Today

- Note about Hill functions: \( \frac{x^n}{kn + x^n} = 1 - \frac{k^n}{kn + x^n} \)
- Shape of graphs
- Intro to optimization
$g(x) = 12x^3 - 12x^2$ has...

(A) a maximum at $x=0$ and a minimum at $x=1/3$.

(B) a minimum at $x=0$ and a maximum at $x=1/3$.

(C) a maximum at $x=0$ and an inflection pt at $x=1/3$.

(D) an inflection pt at $x=0$ and a minimum at $x=1/3$. 
g(x) = 12x^3 - 12x^2 has...

(A) a maximum at x=0 and a minimum at x=1/3.
(B) a minimum at x=0 and a maximum at x=1/3.
(C) a maximum at x=0 and an inflection pt at x=1/3.
(D) an inflection pt at x=0 and a minimum at x=1/3.

g(x) = 12x^2(x-1), also has a zero at x=1.
f(x) = 3x^4 - 4x^3 has...

(A) a maximum at x=0 and a minimum at x=1.
(B) a minimum at x=0 and a maximum at x=1.
(C) a maximum at x=0 and an inflection pt at x=1.
(D) an inflection pt at x=0 and a minimum at x=1.
\[ f(x) = 3x^4 - 4x^3 \] has...

(A) a maximum at \( x=0 \) and a minimum at \( x=1 \).

(B) a minimum at \( x=0 \) and a maximum at \( x=1 \).

(C) a maximum at \( x=0 \) and an inflection pt at \( x=1 \).

(D) an inflection pt at \( x=0 \) and a minimum at \( x=1 \).

\[ f'(x) = g(x) = 12x^2(x-1) \] has a max at \( x=0 \) and a zero at \( x=1 \) with \( f'(1^-) < 0 \) and \( f'(1^+) > 0 \).
\[ f(x) = 3x^4 - 4x^3 \]

- \[ f'(x) = 12 (x^3 - x^2) = 0 \quad \Rightarrow \quad x=0, \ x=1. \]
- \[ f''(x) = 12 (3x^2 - 2x). \]
- \[ \text{SDT: } f''(1) = 1 > 0 \quad \Rightarrow \quad \text{f'(x) is increasing near } x=1. \]
- \[ \text{f'(x) goes from - to 0 to + near } x=1. \]
- \[ \text{f(x) has a minimum at } x=1. \]
- \[ \text{SDT: } f''(0) = 0 \quad \Rightarrow \quad \text{Min/max? Inflection point?} \]
Is $x=0$ an inflection point of $f(x) = 3x^4 - 4x^3$?

(A) Yes because $f''(0) = 0$.

(B) Yes because $f''(0) = 0$ and $f'''(0) < 0$.

(C) No because $f''(-1) = 60$ and $f''(1) = 12$.

(D) Yes because $f''(-1) = 60$ and $f''(1/2) = -3$.

Note: $f'(x) = g(x) = 12x^3 - 12x^2$ from earlier and we agreed that $g(x)$ had a max at $x=0$!
Is \( x=0 \) an inflection point of \( f(x) = 3x^4-4x^3 \)?

(A) Yes because \( f''(0) = 0 \).

(B) Yes because \( f''(0) = 0 \) and \( f'''(0) < 0 \).

(C) No because \( f''(-1) = 60 \) and \( f''(1) = 12 \).

(D) Yes because \( f''(-1) = 60 \) and \( f''(1/2) = -3 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>((\infty,0))</th>
<th>0</th>
<th>((0,2/3))</th>
<th>2/3</th>
<th>((2/3,\infty))</th>
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<tbody>
<tr>
<td>( f''(x) )</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>+</td>
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\( f'''(0)<0 \)

“SDT” applied to \( f'(x) \).

Jumped over a zero of \( f''(x) \)

“FDT” applied to \( f'(x) \).
A table for sketching the graph of
\[ f(x) = 3x^4 - 4x^3 = x^3(3x-4) \]

<table>
<thead>
<tr>
<th>( x )</th>
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<th>(4/3)</th>
<th>((4/3, \infty))</th>
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<tr>
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<td>(-)</td>
<td>0</td>
<td>( +)</td>
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<tr>
<td>( f(x) )</td>
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</table>

\[
\begin{align*}
f''(x) &= 12(3x^2 - 2x) = 12x(3x-2) \\
\end{align*}
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>((-\infty, 0))</th>
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<th>((0, 1))</th>
<th>1</th>
<th>((1, \infty))</th>
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<tbody>
<tr>
<td>( f''(x) )</td>
<td>(+)</td>
<td>0</td>
<td>(-)</td>
<td>0</td>
<td>( +)</td>
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</table>

\[
\begin{align*}
f'''(x) &= 12(3x^3 - x^2) = 12x^2(x-1) \\
\end{align*}
\]
### The whole table

<table>
<thead>
<tr>
<th></th>
<th>(-∞,0)</th>
<th>0</th>
<th>(0,2/3)</th>
<th>2/3</th>
<th>(2/3,1)</th>
<th>1</th>
<th>(1,4/3)</th>
<th>4/3</th>
<th>(4/3,∞)</th>
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<tr>
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- **Not a min/max inflection point**
- **minimum**
- **inflection point**

**f(x) = 3x - 4x^3**
Annotating the graph of $f(x)$ with $f'(x)$ info

- $f$ is increasing: $f' > 0$
- $f$ is decreasing: $f' < 0$
- $f$ is discontinuous: $f'$ jumps past 0

What is the function doing?

What does that mean for the first derivative?
Annotating the graph of $f(x)$ with $f''(x)$ info

What is the first derivative doing?

What does that mean for the second derivative?
From table to graph

The parts list:

1. $f' > 0$, $f'' > 0$
2. $f' < 0$, $f'' > 0$
3. $f' < 0$, $f'' < 0$
4. $f' > 0$, $f'' < 0$

A. $f' = 0$, $f'' > 0$
B. $f' = 0$, $f'' < 0$
C. $f' \neq 0$, $f'' > 0$
D. $f' \neq 0$, $f'' < 0$
From table to graph

The parts list:

<table>
<thead>
<tr>
<th>f’&gt;0</th>
<th>f’&lt;0</th>
<th>f”&gt;0</th>
<th>f’&lt;0</th>
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Graph illustrating various combinations of f’, f”, and f” values.