Today

- Euler’s method for approximating DE solutions
- Logistic equation in many contexts
  - Classic example of the power of mathematics – one unifying description for many apparently unrelated phenomena.
Euler’s method
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$$\text{T}(t) = E + (T_0 - E)e^{-kt},$$
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- A “numerical” method for IVPs.
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- Instead of the actual solution to NLC

\[
T(t) = E + (T_0 - E)e^{-kt},
\]

we find \( T_1, T_2, T_3, \ldots \) which approximate \( T(0.1), T(0.2), T(0.3), \ldots \)
Euler's method for
\[ T'(t) = 0.02 \left( 14 - T(t) \right) \]
with \( T(0)=37 \).

What is the slope of the solution at \( t=0 \)?

(A) -0.02  (C) 0.02
(B) -0.46  (D) -0.28

Forensic calculus!
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(A) -0.02  \quad  (C) 0.02

(B) -0.46  \quad  (D) -0.28
Euler’s method for
\[ T'(t) = k \left( E - T(t) \right) \]
with \( T(0) = T_0 \).

What is the slope of the solution at \( t=0 \)?

(A) \( k(E-T) \)

(B) \( k(E-T_0) \)

(C) \( T'(0) \)

(D) \( kE \)
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What is the equation of the tangent line to \( T(t) \) at \( t=0 \)?

(A) \( y = 37 - 0.46t \)  
(B) \( y = 14 + 23e^{-0.02t} \)  
(C) \( y = 37 - 0.28t \)  
(D) \( y = 37 + 0.46t \)
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What is the equation of the tangent line to \( T(t) \) at \( t=0 \)?

(A) \[ y = T(0) + T'(0)(t-0) \]  (C) \[ y = T_0 + T_0' \ t \]

(B) \[ y = E + (T_0-E)e^{-kt} \]  (D) \[ y = T_0 + k(E-T_0)t \]
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\[ T'(t) = k \left( E - T(t) \right) \]
with \( T(0)=T_0 \).

Use the tangent line at \( t=0 \) to estimate \( T(0.1) \):

(A) \( T_1 = T(0) + 0.1 \ T'(0) \)

(B) \( T_1 = T_0 + 0.1 \ k(E-T_0) \)

(C) \( T_1 = E + (T_0-E)e^{-0.1k} \)

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\[ T(0.1) \approx T_0 + 0.1 \, k \, (E-T_0) \]
Euler's method for

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T_2 = T_1 + 0.1 \ k \ (E-T_1)
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\[ T'(t) = k \left( E - T(t) \right) \]
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T(0.1) \approx T_0 + \Delta t \, k \, (E-T_0) = T_1
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T(0.1) & \approx T_0 + \Delta t \ k \ (E-T_0) = T_1 \\
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Making \( \Delta t \) smaller improves the estimate.

\[ T(0.1) \approx T_0 + \Delta t \cdot k \left( E - T_0 \right) = T_1 \]

\[ T_2 = T_1 + \Delta t \cdot k \left( E - T_1 \right) \]

\[ T_3 = T_2 + \Delta t \cdot k \left( E - T_2 \right) \]
When will Euler’s method underestimate the true solution?

(A) When the derivative of the true solution is positive.

(B) When the derivative of the true solution is negative.

(C) When the second derivative of the true solution is positive.

(D) When the second derivative of the true solution is negative.
When will Euler’s method underestimate the true solution?

(A) When the derivative of the true solution is positive.

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(C) When the second derivative of the true solution is positive.

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Euler’s method

Spreadsheet demo – NLC with a varying environment (fore):

\[ T'(t) = k \ (E(t) - T(t)) \]

with \( T(0) = 37^\circ C \),

\[ k=0.15 \] and

\[ E(t) = 14 + 3 \cos(2 \pi t /24). \]

https://docs.google.com/spreadsheets/d/1JohOzqQ6TeKjt43HGchwoKsRWJ-htMHakaXUMv8ICQ/edit?usp=sharing
Euler's method

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Think about how you would determine time of death if given body temp at two later times.
Rates of change that are proportional to two things
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A chemical reaction with only one reactant occurs at a rate proportional to the how much reactant is present (e.g. radioactive decay):

\[ \frac{dR}{dt} = -kR \]
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\frac{dR_1}{dt} = -kR_1 R_2
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Rates of change that are proportional to two things

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Probability of hitting a blue ball:

\[
\frac{\text{blue area}}{\text{total area}} = \frac{\pi r^2 N}{A} = \text{blue concentration}
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Probability of hitting a blue ball:

\[ \text{blue area/total area} = \frac{\pi r^2 N}{A} \]

\[ \frac{N}{A} = \text{blue concentration} \]
Logistic equation in different contexts...
Rates of change that are proportional to two things
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Infectious disease: bSI (S=susceptible, I=infected)
Rates of change that are proportional to two things

- Infectious disease: \( bSI \) (S=susceptible, I=infected)
- Spread of rumour: \( bNH \) (N = not heard rumour, H = heard rumour)
Rates of change that are proportional to two things

- Infectious disease: \( bSI \) (S=susceptible, I=infected)

- Spread of rumour: \( bNH \) (N = not heard rumour, H = heard rumour)

- Spread of new words: \( bNU \) (use word or not).
Rates of change that are proportional to two things

- Infectious disease: $bSI$ (S=susceptible, I=infected)
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- Spread of new words: $bNU$ (use word or not).
- Spread of new technologies: $bNU$ (use tech or not).
Rates of change that are proportional to two things

- Infectious disease: \( bSI \) (S=susceptible, I=infected)
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- Active oil exploration sites: \( bUD \) (undiscovered and discovered).
Rates of change that are proportional to two things

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- Spread of new technologies: $bNU$ (use tech or not).
- Active oil exploration sites: $bUD$ (undiscovered and discovered).
- Waterlillies in a pond: $bSW$ (waterlillies and space for waterwillies).
...two things that are just different forms of a single thing
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- When X meets Y, there's a chance Y turns into X.
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- When $X$ meets $Y$, there's a chance $Y$ turns into $X$.

- Lose $Y$: $\frac{dY}{dt} = -bXY$ and gain $X$: $\frac{dX}{dt} = bXY$
...two things that are just different forms of a single thing

When X meets Y, there’s a chance Y turns into X.

Lose Y: \( \frac{dY}{dt} = -bXY \) and gain X: \( \frac{dX}{dt} = bXY \)

X+Y= constant = C so Y=C−X.
...two things that are just different forms of a single thing

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- Lose Y: \[ \frac{dY}{dt} = -bXY \] and gain X: \[ \frac{dX}{dt} = bXY \]
- \( X+Y= \text{constant} = C \) so \( Y=C-X \).
- \[ \frac{dX}{dt} = bX(C-X) \]