d(x, y) = \sqrt{(x - c)^2 + y^2} \quad \text{for} \quad (x, y) \in E.$$

Hint: Consider the square of the distance (why is this ok?).

Exercise 4: Find the dimensions of a rectangle with perimeter 100 meters whose area is as large as possible.

Exercise 5: If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

Exercise 6: Determine the point(s) on the curve $y = x^2 + 1$ that are closest to $(0, 2)$ on the interval $-\infty < x < \infty$.
Hint: Consider the square of the distance (why is this ok?).

Exercise 7: At noon, a car B is 10 km north of another car A. Car A travels to the east at 30 km/h and car B travels south at 40 km/h. When will they be the closest to each other?

Exercise 8: Consider a cell of cylindrical shape with circular cross-section. The shape of this cell is determined so as to minimize the sum of the area of the top and bottom (of the cylinder) plus one third of the area of the side of the cylinder, while the volume is fixed. Determine the ratio:

$$\frac{\text{the height of the cylinder}}{\text{the diameter of the cross-section}}$$

Explain your answer.