Lecture 31 (Nov. 20, 2013)

Learning Goal: Newton's method

- (Continue) Linear Approximation

Given $(x_0, f(x_0))$ and $f'(x_0)$, $f(x)$ is estimated by $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ for x near $x = x_0$

Example 1: Estimate $f(x) = e^{\sin(x)}$ near $x=0$ by linear approximation

$x_0 = 0$, $f(x_0) = 1$, $f'(x_0) = e^{\sin(x)} \cdot \cos(x) \big|_{x=x_0} = 1$

$f(x) \approx f(x_0) + f'(x_0)(x - x_0) = 1 + 1 \cdot (x - 0) = x + 1$

Example 2: Estimate $\sqrt{8}$ based on ① $\sqrt{4} = 2$; ② $\sqrt{9} = 3$

① set $f(x) = \sqrt{x}$, $x_0 = 4$, $f(x_0) = 2$, $f'(x_0) = \frac{1}{2\sqrt{x}} \big|_{x=x_0} = \frac{1}{4}$

then $f(8) = \sqrt{8} \approx f(x_0) + f'(x_0)(x - x_0) = 2 + \frac{1}{4}(8-4) = 3$

② set $f(x) = \sqrt{x}$, $x_0 = 9$, $f(x_0) = 3$, $f'(x_0) = \frac{1}{2\sqrt{x}} \big|_{x=x_0} = \frac{1}{6}$

then $f(8) = \sqrt{8} \approx f(x_0) + f'(x_0)(x - x_0) = 3 + \frac{1}{6}(8-9) = \frac{17}{6} \approx 2.8333$

Actual $\sqrt{8} \approx 2.828427$

Notice: ① both approximations overestimate $\sqrt{8}$

$f''(x) = \left(\frac{1}{2\sqrt{x}}\right)' = -\frac{1}{4} x^{-\frac{3}{2}} < 0 \Rightarrow$ concave down $\Rightarrow$ overestimate

② smaller $x - x_0$ provides a better approximation of $f(x)$

Example 3: Estimate $\sqrt{0.9}$ based on ① $f(x) = \sqrt{x}$; ② $f(x) = \sqrt{1-x}$

① $f(x) = \sqrt{x}$, which indicates $\sqrt{0.9} = f(x = 0.9)$

choose $x_0 = 1$, then $f(x_0) = 1$, $f'(x_0) = \frac{1}{2}$

then $\sqrt{0.9} = f(0.9) \approx 1 + \frac{1}{2}(0.9 - 1) = 1 - 0.05 = 0.95$

② $f(x) = \sqrt{1-x}$, which indicates $\sqrt{0.9} = f(x = 0.1)$

choose $x_0 = 0$, then $f(x_0) = 1$, $f'(x_0) = -\frac{1}{2}$

then $\sqrt{0.9} = f(0.1) \approx 1 + (-\frac{1}{2})(0.1 - 0) = 1 - 0.05 = 0.95$

- Newton's method: find the roots of $f(x) = 0$

e.g. ① $f(x) = x^2 - 8$; ② $f(x) = e^x - 1$; ③ $f(x) = x^3 + 5x^2 - 2$

In general, to find the root of $f(x) = 0$, we start from some point (x_0, y_0) on $f(x)$

near $x = x_0$, $f(x)$ is approximated by $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ using linear approximation

then the root of $f(x) = 0$ can be estimated from $f(x_0) + f'(x_0)(x - x_0) = 0$

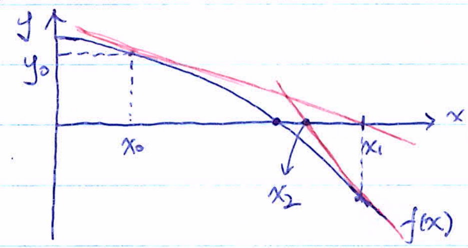
$\Rightarrow x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

To improve the approximation, we can repeat the procedure by linearly approximating $f(x)$ near $x = x_1$: $f(x) \approx f(x_1) + f'(x_1)(x - x_0) = 0$

$\Rightarrow x_2 = x_1 - \dfrac{f(x_0)}{f'(x_0)}$

$\vdots$

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}, \quad k = 1, 2, 3, \cdots$$



Example 4 Estimate $\sqrt{8}$ by Newton's method with $x_0 = 2$

$\Leftrightarrow$ find the positive root of $f(x) = x^2 - 8 = 0 \quad \Rightarrow f'(x) = 2x$

$x_0 = 2, \quad f(x_0) = -4, \quad f'(x_0) = 4$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 2 - \left(\dfrac{-4}{4}\right) = 3, \quad f(x_1) = 1, \quad f'(x_1) = 6$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 3 - \dfrac{1}{6} = \dfrac{17}{6}, \quad f(x_2) = \left(\dfrac{17}{6}\right)^2 - 8, \quad f'(x_2) = \dfrac{17}{3}$

$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = \dfrac{17}{6} - \dfrac{\left(\frac{17}{6}\right)^2 - 8}{\frac{17}{3}} \approx 2.828431$ where we know $\sqrt{8} \approx 2.828427$