# Lecture 22 (Oct. 28, 2013)

- Learning Goals: ① Solution to a DE
  ② DE $\frac{dy}{dx} = ky$, $k$ - constant
  ③ Application: population growth

- Solution to a DE: a function that satisfies DE

Verify a solution: Plug the function into LHS & RHS of DE, LHS $\overset{?}{=}$ RHS

Example I: Which of the following function is a solution to $\frac{dy}{dx} = \frac{1}{y}$ ?

① $y = \ln(2x)$ ; ② $y = \sqrt{2x}$ ; ③ $y = \frac{1}{x}$

① $LHS = \frac{dy}{dx} = \frac{1}{x}$ ; $RHS = \frac{1}{y} = \frac{1}{\ln(2x)}$ $\Rightarrow LHS \neq RHS$

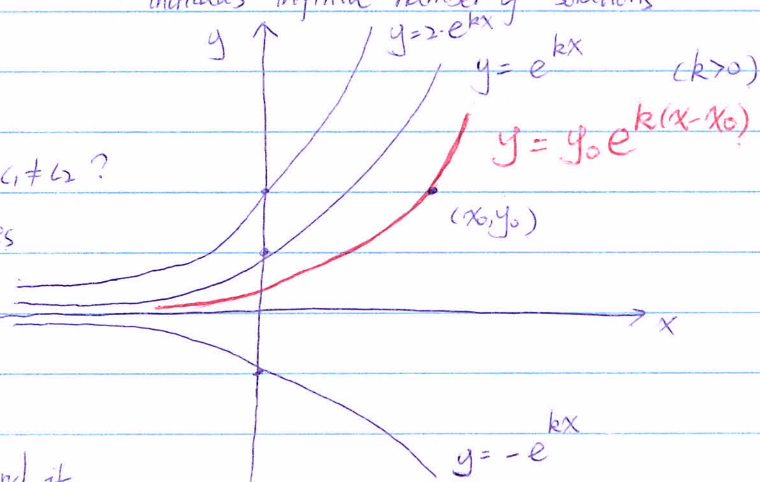
② $LHS = \frac{dy}{dx} = \sqrt{2} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{2x}}$, $RHS = \frac{1}{y} = \frac{1}{\sqrt{2x}}$ $\Rightarrow LHS = RHS \Rightarrow$ it's a solution

③ $LHS = \frac{dy}{dx} = -\frac{1}{x^2}$ ; $RHS = \frac{1}{y} = x$ $\Rightarrow LHS \neq RHS$

- DE $\frac{dy}{dx} = k \cdot y$, $k$ - constant

Solutions to this DE are $y = c \cdot e^{kx}$, $c \in \mathbb{R}$ (recall the example from last Friday)
↳ indicates infinite number of solutions

Sketch for $c = 1, 2, -1$




Q: Is there $(x_0, y_0)$ that satisfies
$y = c_1 e^{kx}$ and $y = c_2 e^{kx}$ with $c_1 \neq c_2$ ?
$\Leftrightarrow$ is there $(x_0, y_0)$ that satisfies
$y_0 = c_1 e^{kx_0} = c_2 e^{kx_0}$
where $c_1 \neq c_2$ ?

A: No!!!

$\Rightarrow$ Only one solution can be found if extra condition "the solution passes through the point $(x_0, y_0)$" is given

$y_0 = c e^{kx_0}$ $\Rightarrow$ find $c = y_0 e^{-kx_0}$ and the solution of DE becomes $y = y_0 e^{k(x - x_0)}$

One particular point $(0, y_0)$ provides $c = y_0$, called "initial value"
and the solution is $y = y_0 e^{kx}$

Meaning of $\frac{dy}{dx} = ky$: $\frac{dy}{dx}$ - slope of the tangent line $\Leftrightarrow$ instantaneous rate of change

$k \cdot y$ - linear, proportional to $y$

assume $k>0$, $y_1 > 0$ $\Rightarrow \left.\frac{dy}{dx}\right|_{y=y_1} = ky_1 > 0$ $\Rightarrow$ function is increasing

after small increment $\Delta x > 0$, $y_2 > y_1 \Rightarrow \left.\frac{dy}{dx}\right|_{y=y_2} = ky_2 > ky_1$

$\Rightarrow y$ is increasing with a steeper tangent line at $y_2$ $\Rightarrow \cdots$

Notice: this behaviour is consistent with the sketch $y = c \cdot e^{kx}$ with $\begin{cases} y_1 > 0 \\ k > 0 \end{cases}$

• Population Growth:

Assume $N(t)$ is the number of individuals at time $t$

$r$ is the average per capita birth rate $= \dfrac{\#\text{ of birth per year}}{\text{population size}}$

$m$ is the average per capita death rate $= \dfrac{\#\text{ of death per year}}{\text{population size}}$

$r, m$ - positive constants.

Then $r \cdot N(t)$ - average birth rate = instantaneous rate at time $t$

$m \cdot N(t)$ - death rate = # of individuals is moved out of the population at time $t$

$\frac{dN}{dt}$ - instantaneous rate of change of the population, including the increace (birth) and decrease (death)

$\Rightarrow \frac{dN}{dt} = r \cdot N - m \cdot N = (r-m) \cdot N = k \cdot N$

Assume $N(t=0) = N_0$ is the initial population size, we know $N(t) = N_0 e^{(r-m)t}$

$k = r - m$ "net growth rate"

doubling time $= \frac{\ln 2}{k} = \frac{\ln 2}{r-m}$

The population $N(t)$ increases only if $r > m$.