Today

- Introduction to Differential Equations
- Linear DE ($y' = ky$)
- Nonlinear DE (e.g. $y' = y(1-y)$)
- Qualitative analysis (phase line)

Differential equations (DE)

Carbon dating: The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write down a DE for the amount of C-14 at time t.

(A) $C'(t) = k\ C(t)$ where $k>0$.

(B) $C'(t) = k\ C(t)$ where $k<0$.

(C) $C(t) = C_0 e^{kt}$.

(D) $C'(t) = C_0 e^{-kt}$.

Differential equations (DE)

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(A) $C'(t) = k\ C(t)$ where $k>0$. <---solution grows!

(B) $C'(t) = k\ C(t)$ where $k<0$.

(C) $C(t) = C_0 e^{kt}$. <---if $k<0$, this might be the solution but it's not a DE.

(D) $C'(t) = C_0 e^{-kt}$. <---this is not a DE either.

Differential equations (DE)

Carbon dating: The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write down a DE for the amount of C-14 at time t.

$$C'(t) = -r\,C(t) \quad \text{where } r > 0.$$

Solution:

(A) $C(t) = e^{-rC}$

(B) $C(t) = 17e^{-rt}$

(C) $C(t) = -rC^2/2$

(D) $C(t) = 5e^{rt}$

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- In fact, $C(t) = C_0e^{-rt}$ is a solution for all values of C_0 – show on board.
- DEs are often given with an initial condition (IC) e.g. $C(0)=17$ which can be used to determine C_0.
- DE + IC is called an Initial Value Problem (IVP)

Summary of what you should be able to do

- Take a word problem of the form "Quantity blah changes at a rate proportional to how much blah there is" and write down the DE:
  - $Q'(t) = k\ Q(t)$
- Write down the solution to this equation:
  - $Q(t) = Q_0 e^{kt}$
- Determine k and Q_0 from given values or %ages of Q at two different times (i.e. data).
- Determine half-life/doubling time from data or k.

DEs – a broad view

- We have talked about linear DEs so far:
  - $y' = ky$
- A linear DE is one in which the y' and the y appear linearly (more later about $y' = a + ky$).
- Some nonlinear equations:

$$v' = g - v^2, \quad y' = -\sin(y), \quad (h')^2 = bh.$$

object falling through air	pendulum under water	water draining from a vessel

DEs - a broad view

- Where do nonlinear equations come from?
- Population growth:

$$N' = bN - dN = kN \qquad \text{(linear)}$$

where b is per-capita birth rate, d is per-capita death rate and k=b-d.

- Suppose the per-capita death rate is not constant but increases with population size (more death at high density) so d = cN.

$$N' = bN - (cN)N = bN - cN^2$$

DEs - a broad view

$$\frac{dN}{dt} = bN - cN^2$$

This is called the logistic equation, usually written as

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

where r=b and K=1/c. This is a nonlinear DE because of the N^2.

Qualitative analysis

- Finding a formula for a solution to a DE is ideal but what if you can't?
- Qualitative analysis - extract information about the general solution without solving.
    - Steady states
    - Slope fields
    - Stability of steady states
    - Plotting y' versus y (state space/phase line)

$$x' = x(1 - x)$$

velocity position

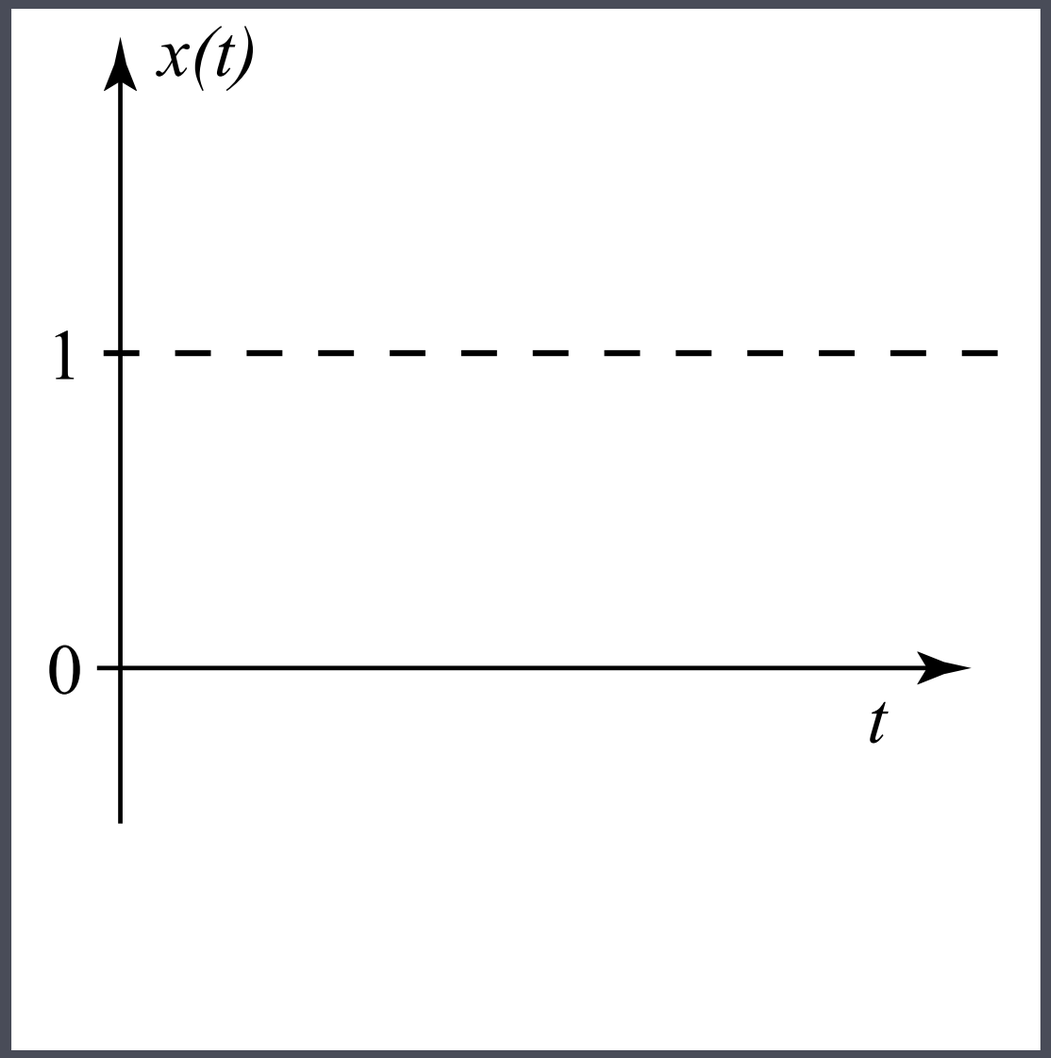
- Steady state. Where can you stand so that the DE tells you not to move?

(A) x=-1

(B) x=0

(C) x=1/2

(D) x=1



A steady state is a constant solution.

$$y' = -y(y-1)(y+1)$$

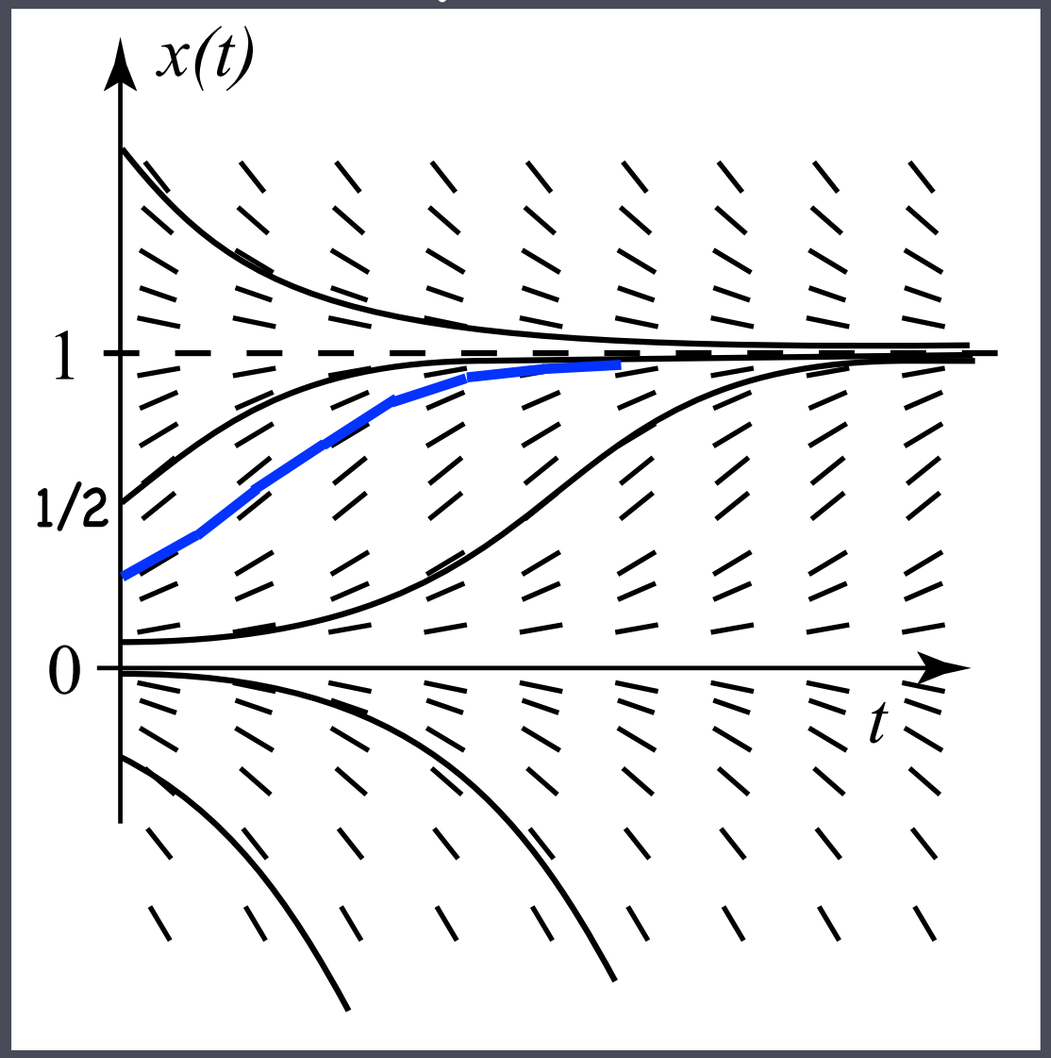
- What are the steady states of this equation?

$$x' = x(1 - x)$$

velocity

position

Slope field



- Slope field.
- At any t, don't know x yet so plot all possible x′ values
- Now draw x(t) for all t.
- Solution curves must be tangent to slope field everywhere.

When x(t) = 1/2 for all t, x′?

(A) 0

(B) 1/4

(C) 1/2

(D) 1

$$y' = -y(y-1)(y+1)$$

- What are the steady states of this equation?
- Draw the slope field for this equation.
    - Include the steepest slope element in each interval between steady states and two others (roughly).