Today

- Introduction to Differential Equations
- Linear DE (\( y' = ky \) )
- Nonlinear DE (e.g. \( y' = y (1-y) \) )
- Qualitative analysis (phase line)
Carbon dating: The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write down a DE for the amount of C-14 at time $t$.

(A) $C'(t) = k \, C(t)$ where $k>0$.

(B) $C'(t) = k \, C(t)$ where $k<0$.

(C) $C(t) = C_0 e^{kt}$.

(D) $C'(t) = C_0 e^{-kt}$.
Differential equations (DE)

Carbon dating: The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write down a DE for the amount of C-14 at time \( t \).

(A) \( C'(t) = k \, C(t) \) where \( k > 0 \). \quad \text{<---solution grows!}

(B) \( C'(t) = k \, C(t) \) where \( k < 0 \).

(C) \( C(t) = C_0 e^{kt} \). \quad \text{<---if } k < 0, \text{ this might be the solution but it's not a DE.}

(D) \( C'(t) = C_0 e^{-kt} \). \quad \text{<---this is not a DE either.}
Carbon dating: The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write down a DE for the amount of C-14 at time \( t \).

\[
C'(t) = -r \ C(t) \quad \text{where } r > 0.
\]

Solution:

(A) \( C(t) = e^{-rt} \)

(B) \( C(t) = 17e^{-rt} \)

(C) \( C(t) = -rC^2/2 \)

(D) \( C(t) = 5e^{rt} \)
Differential equations (DE)

Carbon dating: The amount of Carbon-14 in a sample decreases at a rate proportional to how much C-14 is present. Write down a DE for the amount of C-14 at time $t$.

Solution:

(A) $C(t) = e^{-rt}$

(B) $C(t) = 17e^{-rt}$

(C) $C(t) = -r\frac{C^2}{2}$

(D) $C(t) = 5e^{rt}$

$C'(t) = -rC(t)$ where $r > 0$.

In fact, $C(t) = C_0e^{-rt}$ is a solution for all values of $C_0$ - show on board.

DEs are often given with an initial condition (IC) e.g. $C(0)=17$ which can be used to determine $C_0$.

DE + IC is called an Initial Value Problem (IVP)
Summary of what you should be able to do

- Take a word problem of the form “Quantity blah changes at a rate proportional to how much blah there is” and write down the DE:
  \[ Q'(t) = k \ Q(t) \]
- Write down the solution to this equation:
  \[ Q(t) = Q_0 e^{kt} \]
- Determine \( k \) and \( Q_0 \) from given values or %ages of \( Q \) at two different times (i.e. data).
- Determine half-life/doubling time from data or \( k \).
DEs – a broad view

We have talked about linear DEs so far:

- \( y' = ky \)

A linear DE is one in which the \( y' \) and the \( y \) appear linearly (more later about \( y' = a + ky \)).

Some nonlinear equations:

- \( v' = g - v^2 \), \( y' = -\sin(y) \), \( (h')^2 = bh \).

Object falling through air, pendulum under water, water draining from a vessel.
Where do nonlinear equations come from?

Population growth:

\[ N' = bN - dN = kN \quad \text{(linear)} \]

where \( b \) is per-capita birth rate, \( d \) is per-capita death rate and \( k = b - d \).

Suppose the per-capita death rate is not constant but increases with population size (more death at high density) so \( d = cN \).

\[ N' = bN - (cN)N = bN - cN^2 \]
DEs – a broad view

\[
\frac{dN}{dt} = bN - cN^2
\]

This is called the logistic equation, usually written as

\[
\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right)
\]

where \(r=b\) and \(K=1/c\). This is a nonlinear DE because of the \(N^2\).
Qualitative analysis

Finding a formula for a solution to a DE is ideal but what if you can’t?

Qualitative analysis – extract information about the general solution without solving.

- Steady states
- Slope fields
- Stability of steady states
- Plotting y’ versus y (state space/phase line)
A steady state is a constant solution.

\[ x' = x(1 - x) \]

Steady state. Where can you stand so that the DE tells you not to move?

(A) \( x = -1 \)
(B) \( x = 0 \)
(C) \( x = \frac{1}{2} \)
(D) \( x = 1 \)
\[ y' = -y(y-1)(y+1) \]

What are the steady states of this equation?
\[ x' = x(1 - x) \]

- **Slope field.**
- At any \( t \), don’t know \( x \) yet so plot all possible \( x' \) values
- Now draw \( x(t) \) for all \( t \) where \( x(t) = \frac{1}{2} \) what is \( x' \)?
  - (A) 0
  - (B) \( \frac{1}{4} \)
  - (C) \( \frac{1}{2} \)
  - (D) 1
- Solution curves must be tangent to slope field everywhere.
$y' = -y(y-1)(y+1)$

What are the steady states of this equation?

Draw the slope field for this equation.

Include the steepest slope element in each interval between steady states and two others (roughly).