MATH 102: INTRODUCTION TO DIFFERENTIAL EQUATIONS

Exponential growth and decay, population growth, etc.

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WHAT IS A DIFFERENTIAL EQUATION?

A differential equation is an equation that contains derivatives.

Why are they useful?

- Often used to model rates of change
- Examples
  - How a population size is changing over time
  - How a radioactive substance is decaying
  - How a cup of coffee is cooling down

EX: CONSTANT PER-CAPITA POPULATION GROWTH

We are wanting to model the growth of a population with respect to time.

We are told that the rate of change of the population is proportional to the current population size.

$$\frac{dP}{dt} \propto P$$

and that there is a constant per-capita rate of growth.

$$\frac{dP}{dt} = kP$$

Q: Now, let's say we are told that this per-capita growth rate is 1.5 so that $\frac{dP}{dt} = 1.5P$.

What is a solution P(t) to the differential equation?

A. $P(t) = e^{1.5p}$

B. $P(t) = \frac{1.5p^2}{2}$

C. $P(t) = e^{1.5t}$

D. $P(t) = 1.5$

*Many differential equations rely on the idea of exponential growth and decay

Now, let's say we are asked to solve the same differential equation, $\frac{dP}{dt} = 1.5P$, but are now told that the population at time 0, P(0)=1000.

Does our solution $P(t) = e^{1.5t}$ still work?

$$P(0) = e^{1.5(0)} = 1$$

NO!

What can we do to fix this?

- Add 1000 as a coefficient

$P(t) = 1000e^{1.5t}$

Does this satisfy our differential equation?

$\frac{dP}{dt} = 1.5(1000e^{1.5t}) = 1.5P$

Yes!

These types of problems are called **Initial Value Problems.**

- given $\frac{dF}{dt} = kF$
- Told that $F(0) = F_0$
- Solution will be $F(t) = F_0e^{kt}$

EX: GROWTH OF A MOUSE POPULATION

SOLVING EXPONENTIAL EQUATIONS USING 2 POINTS

Your house has a mouse infestation. 2 months ago there were 3 mice in your house. Now, there are 24.

Assume that the rate of change of the mouse population is proportional to the current population size. (ie: $\frac{dP}{dt} = kP$)

What solution equation can we use to model this situation?

Doubling Time

Q: When was the mouse population exactly double its initial size?

A. $t = 2\frac{ln(2)}{ln(8)}$

B. $t = 2\frac{ln(8)}{ln(2)}$

C. $t = \frac{24}{ln(2)}$

D. $t = 24ln(2)$

This is known as the **doubling time**.

$$\tau_{double} = \frac{\ln(2)}{k}$$

Half Life

- **Half-life** is exactly how long it takes for a substance to decrease to ½ of its amount.

Have initial substance amount A_0, and want to find when the substance amount is $\frac{1}{2}A_0$.

$$\frac{1}{2}A_0 = A_0 e^{k\tau}$$

$$\frac{1}{2} = e^{k\tau}$$

$$\tau_{half} = \frac{\ln(\frac{1}{2})}{k} = \frac{-\ln(2)}{k}$$

EX: EXPONENTIAL BREAKDOWN OF DRUG IN THE BODY

A patient takes 600 mg of ibuprofen which has a half-life of about 2 hours. Assume that the rate at which the substance is being broken down is proportional to the amount remaining.

Q: How long after the patient has taken the drug will only 200 mg remain in his system?

A. 2.00 hrs
B. 5.62 hrs
C. 3.17 hrs
D. 3.00 hrs

POPULATION GROWTH REVISITED

In 2013, a population with a starting size of 1000 people had 30 births and 20 deaths. Assume that the population growth continues this way and is proportional to the current population size.

How can we create a differential equation describing the rate of population growth?

Know that population growth can be modelled as

$$\frac{dN}{dt} = kN$$

How can we determine k from the data given?

Change in Population Size

- (change in population size per year) = (# of births per year) - (# of deaths per year)

- k = rate at which population is changing

- Assume all individuals in the population each have an average per-capita birth rate and per-capita death rate.

- $\mathrm{b} = \mathrm{per} - \mathrm{capita\ birth\ rate} = \frac{\#\ of\ births\ per\ year}{population\ size}$

- $\mathrm{m} = \mathrm{per} - \mathrm{capita\ death\ rate} = \frac{\#\ of\ deaths\ per\ year}{population\ size}$

In 2013 a population with a starting size of 1000 people had 30 births and 20 deaths. Assume that the population growth continues this way and is proportional to the current population size. How can we create a differential equation describing the rate of population growth?

$$\frac{dN}{dt} = bN - mN$$

$$\frac{dN}{dt} = (b - m)N$$

$$\frac{dN}{dt} = kN \qquad \text{where } \mathrm{k} = \mathrm{b} - \mathrm{m}$$

In 2013 a population with a starting size of 1000 people had 30 births and 20 deaths. Assume that the population growth continues this way and is proportional to the current population size. How can we create a differential equation describing the rate of population growth?

$$\frac{dN}{dt} = 0.01N.$$

Q: What is the solution to this differential equation?

A. $N(t) = 1000e^{0.01t}$
B. $N(t) = 0.01e^{1000t}$
C. $N(t) = e^{0.01t}$
D. $N(t) = 0.005N^2$

Q: How large will the population be at the beginning of the year 2020?

A. 1100
B. 1062
C. 1029
D. 1073