Announcements

- OSH 3 due today!
- Office Hours before Midterm with Cole: Monday October 17 from 3 to 6pm in Math Annex 1118
Today…

Optimization Examples

- Today: Optimization with constraints

1. One example with clicker questions
2. One complete example (instructor only)
3. One practice problem for you
Recall: from Week 5 Lecture 2

Find the cylinder of maximal volume that would fit inside a sphere of radius $R$.

- $V = \pi r^2 h$ (objective function: quantity to optimize)
- $R^2 = r^2 + \left(\frac{h}{2}\right)^2$ (constraint: relates independent variables with problem parameters)
Cylinder inside a sphere

- Use constraint to re-write objective function in terms of only one variable:

$$R^2 = r^2 + \left( \frac{h}{2} \right)^2 \quad \& \quad V = \pi r^2 h$$

$$\Rightarrow V(h) = \pi \left( R^2 - \frac{h^2}{4} \right) h$$

- Now we can use sketching and calculus tools to optimize the objective function!
Cylinder inside a sphere

\[ V(h) = \pi \left( R^2 - \frac{h^2}{4} \right) h \]

\( V(h) = 0? \)

- If \( h = 0 \), then \( V = 0 \).
- If \( h = 2R \), then \( V = 0 \).
- If \( 0 < h < 2R \), then \( V > 0 \).
Cylinder inside a sphere

Q1. The derivative of

\[ V(h) = \pi \left( R^2 - \frac{h^2}{4} \right) h \]

is

A. \( V'(h) = \pi \left( 2Rh - \frac{3h^2}{4} \right) \)

B. \( V'(h) = \pi \left( R^2 - \frac{3h^2}{4} \right) \)

C. \( V'(h) = \pi \left( R^2 - \frac{h^2}{12} \right) \)

D. \( V'(h) = \pi \left( R^2 + 2Rh - \frac{3h^2}{4} \right) \)
Cylinder inside a sphere

\[ V'(h) = \pi \left( R^2 - \frac{3h^2}{4} \right) \]

Q2. The Critical Points of \( V(h) \) are

A. \( h = \frac{2R}{3} \)
B. \( h = \frac{4R}{3} \)
C. \( h = \frac{4R}{\sqrt{3}} \)
D. \( h = \frac{R}{\sqrt{3}} \)
E. \( h = \frac{2R}{\sqrt{3}} \)
Cylinder inside a sphere

\[ V'(h) = \pi \left( R^2 - \frac{3h^2}{4} \right) = 0 \]

if and only if

\[ h = \frac{\pm 2R}{\sqrt{3}}, \]

but we ignore negative \( h \) (height should be positive).
Cylinder inside a sphere

Q3. \( h = \frac{2R}{\sqrt{3}} \) is a

A. Local maximum
B. Local minimum
C. Neither

Use the second derivative test.
The second derivative is

\[
V''(h) = -\pi \frac{3h}{2} < 0
\]

for \( h > 0 \).
Cylinder inside a sphere

Sketch:

We found that the local maximum is at $h = \frac{2R}{\sqrt{3}}$. 
Cylinder inside a sphere

Q4. The corresponding radius of the cylinder is

A. \( r = \frac{\sqrt{2}}{\sqrt{3}} R \)

B. \( r = \frac{\sqrt{2}}{3} R \)

C. \( r = \frac{2}{3} R \)

D. \( r = \frac{2}{\sqrt{3}} R \)

E. Yikes!

Substitute \( h = \frac{2R}{\sqrt{3}} \) into constraint:

\[
r^2 = R^2 - \frac{h^2}{4} = \frac{2}{3} R^2
\]
Cylinder inside a sphere

Q5. The cylinder with maximal volume has dimensions \( r = \frac{\sqrt{2}}{\sqrt{3}} R \) and \( h = \frac{2R}{\sqrt{3}} \). What is the volume of this cylinder?

A. \( V = \pi \frac{4}{3} R^3 \)
B. \( V = \pi \frac{4}{9} R^3 \)
C. \( V = \pi \frac{2}{3\sqrt{3}} R^3 \)
D. \( V = \pi \frac{4}{3\sqrt{3}} R^3 \)
E. \( V = \pi \frac{4}{3\sqrt{3}} R^2 \)

\[
V = \pi r^2 h = \pi \frac{2}{3} R^2 \left( \frac{2R}{\sqrt{3}} \right) = \frac{4\pi}{3\sqrt{3}} R^3
\]

Interestingly, this is \( \frac{1}{\sqrt{3}} \) of the volume of the sphere.
Baculovirus

What is the minimal surface area of a cylindrical cell of a fixed volume $K$?

- Baculovirus:

Baculovirus

What is the minimal surface area of a cylindrical cell of a fixed volume?

- Baculovirus:

Baculovirus

- Baculovirus must transport a fixed volume, \( K \), of viral DNA and protein into an uninfected cell’s nucleus.
- To maximize virus production within infected cells, a minimal amount of material is to be used to construct the membrane of the baculovirus.
- Assuming that the baculovirus has a cylindrical shape, what is the minimal surface area of the virus, given that the virus’ volume must be \( K \)?
Baculovirus

- Volume of cylinder must be $K$ (constraint):

\[ K = \pi r^2 L \]

- Surface area of cylinder (objective function):

\[ S = 2\pi rL + 2\pi r^2 \]

  outer surface 2 circular ends

- Re-write

\[ L = \frac{K}{\pi r^2} \]
Baculovirus

- Rewrite the constraint:

\[ K = \pi r^2 L \Rightarrow L = \frac{K}{\pi r^2} \]

- Rewrite the surface area as a function of only \( r \):

\[
S = 2\pi r L + 2\pi r^2 \\
S(r) = 2\frac{K}{r} + 2\pi r^2
\]
Baculovirus

- Our problem is to now find the minimum value of

\[ S(r) = 2 \frac{K}{r} + 2\pi r^2. \]

- Look for CPs:

\[ S'(r) = -2 \frac{K}{r^2} + 4\pi r = 0 \]

\[ \Rightarrow 2 \frac{K}{r^2} = 4\pi r \Rightarrow r^3 = \frac{K}{2\pi} \Rightarrow r = \left( \frac{K}{2\pi} \right)^{\frac{1}{3}} \]
Baculovirus

- Is our CP a local max or min?
- Calculate the second derivative:

\[ S''(r) = 4 \frac{K}{r^3} + 4\pi > 0 \]

so any CP is automatically a minimum (by the Second Derivative Test).
To sketch

\[ S(r) = 2\frac{K}{r} + 2\pi r^2,\]

note that

- there is one local minima at \( r = \left( \frac{K}{2\pi} \right)^{\frac{1}{3}} \)
- the second derivative, \( S''(r) > 0 \), so \( S(r) \) is concave up
- For \( r \ll 1 \), \( S(r) \approx 2\frac{K}{r} \).
- For \( r \gg 1 \), \( S(r) \approx 2\pi r^2 \).
- We’re only worried about positive radius \( (r > 0) \).
Baculovirus Sketch (with $K = 1$)
Baculovirus

- $r = \left(\frac{K}{2\pi}\right)^{\frac{1}{3}}$ is the radius of the cylinder that has minimal surface area with volume $K$.
- What’s the length? Use the constraint!

\[
K = \pi r^2 L \Rightarrow L = \frac{K}{\pi r^2}
\]

- Using $r = \left(\frac{K}{2\pi}\right)^{\frac{1}{3}}$, we find the length of the cylinder is

\[
L = \left(\frac{4K}{\pi}\right)^{\frac{1}{3}}.
\]
Based on our previous results, the cylinder with volume $K$ and minimal surface area has

$$\frac{L}{r} = 2.$$ 

Is that true for Baculovirus?
For Baculovirus, \( \frac{L}{r} \neq 2 \).

Baculovirus likely evolved other structures to be long and skinny so it can efficiently enter the nucleus through the nuclear pore complex.
Summary & Practice Problem

1. Identify **constraint** and **objective function**
2. Use the **constraint** to write the **objective function** as a function of only **one** variable.
3. Use calculus to find the extrema.
Practice Problem for Students (if extra time)

The sum of two positive numbers is 20. Find the numbers if

1. their product is a maximum.
2. the sum of their squares is a minimum.
3. the product of the square of one and the cube of the other is at a maximum.
Answers

1. B
2. E
3. A
4. A
5. D
Related Exam Problems

1. A box with a square base is to be made so that its diagonal has length 1. What height $y$ would make the volume maximal?
2. What is the maximal volume?