Today

- Euler's method (cont)
- Qualitative analysis of differential equations
    - Steady states
    - Slope fields
    - Stability of steady states
    - Velocity (y') versus position (y)

Euler’s method (cont)

- Spreadsheet demo

DEs - a broad view

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object falling through air	pendulum under water	water draining from a vessel

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where r=b and K=1/c. This is a nonlinear DE because of the N^2.

Qualitative analysis

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- Qualitative analysis - extract information about the general solution without solving.
    - Steady states
    - Slope fields
    - Stability of steady states
    - Plotting y′ versus y (state space/phase line)

$$\underset{\substack{\uparrow \\ \text{velocity}}}{x'} = x(1 - \underset{\substack{\uparrow \\ \text{position}}}{x})$$

- **Steady state**. Where can you stand so that the DE tells you not to move?

(A) x=-1

(B) x=0

(C) x=1/2

(D) x=1

This is the logistic eq with r=1, K=1.

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velocity position

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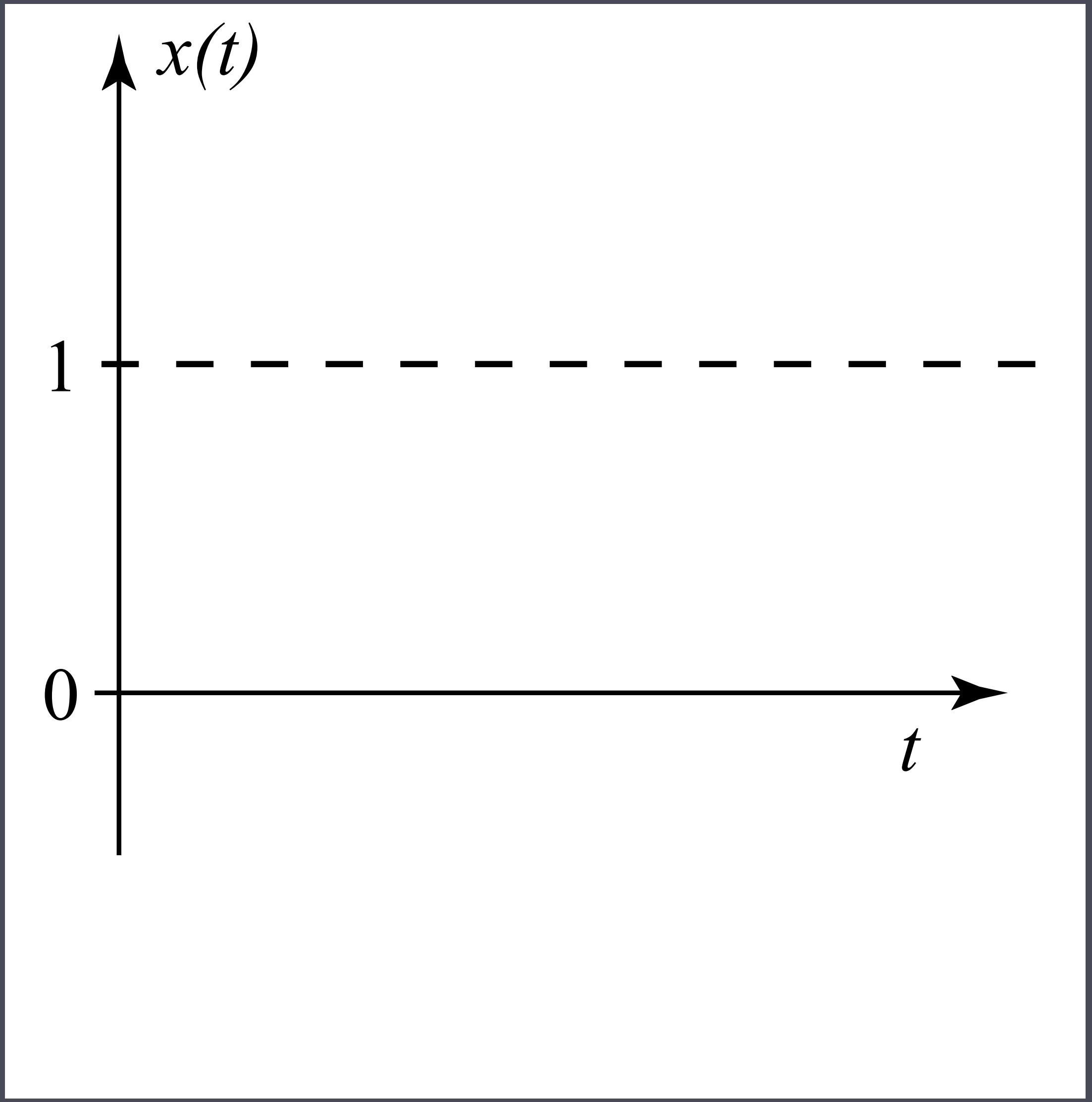


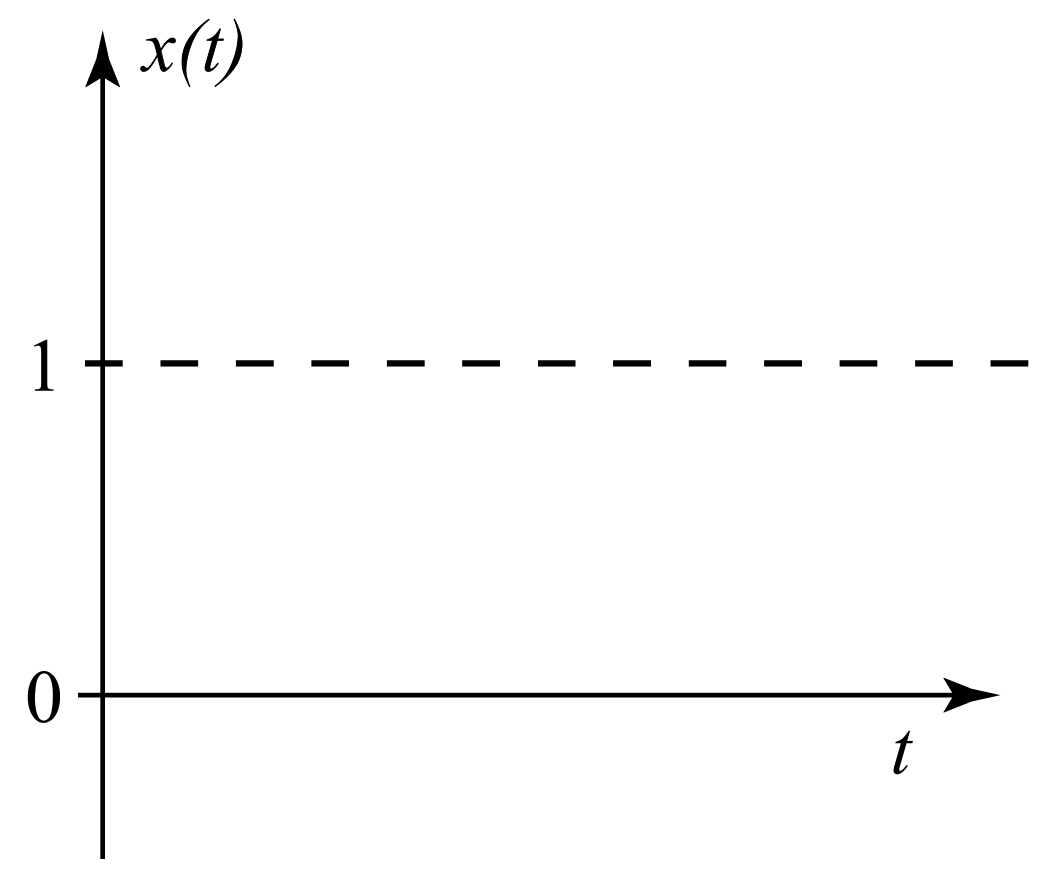
A steady state is a constant solution.

$$x' = x(1 - x)$$

velocity

position

Slope field



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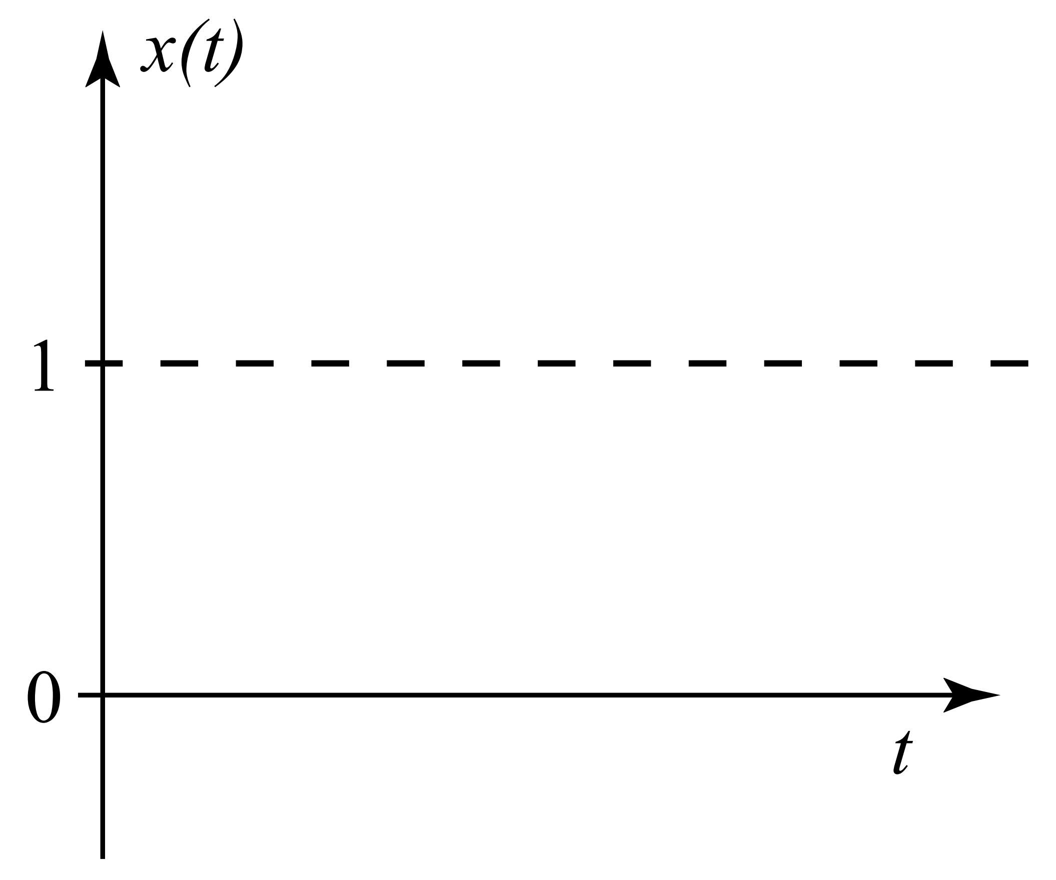
Slope field



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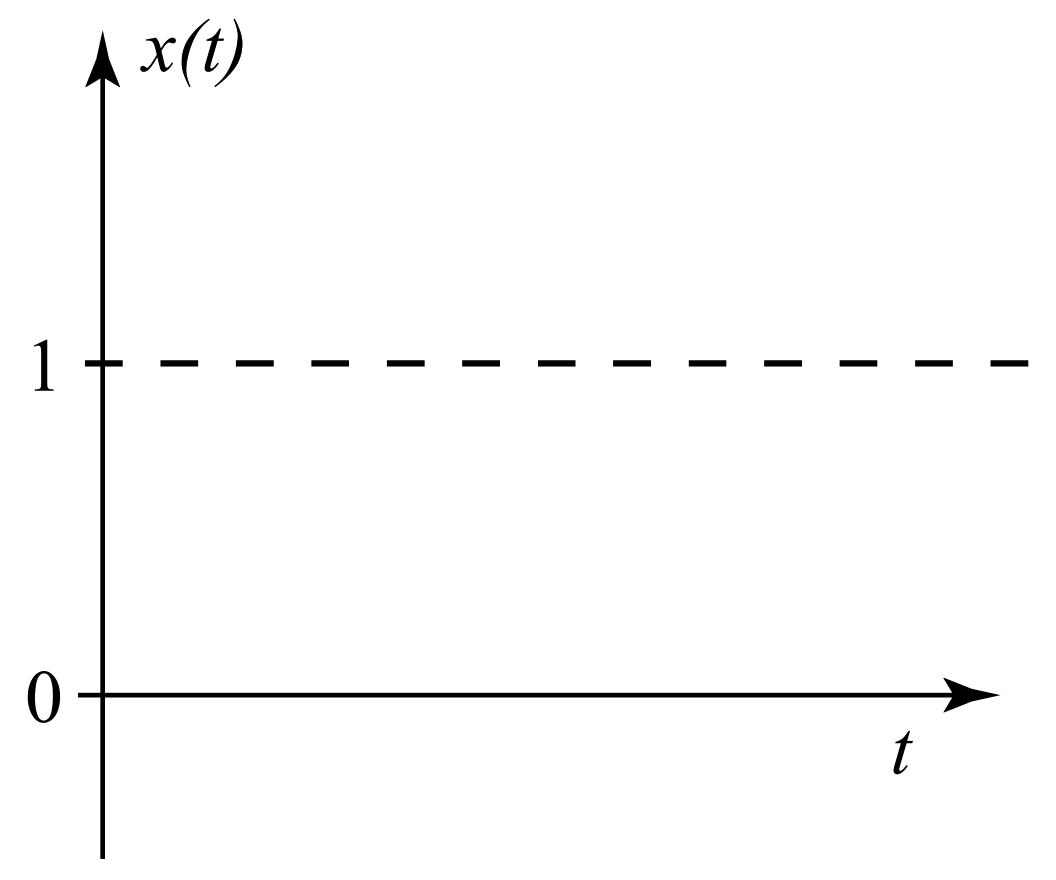
- At any t, don't know x yet so plot all possible x' values

Slope field



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  - When x(t)=1/2 what is x'?

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    (B) 1/4

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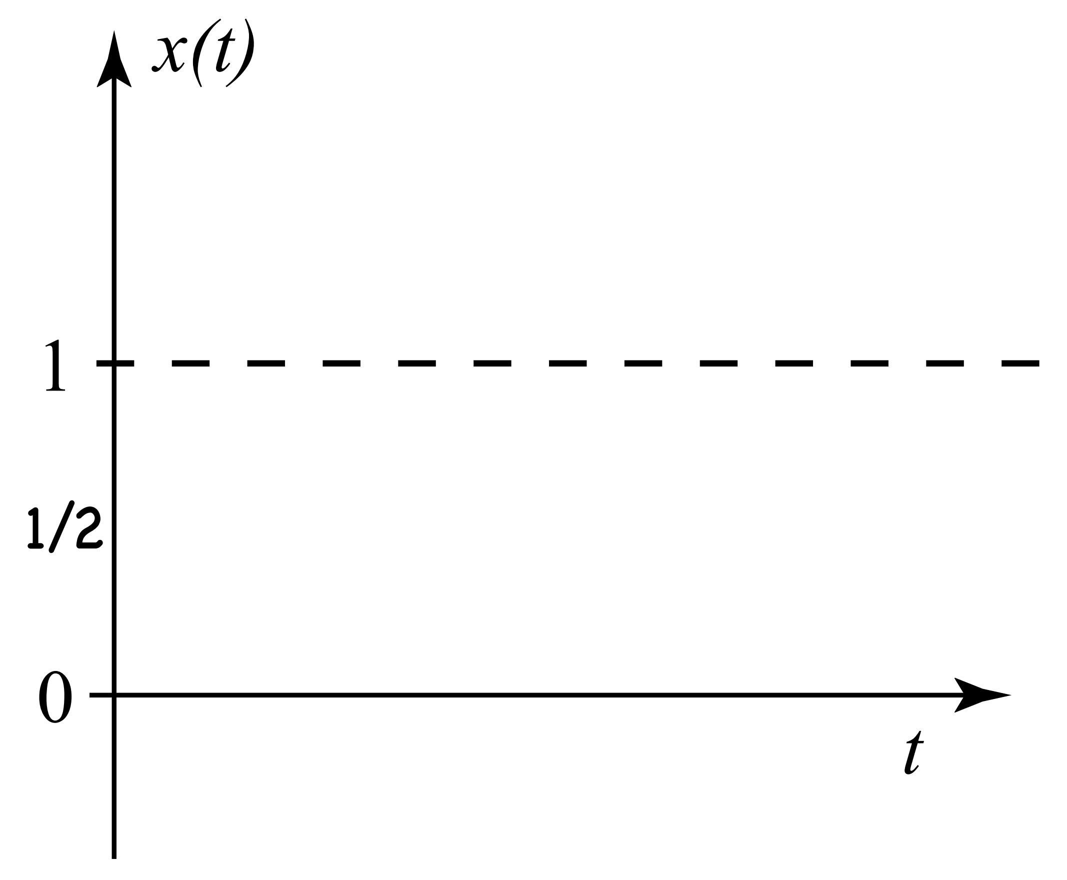
Slope field



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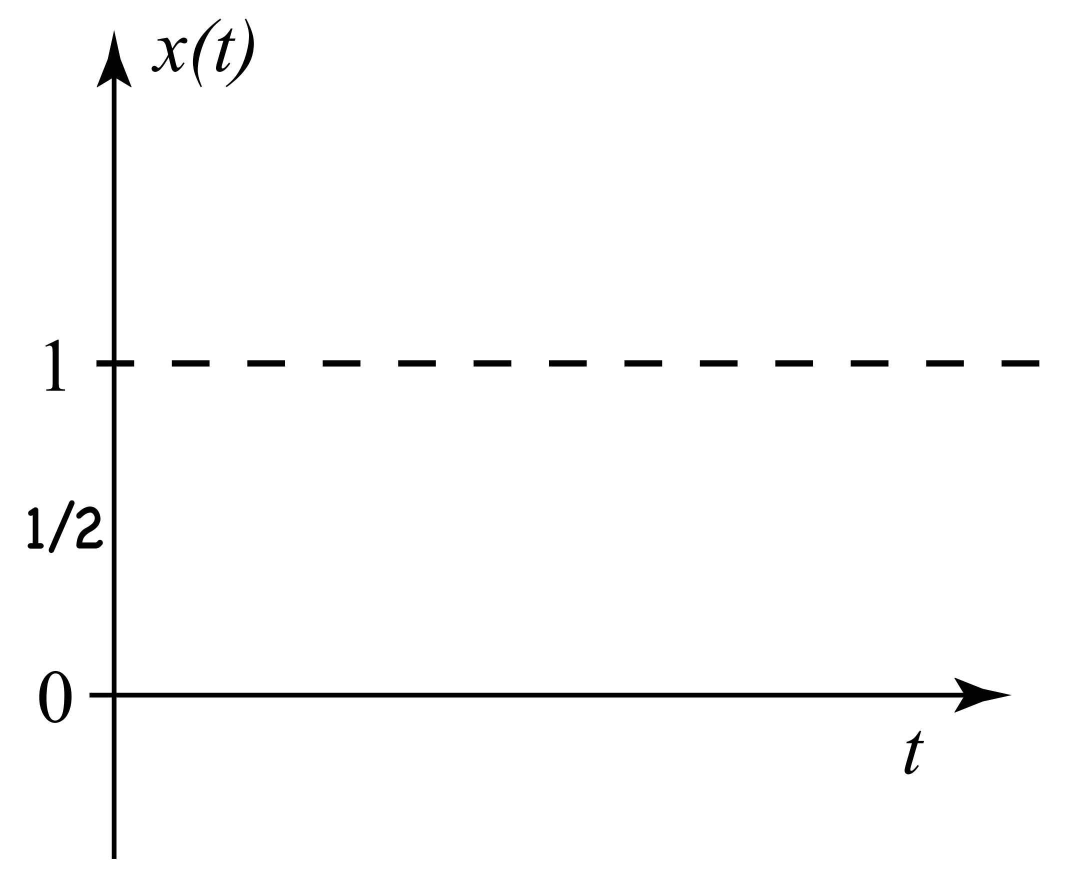

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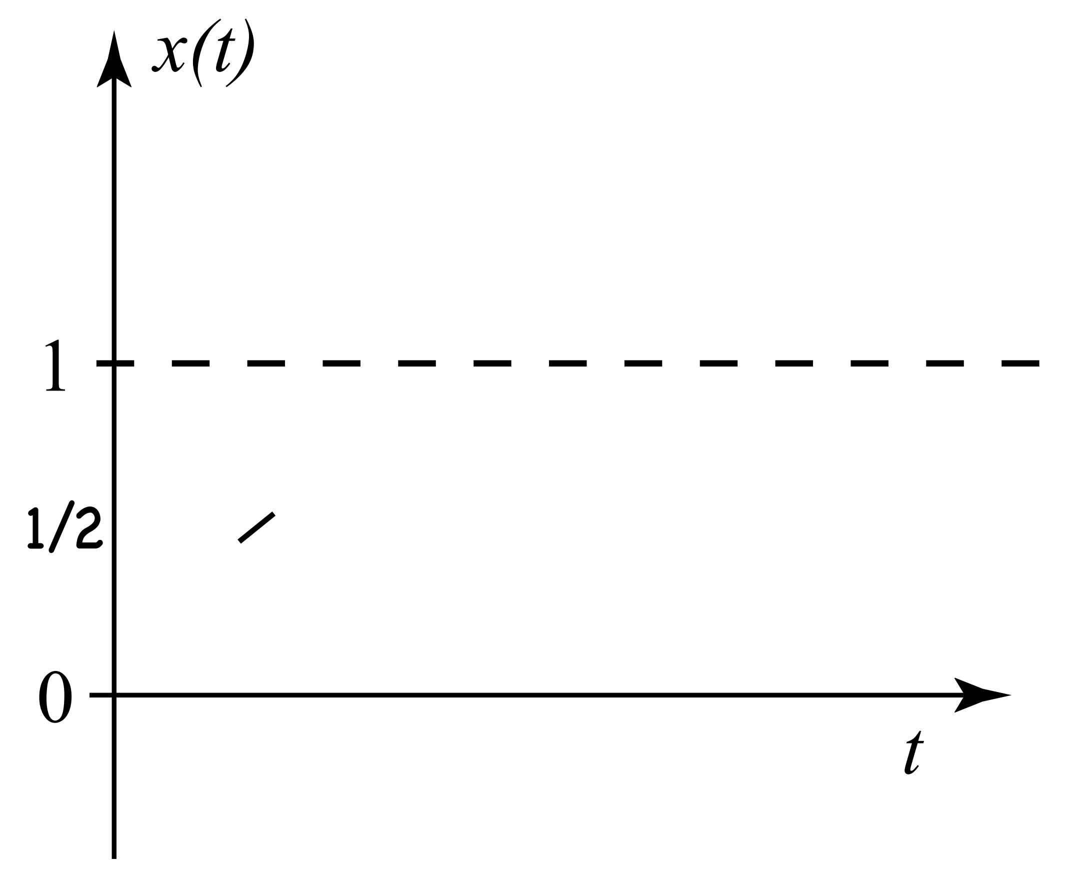

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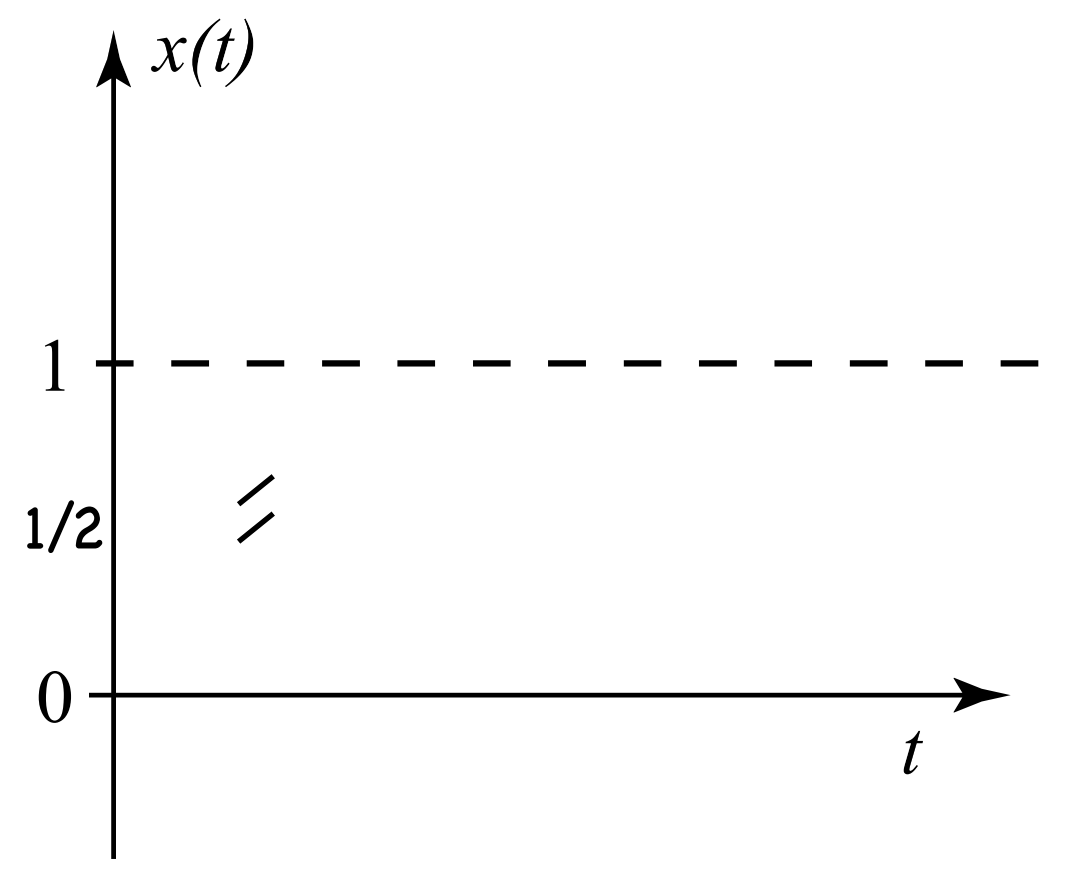

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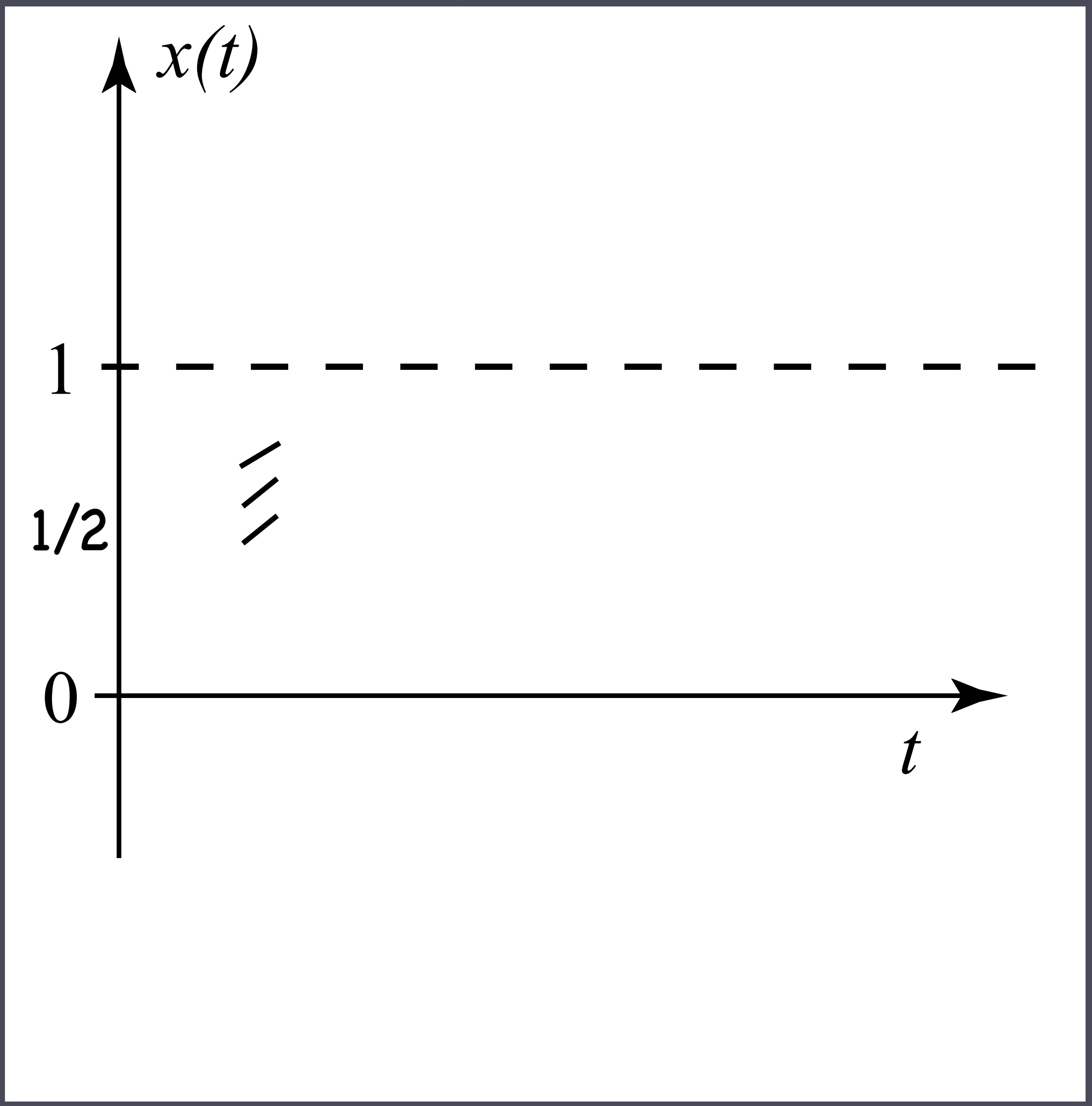
$$\underset{\substack{\uparrow \\ \text{velocity}}}{x'} = x(1-\underset{\substack{\uparrow \\ \text{position}}}{x})$$

- **Slope field**.
- At any t, don’t know x yet so plot all possible x’ values

Slope field



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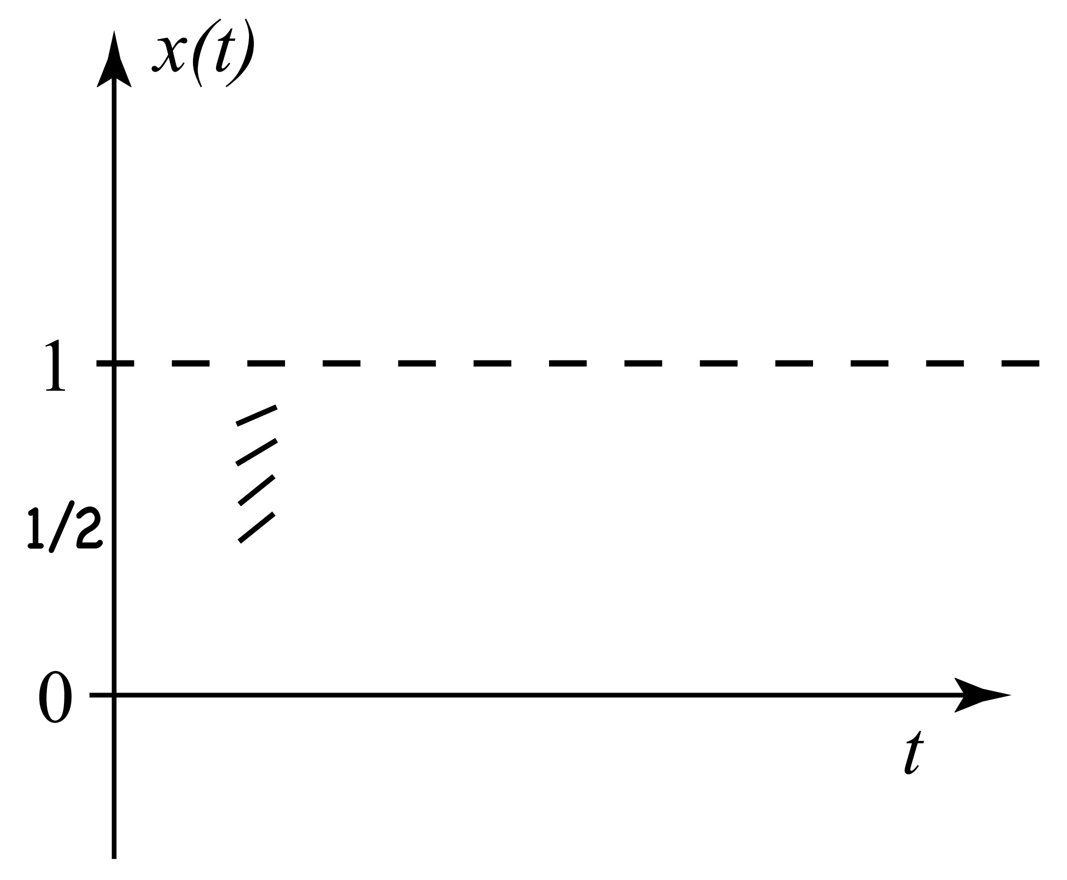

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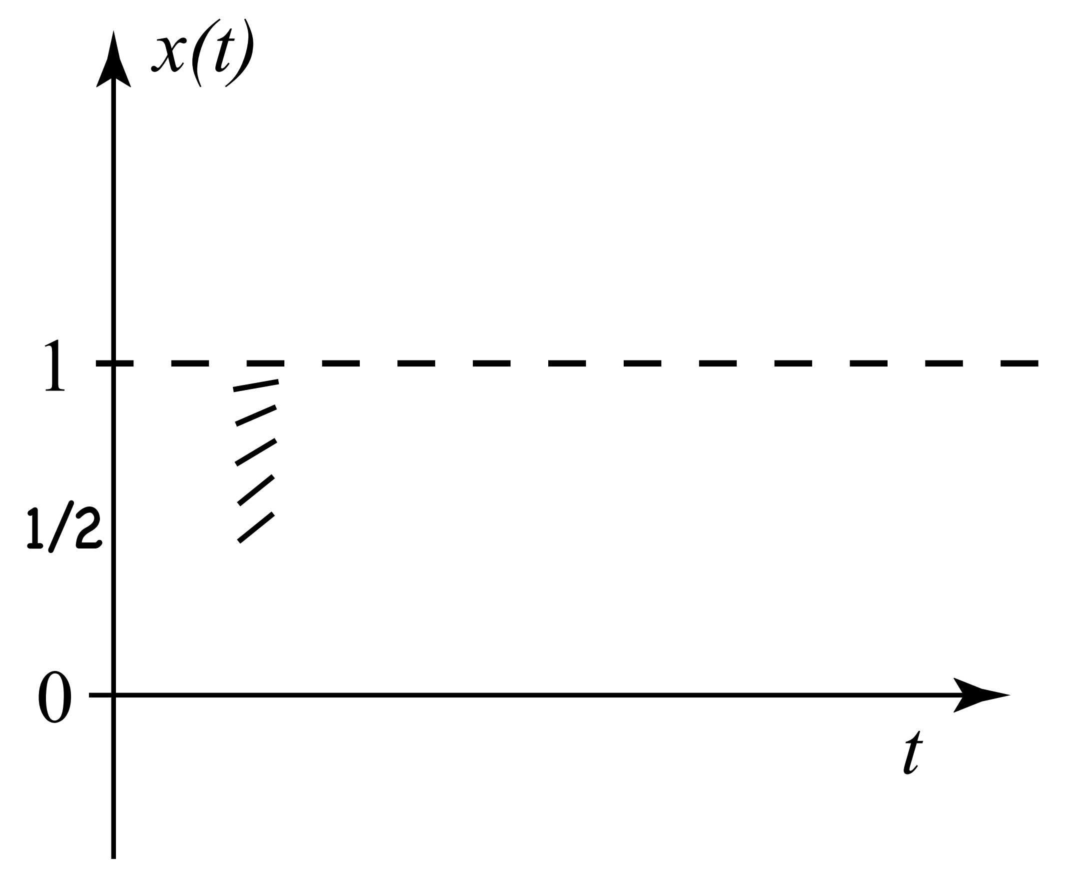

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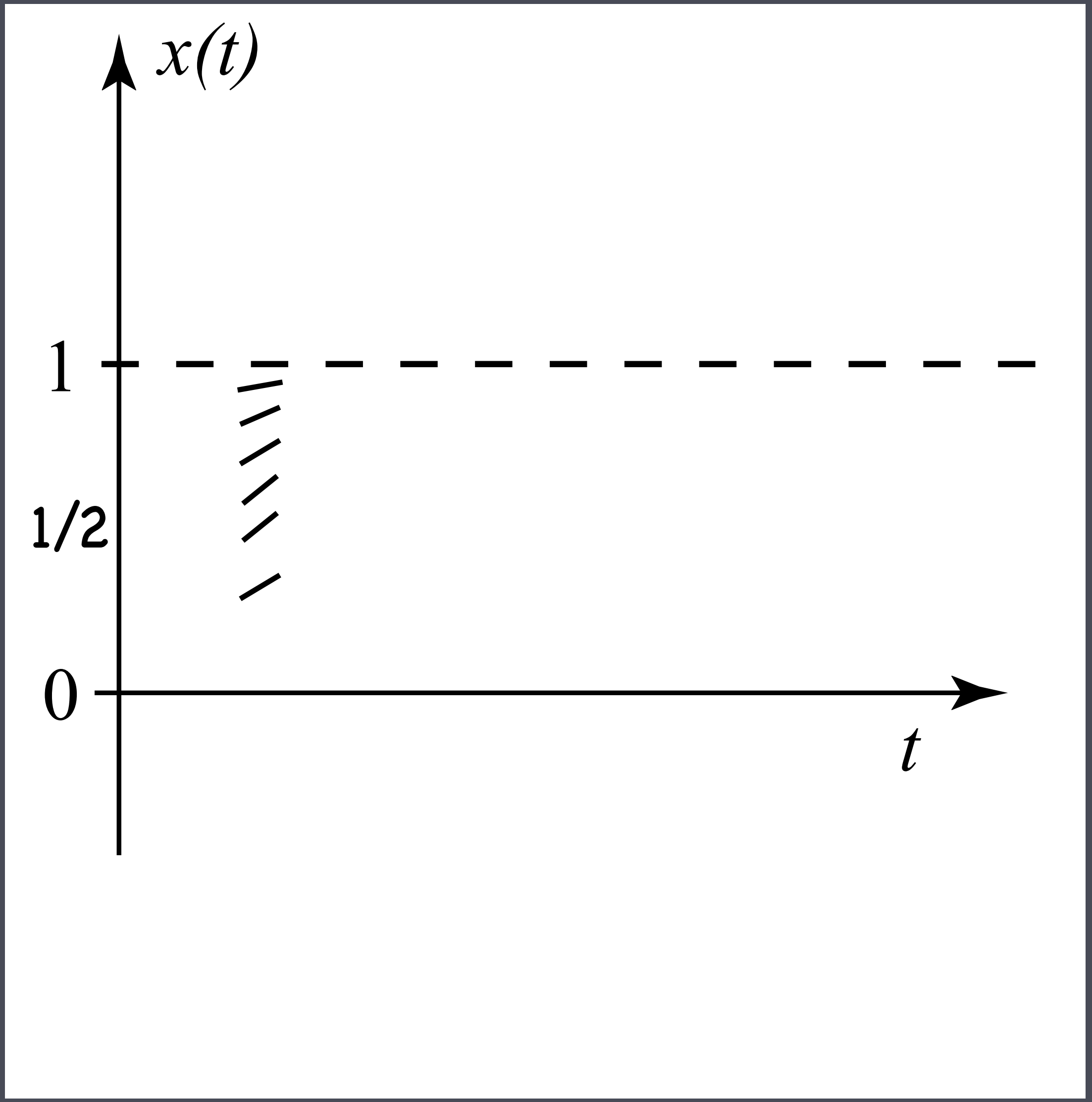

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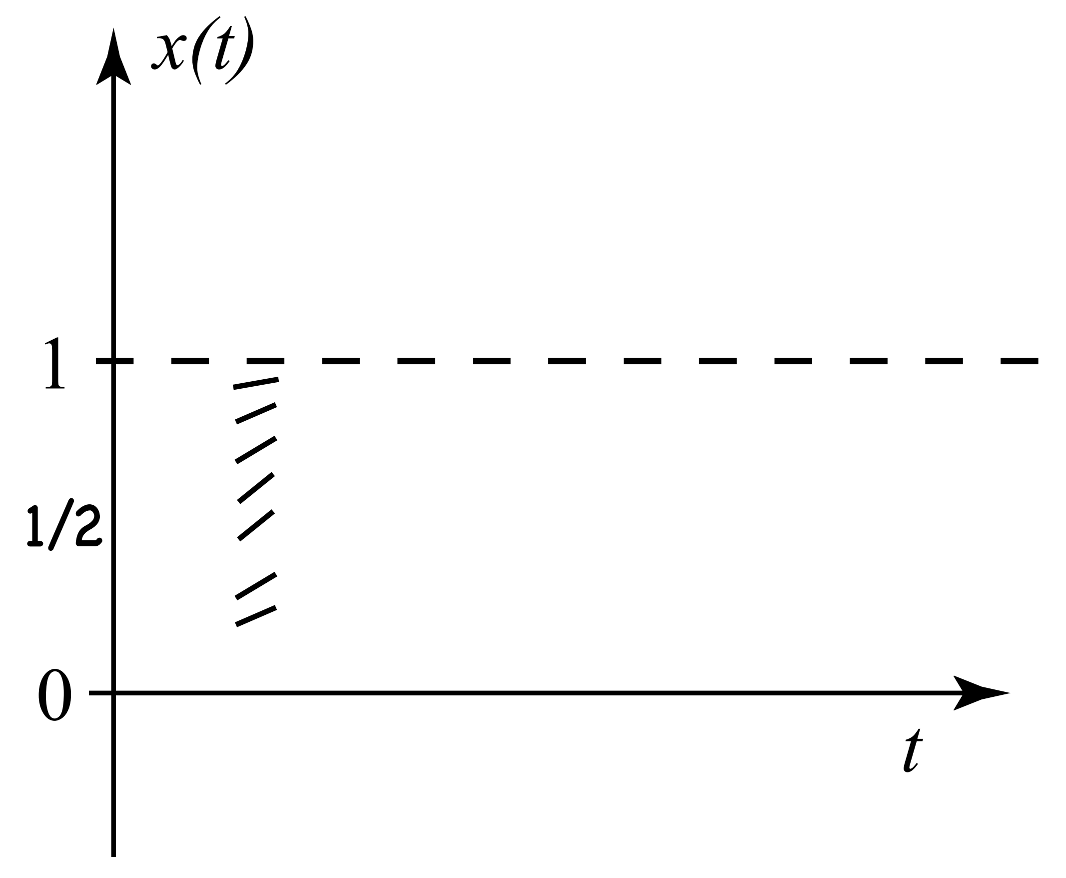

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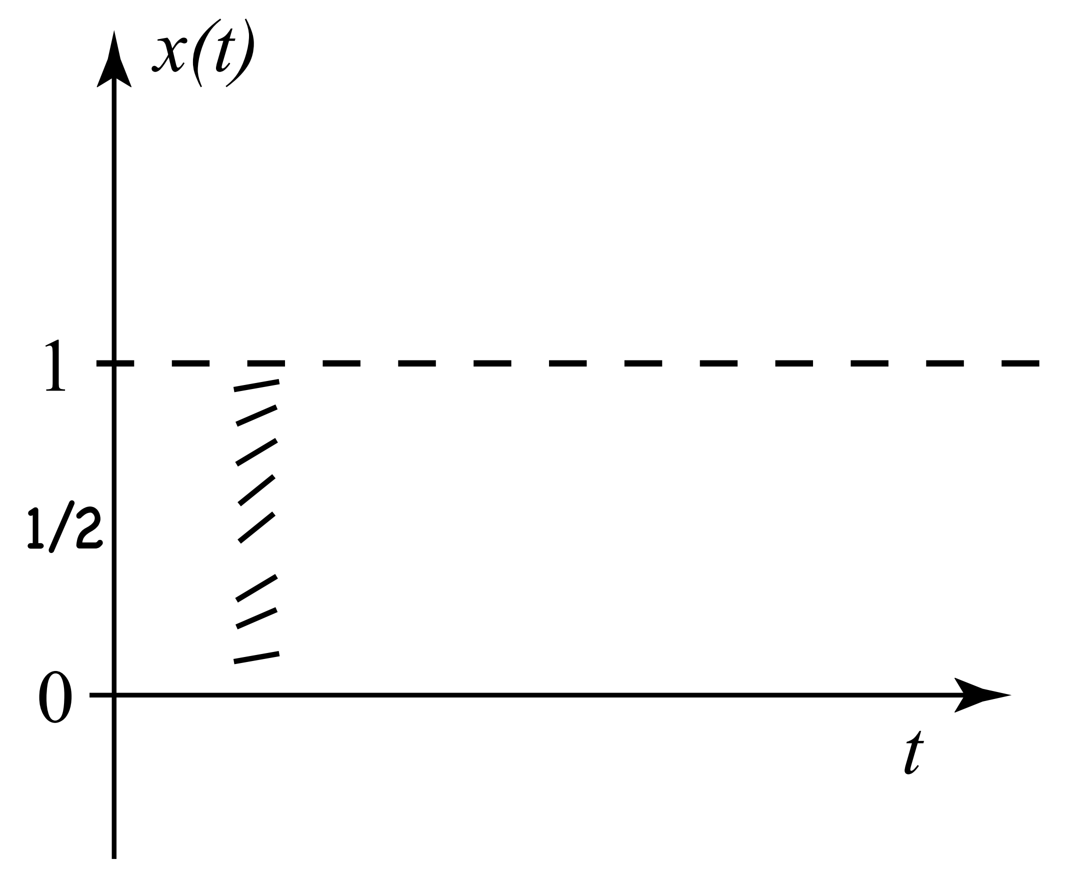

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$$\underset{\substack{\uparrow \\ \text{velocity}}}{x'} = x(1 - \underset{\substack{\uparrow \\ \text{position}}}{x})$$

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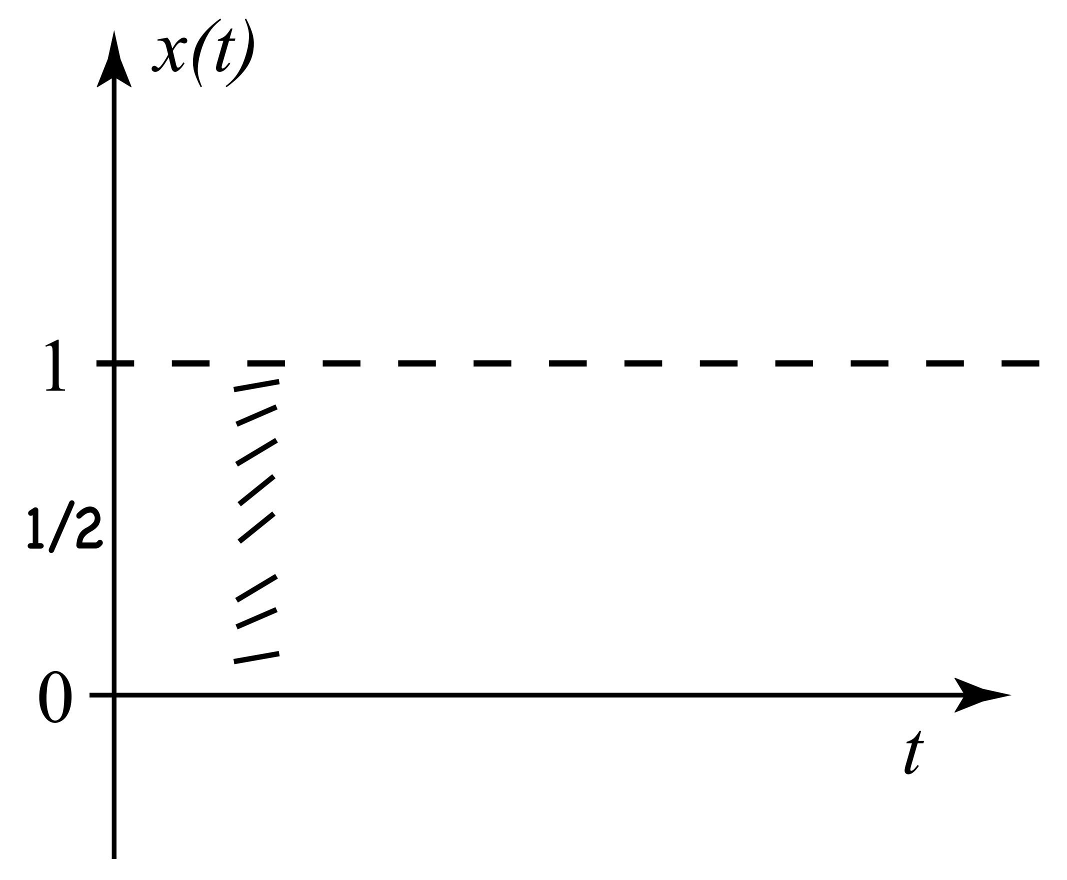

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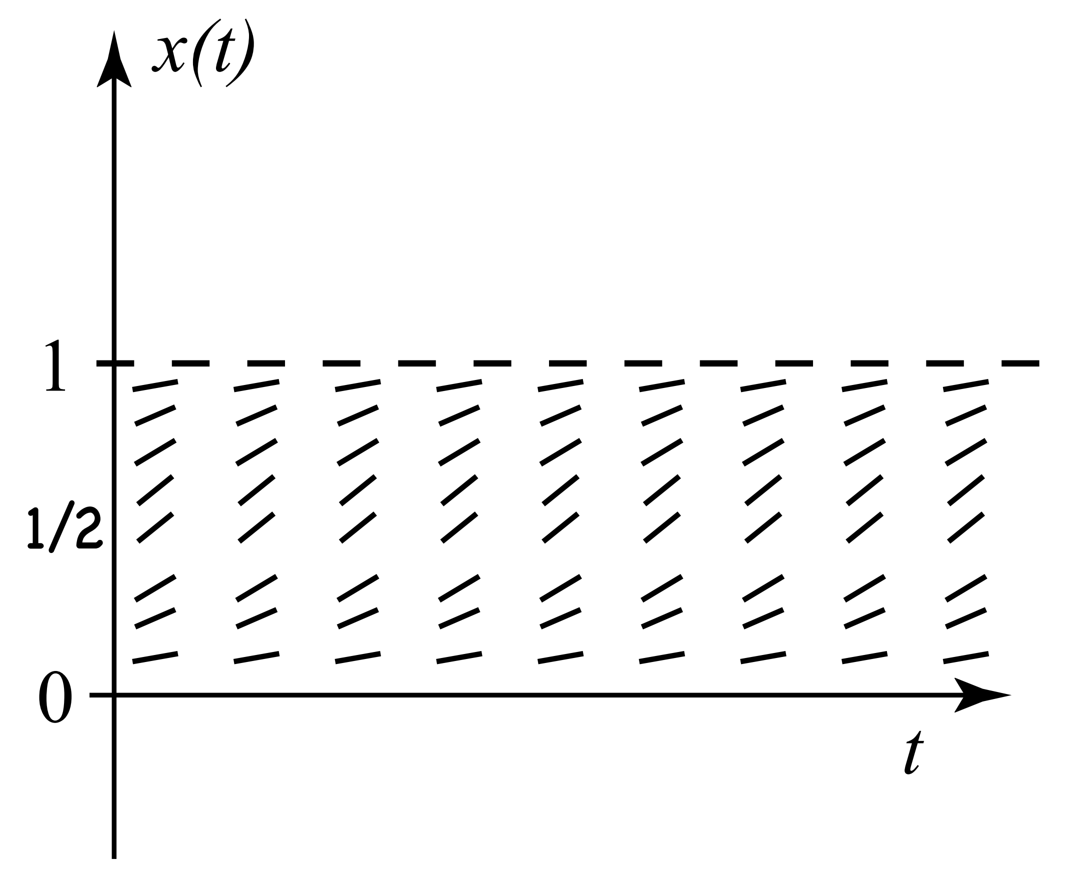

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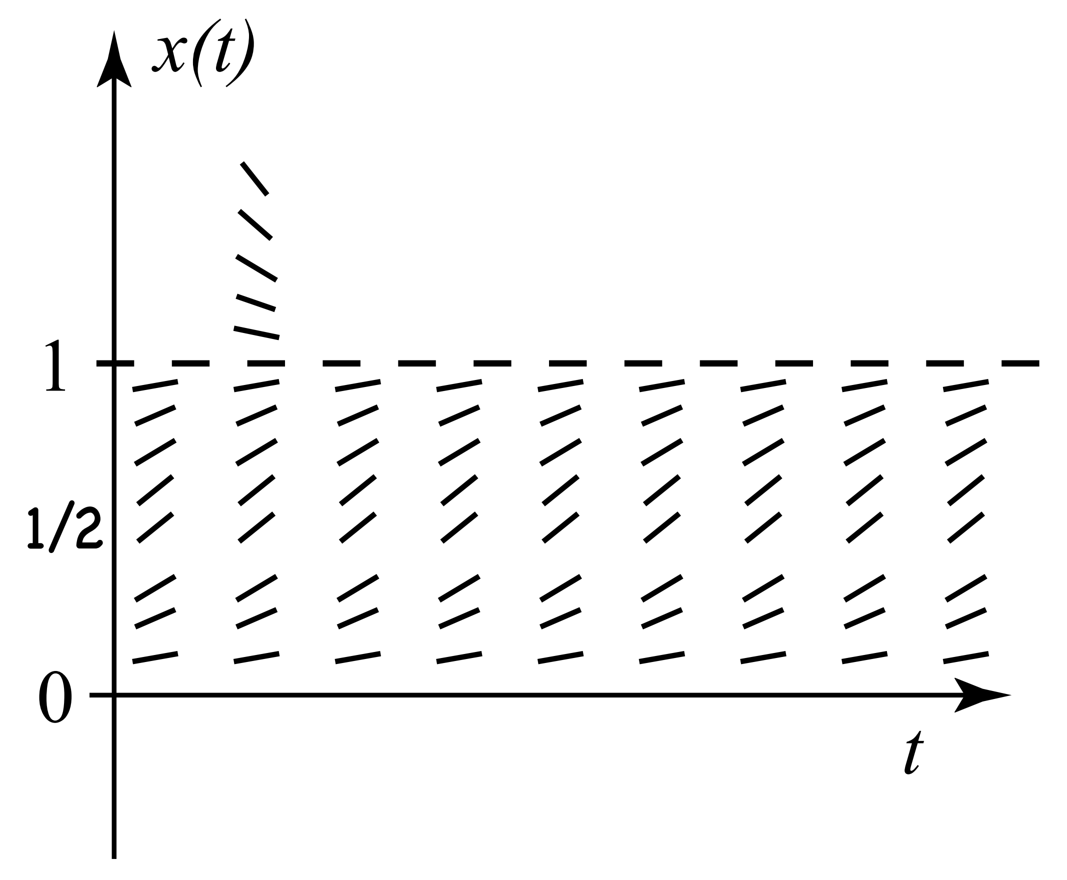

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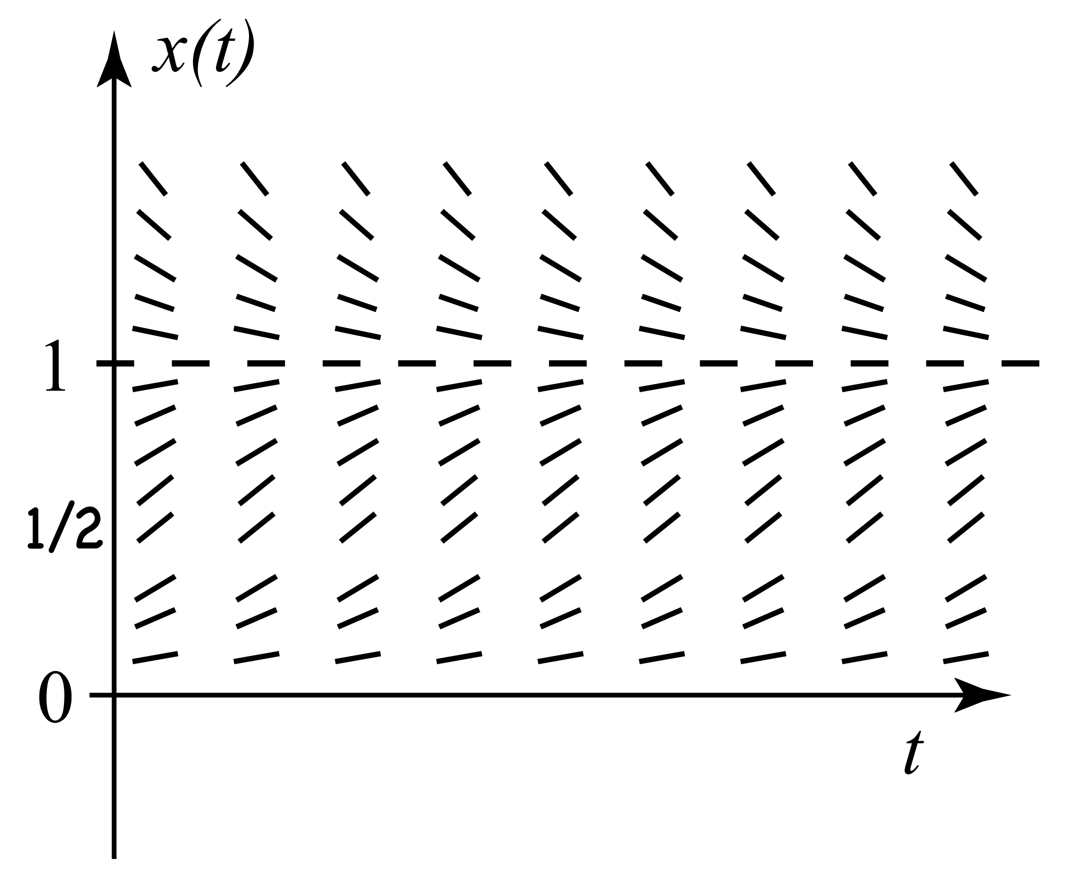

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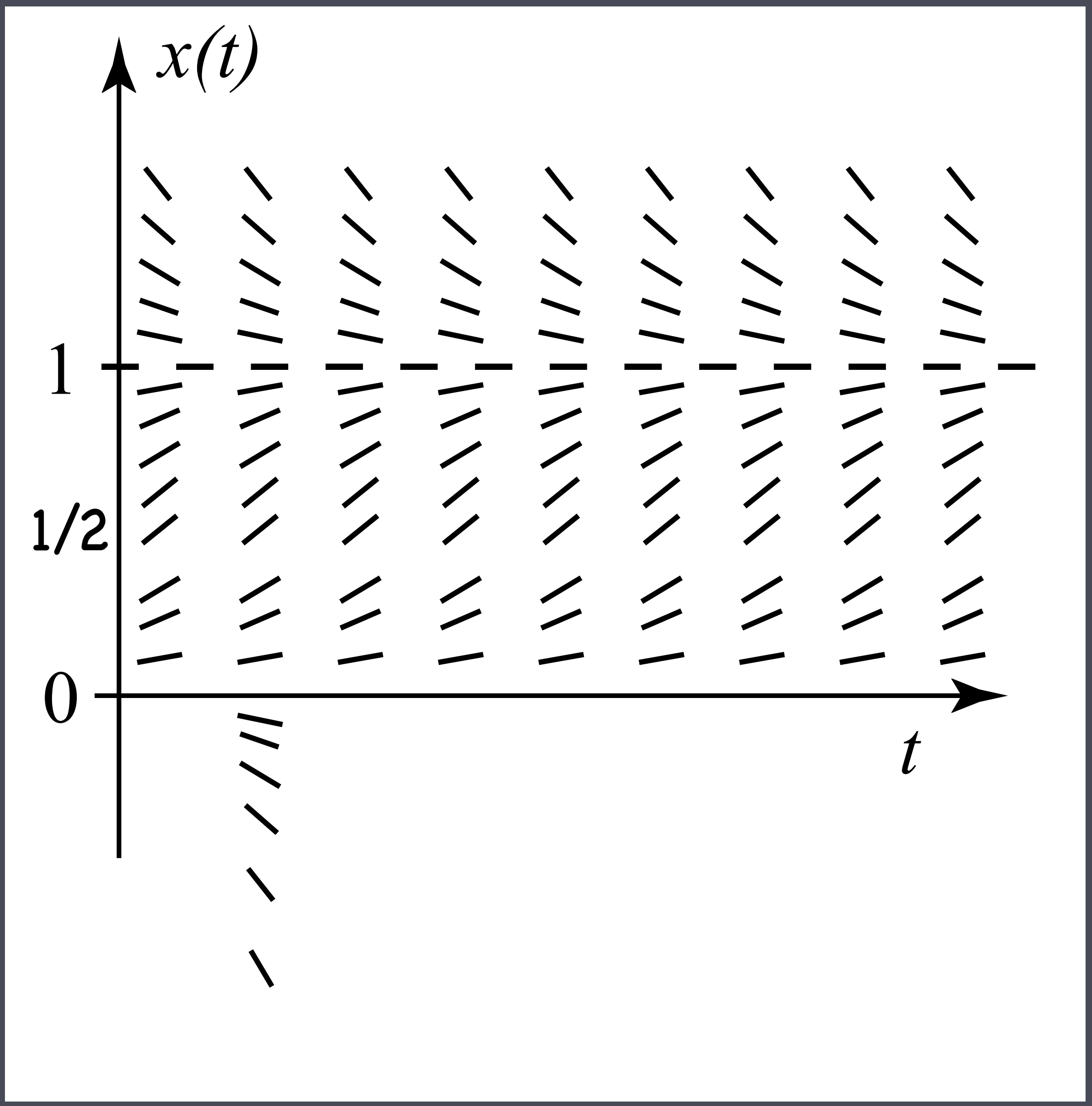

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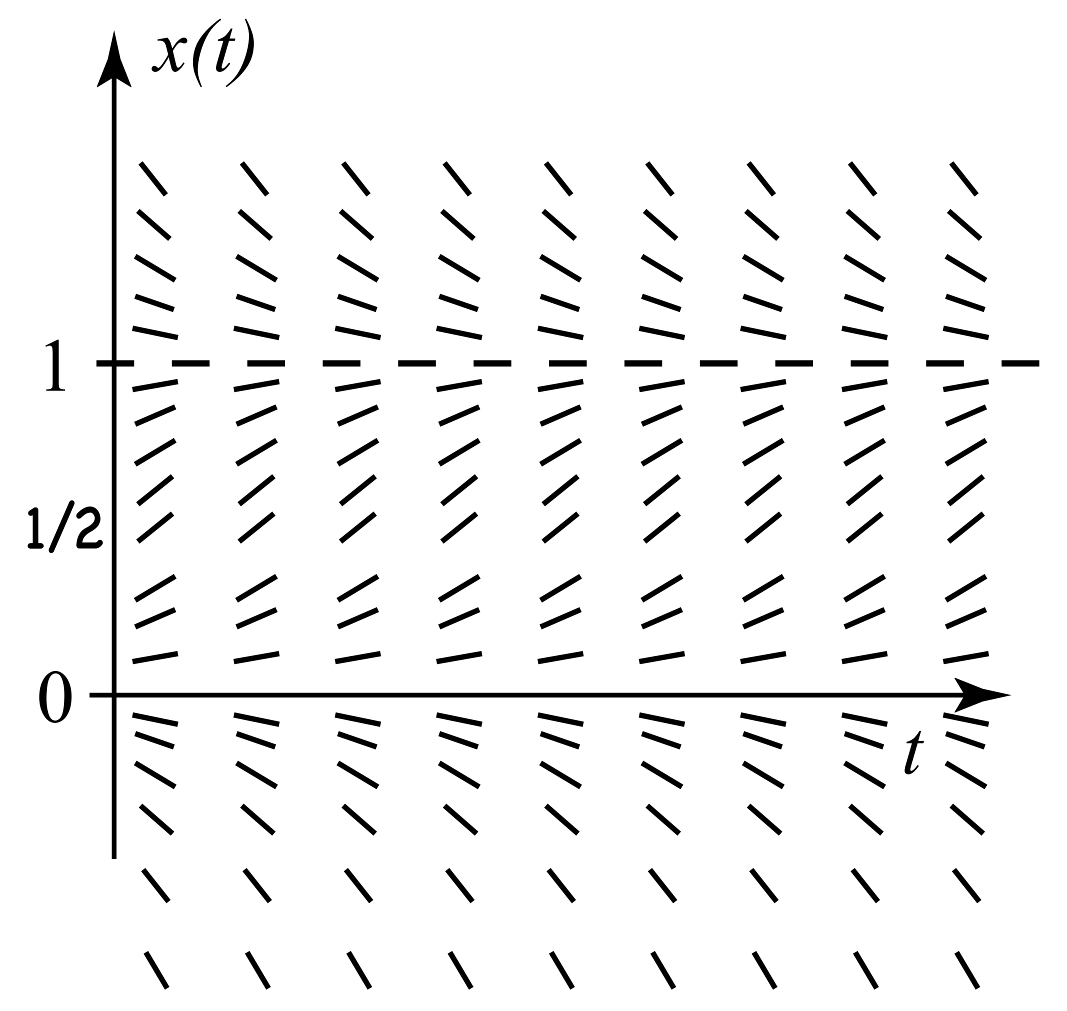

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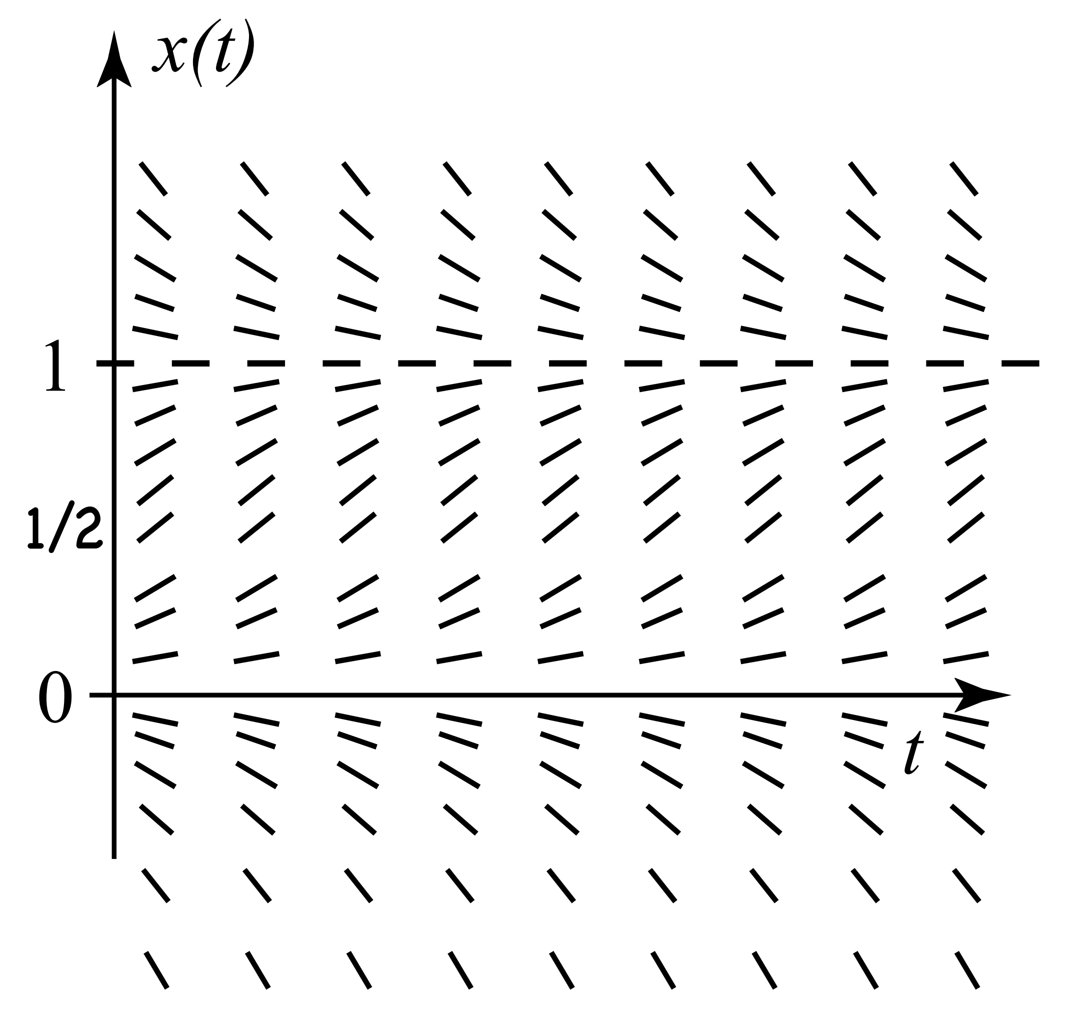

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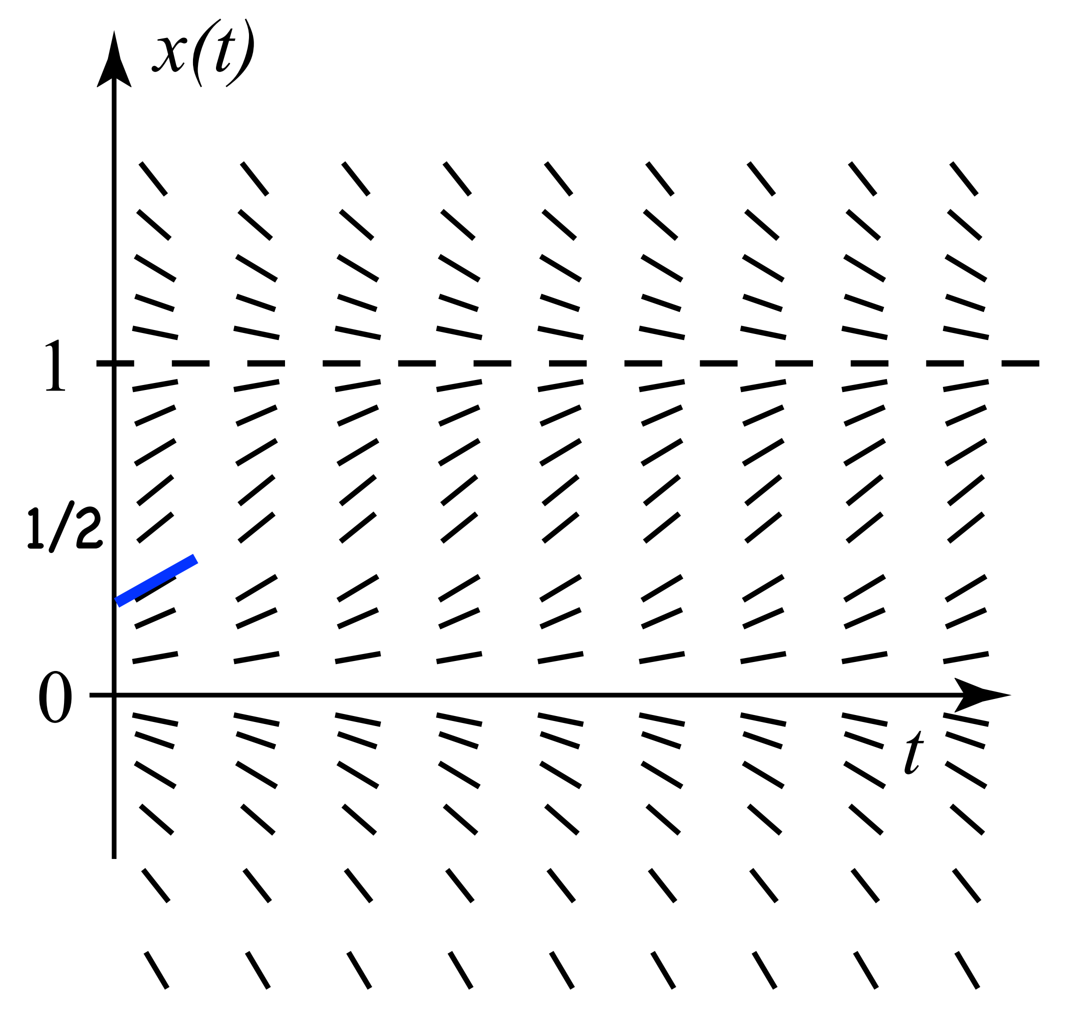

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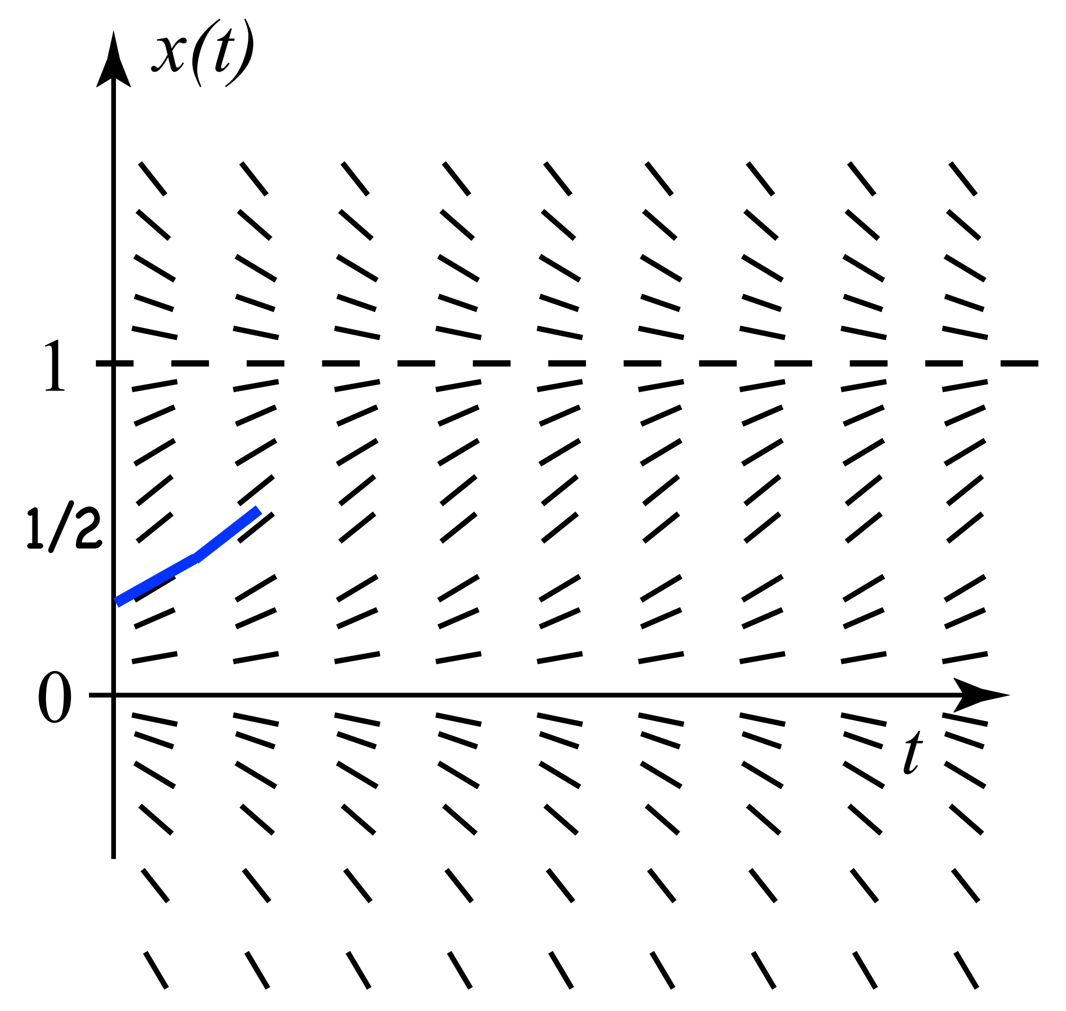

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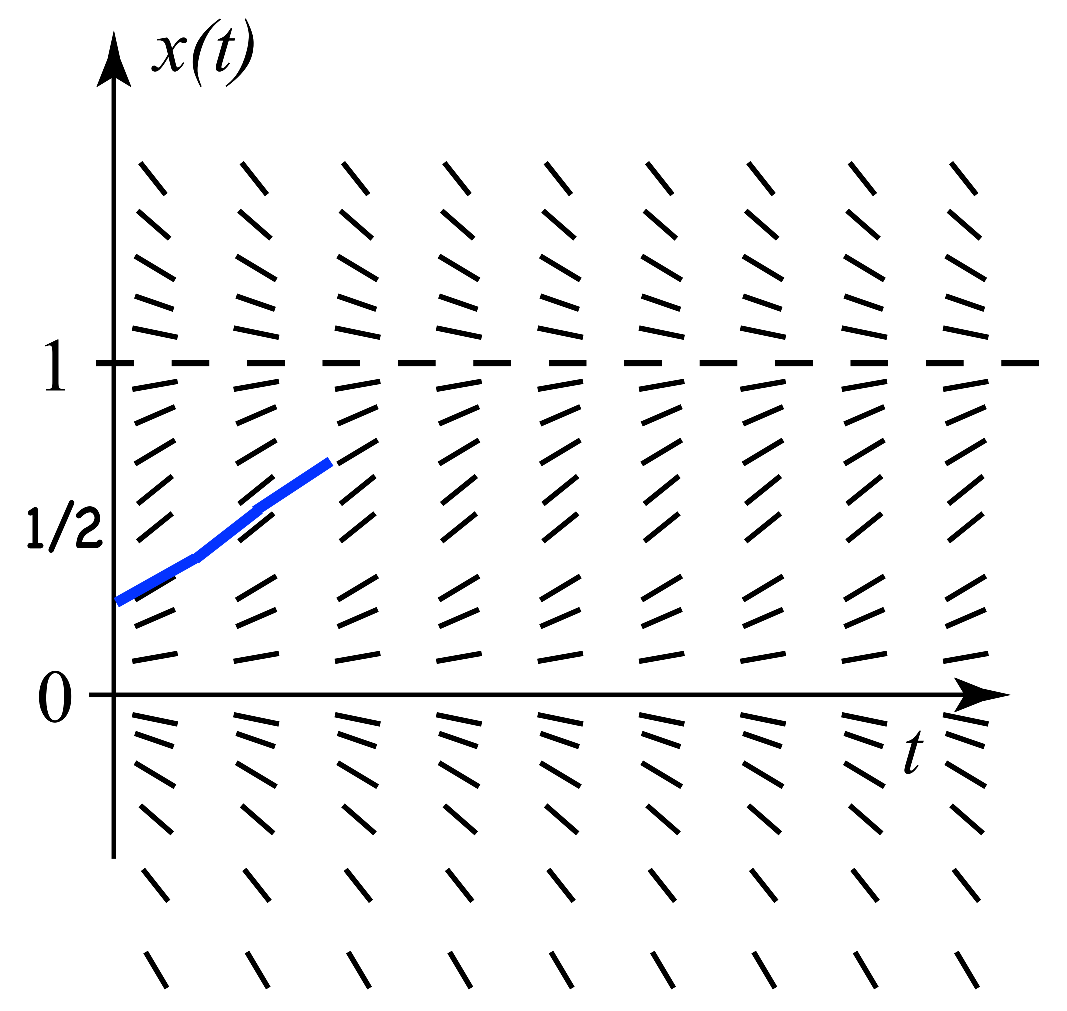

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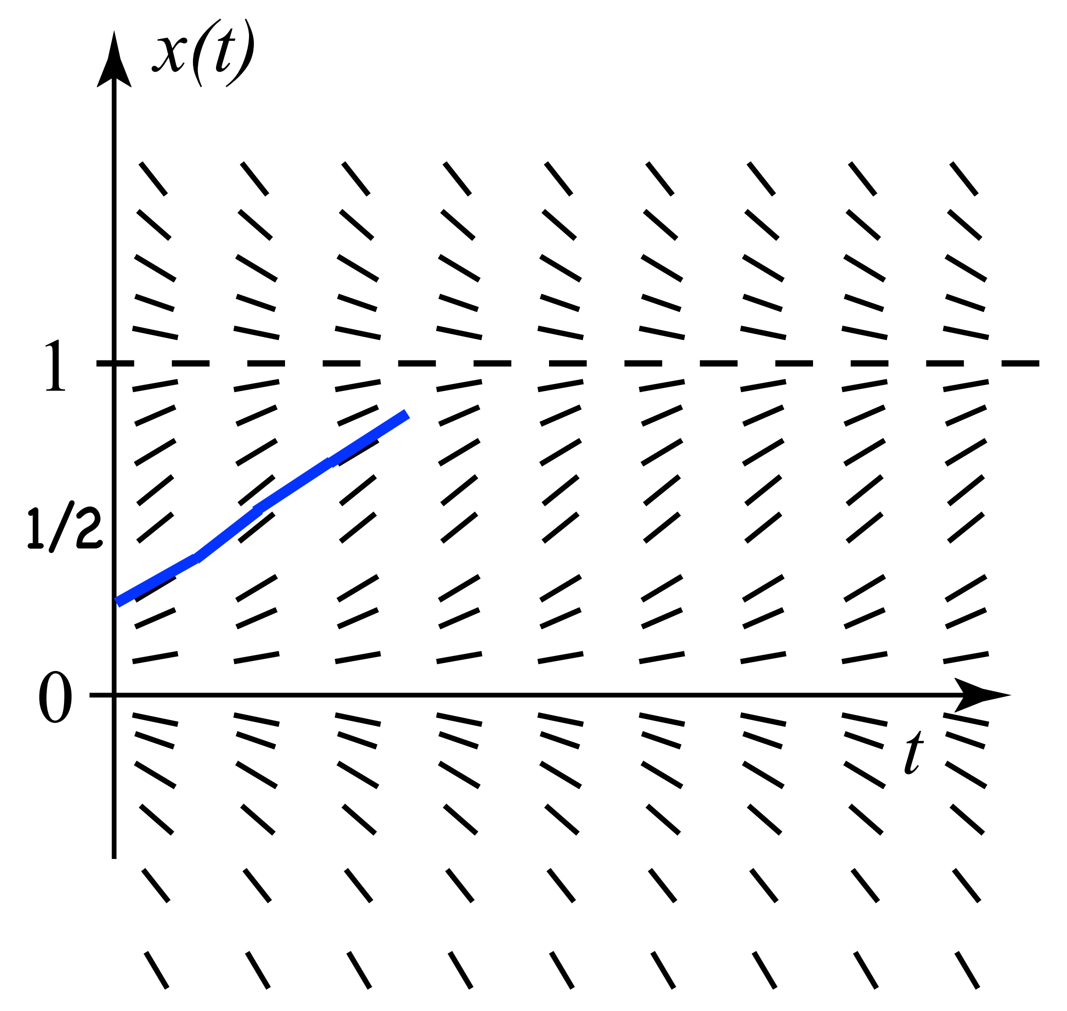

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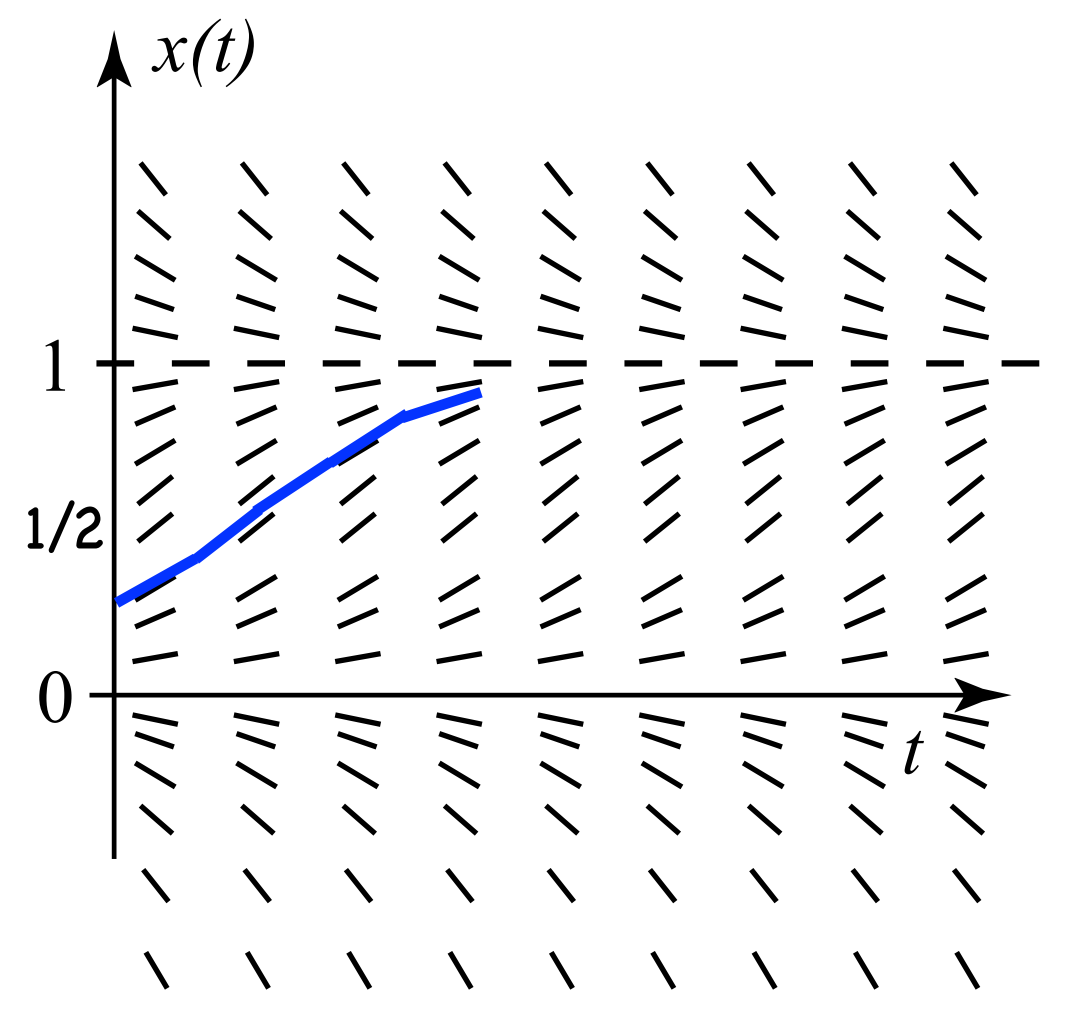

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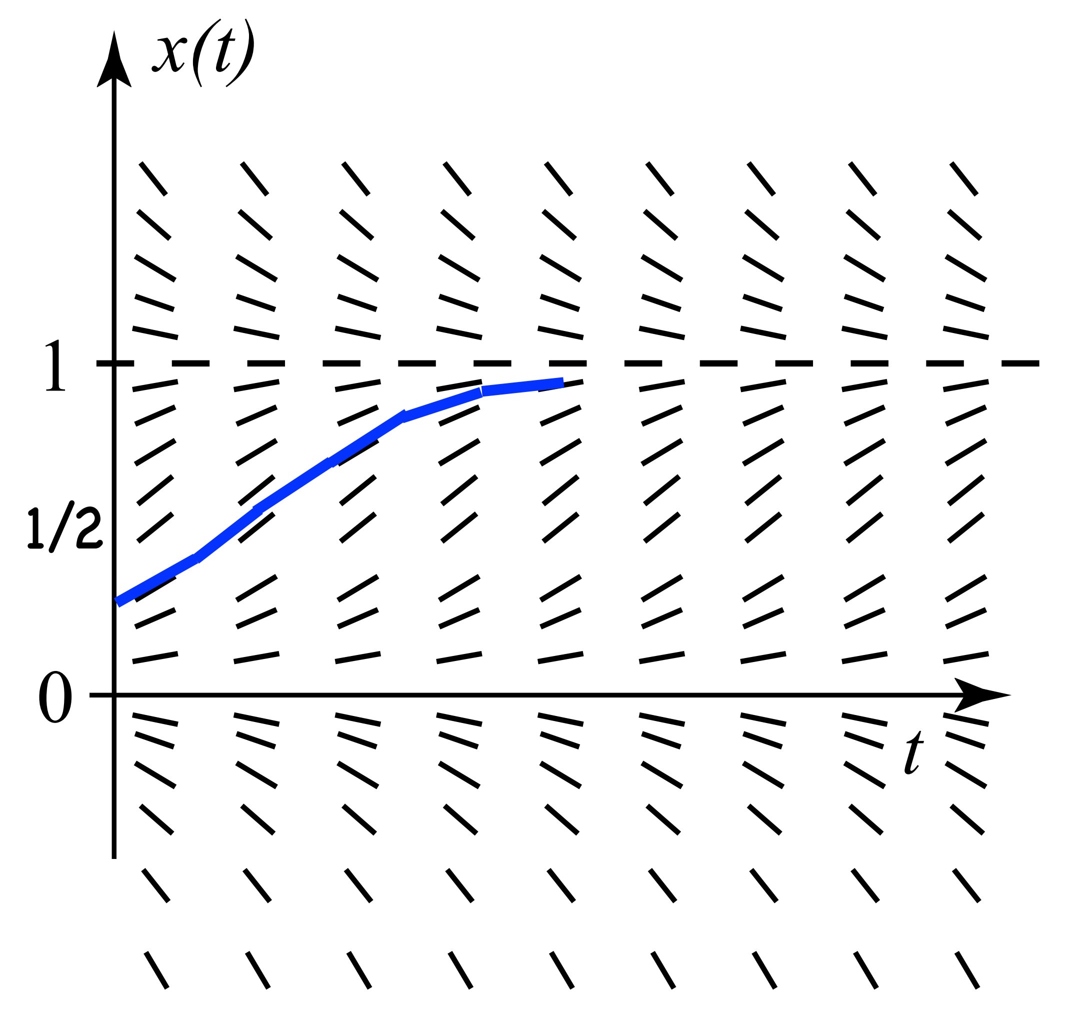

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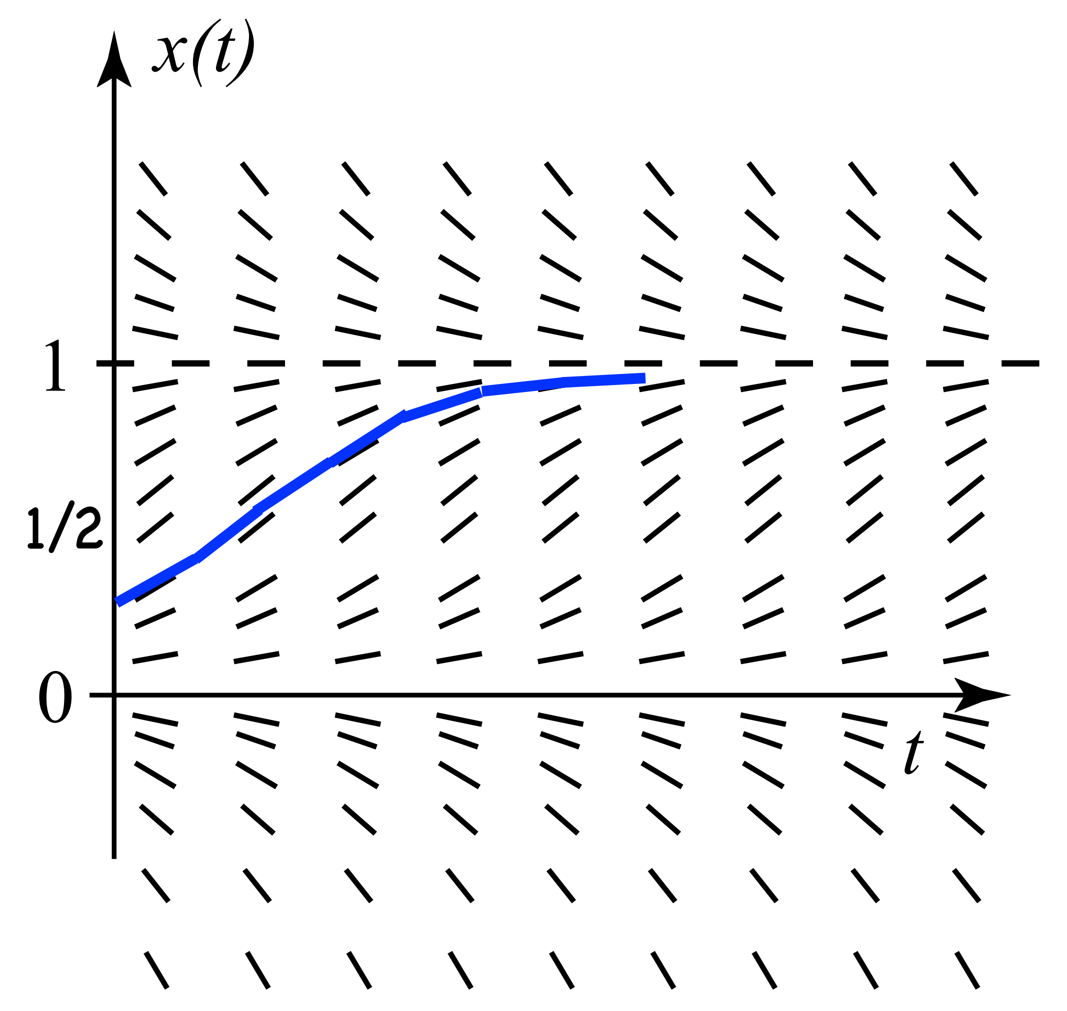

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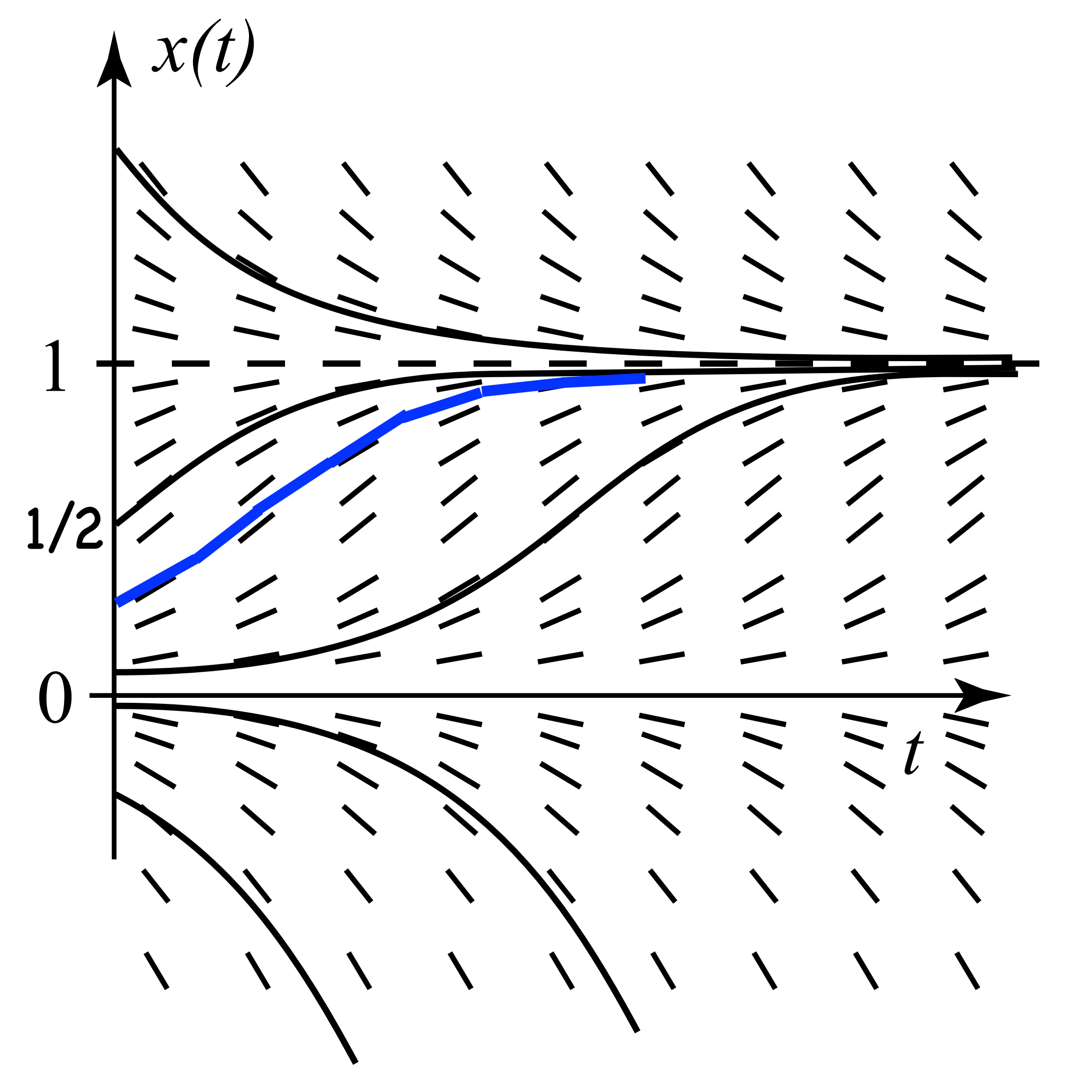
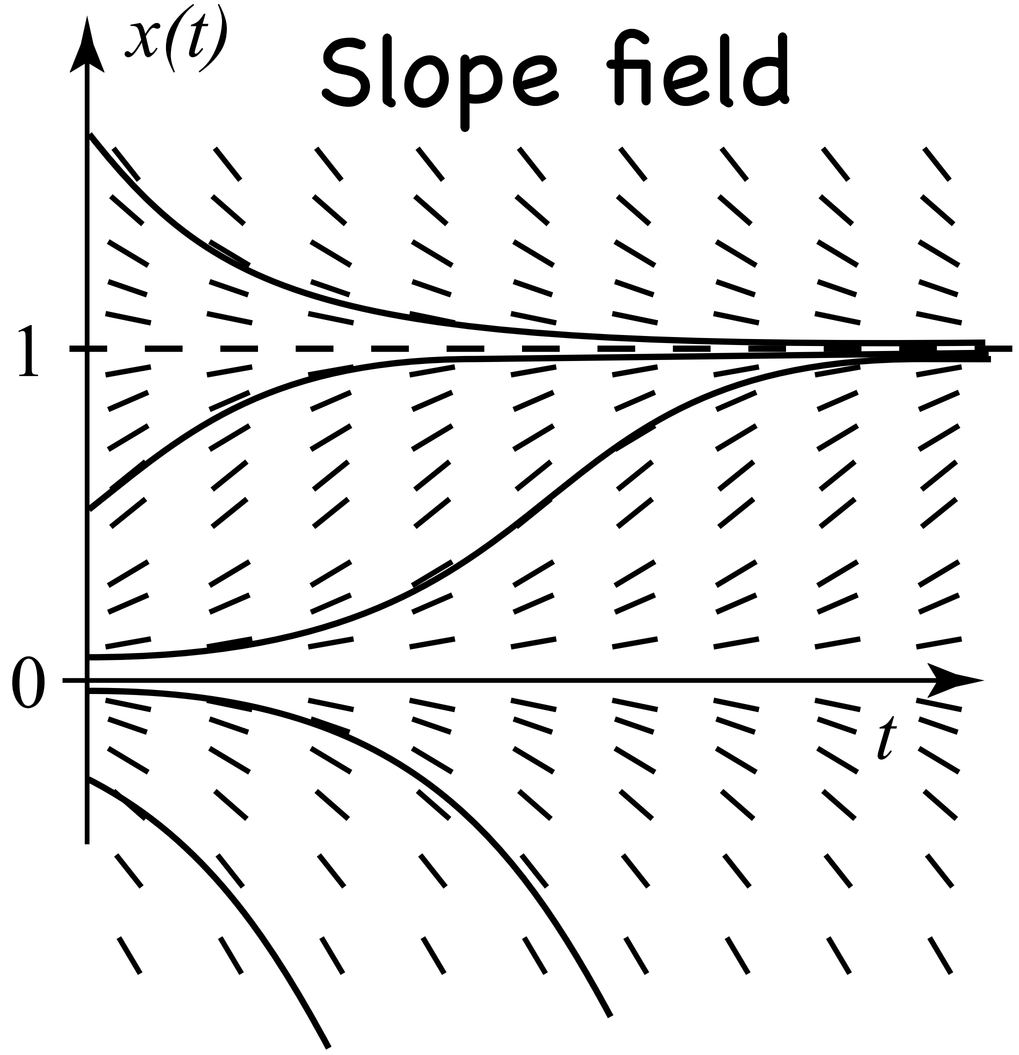


Velocity versus position

Velocity (x′) vs. position (x)



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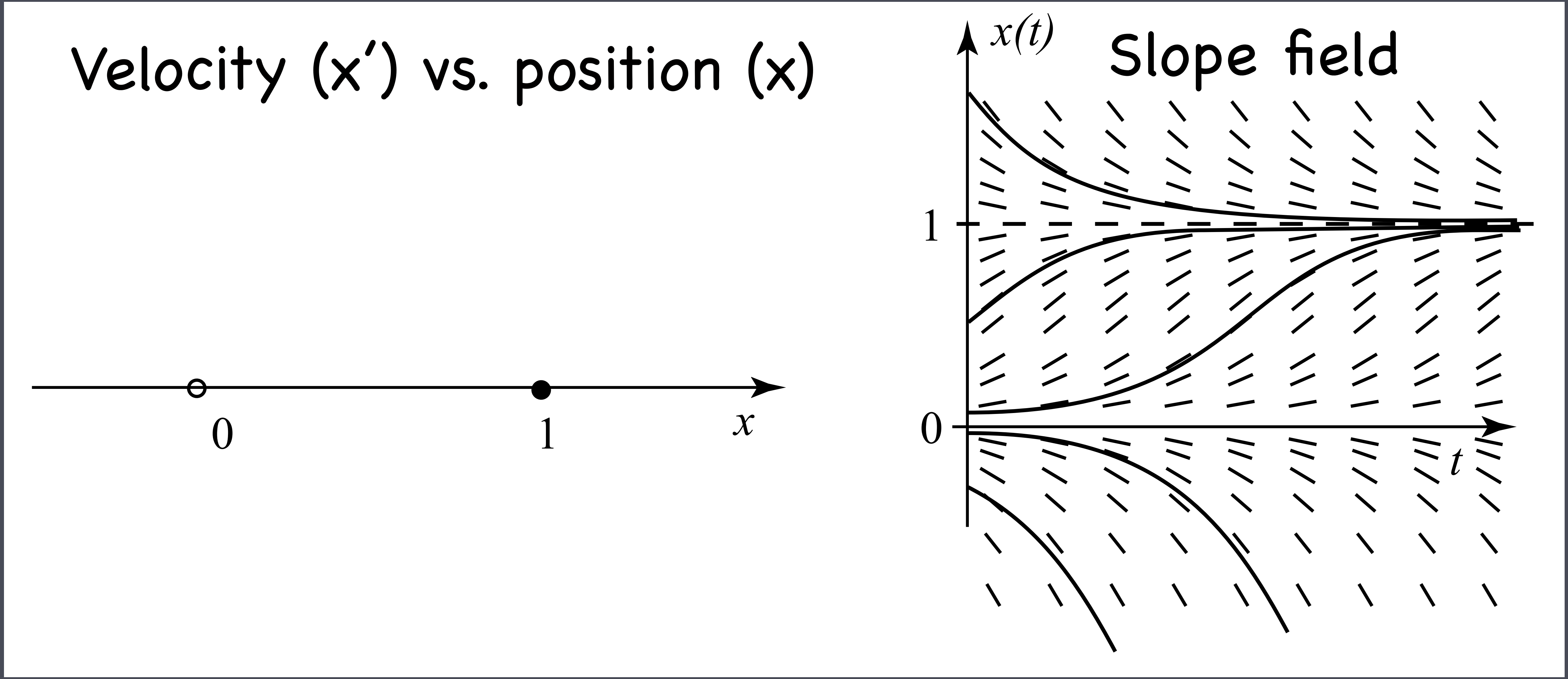
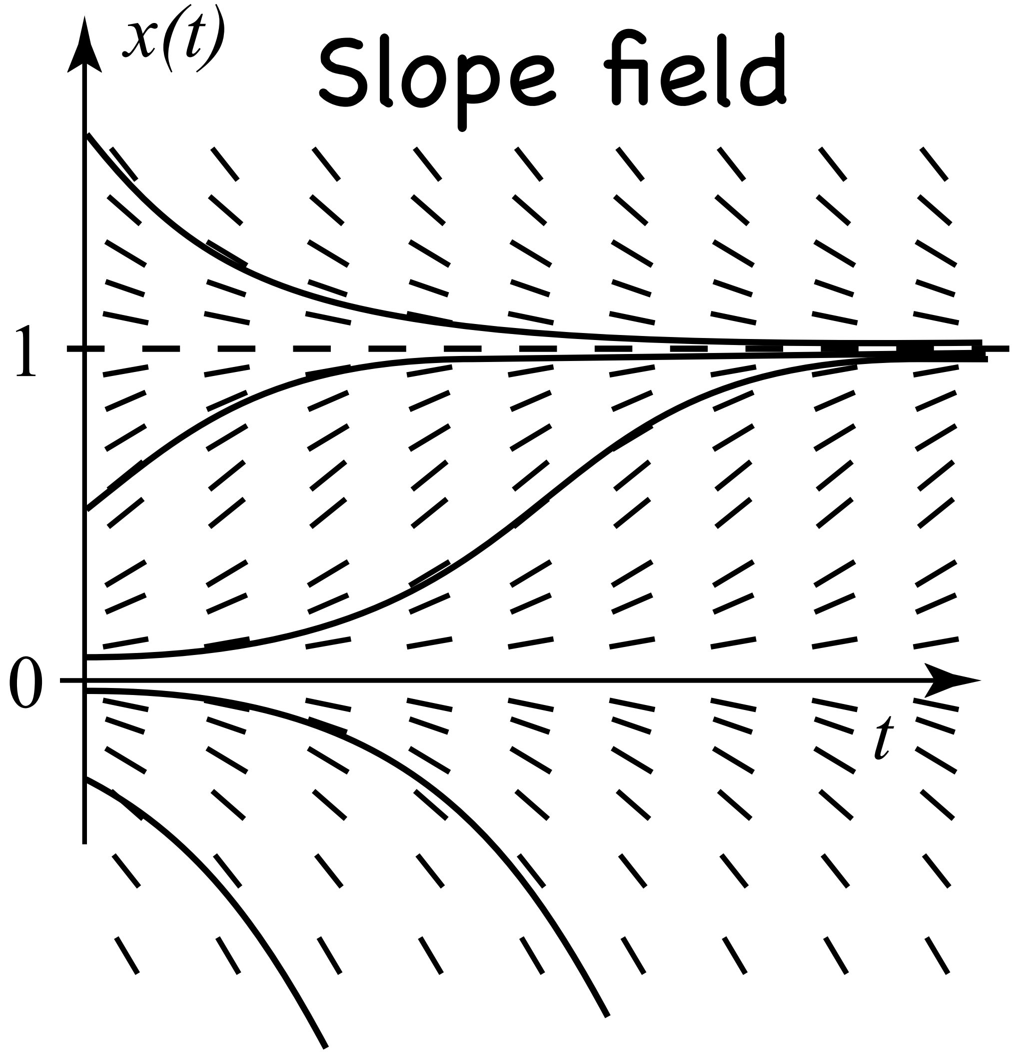


Velocity versus position

Velocity (x′) vs. position (x)

$x' = f(x) = x(1-x)$


Slope field

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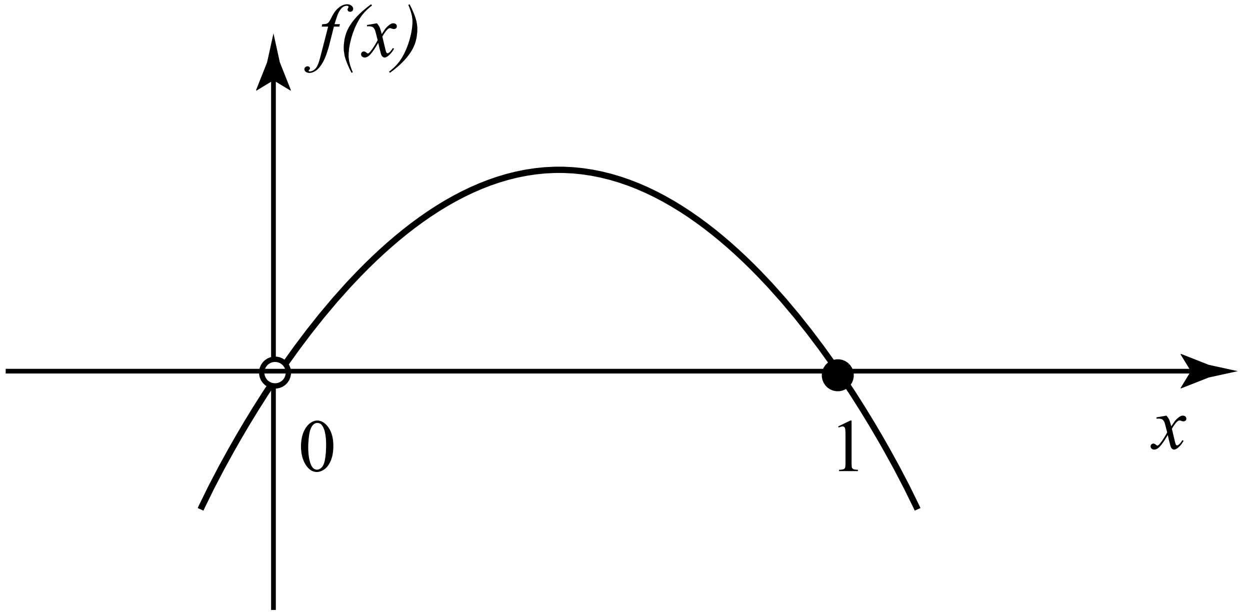

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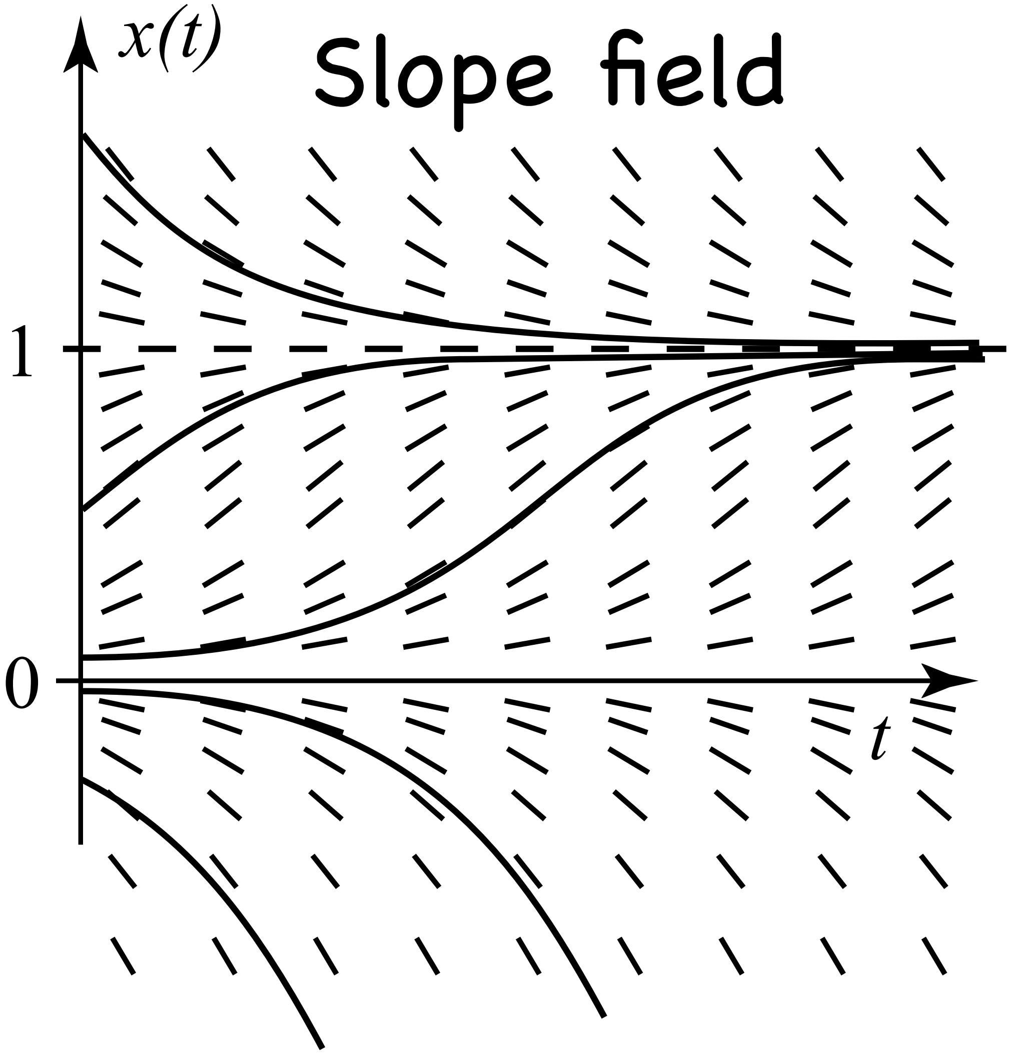
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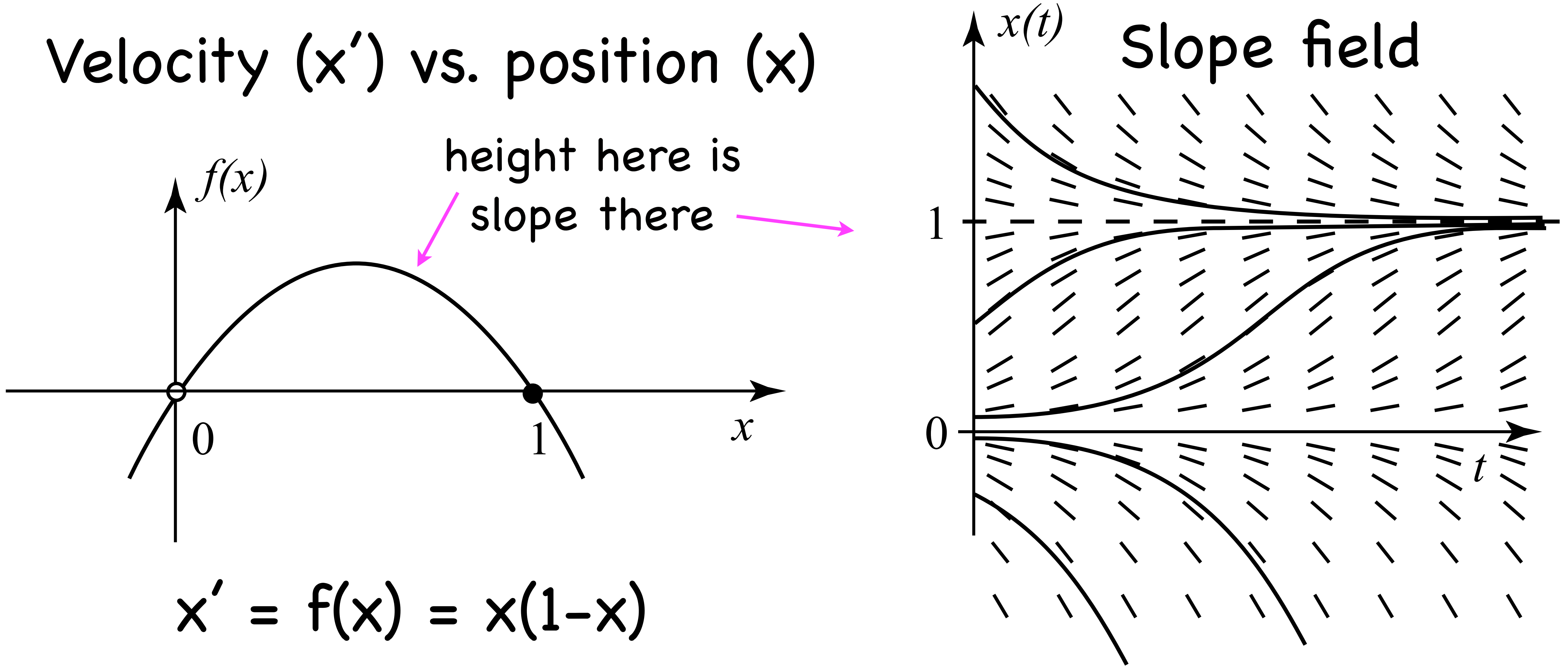


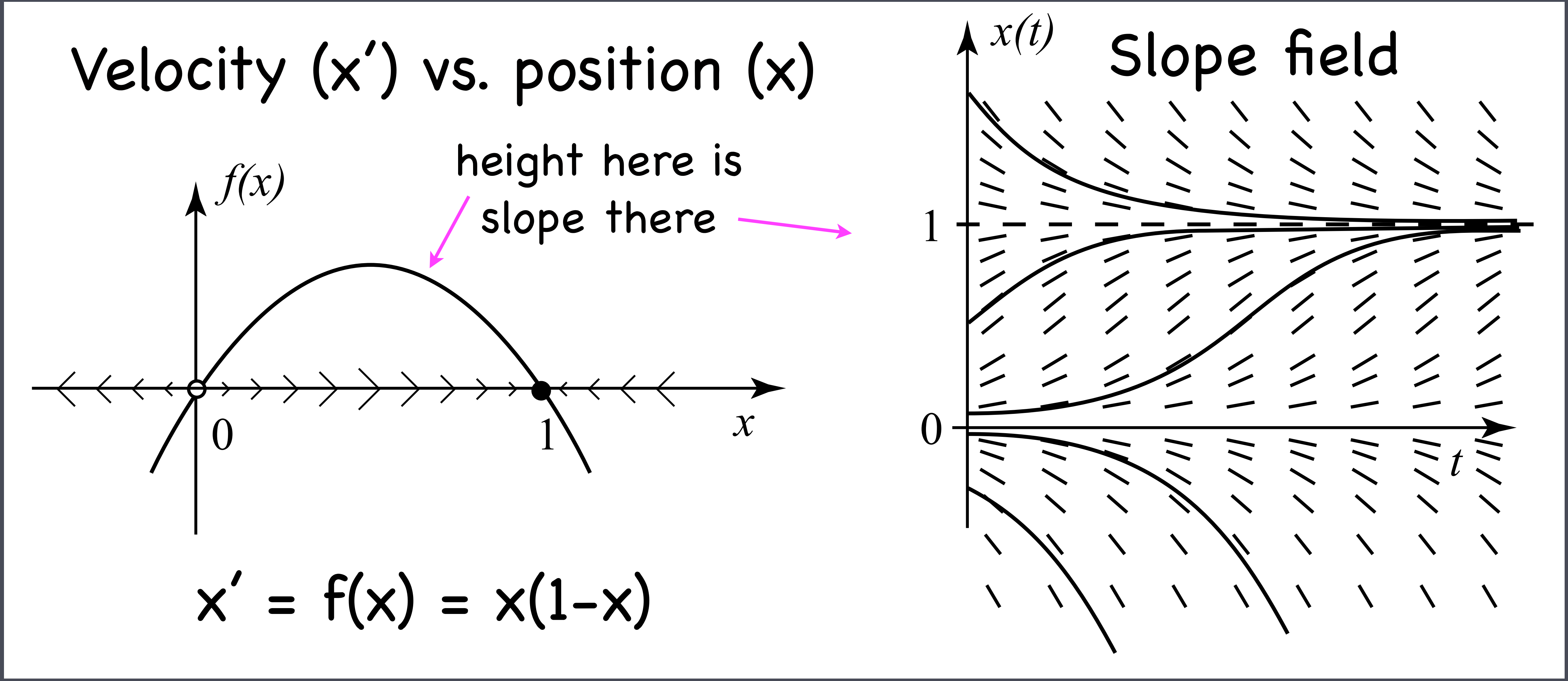
Velocity (x′) vs. position (x)

height here is slope there

Slope field

$$x' = f(x) = x(1-x)$$

Velocity versus position



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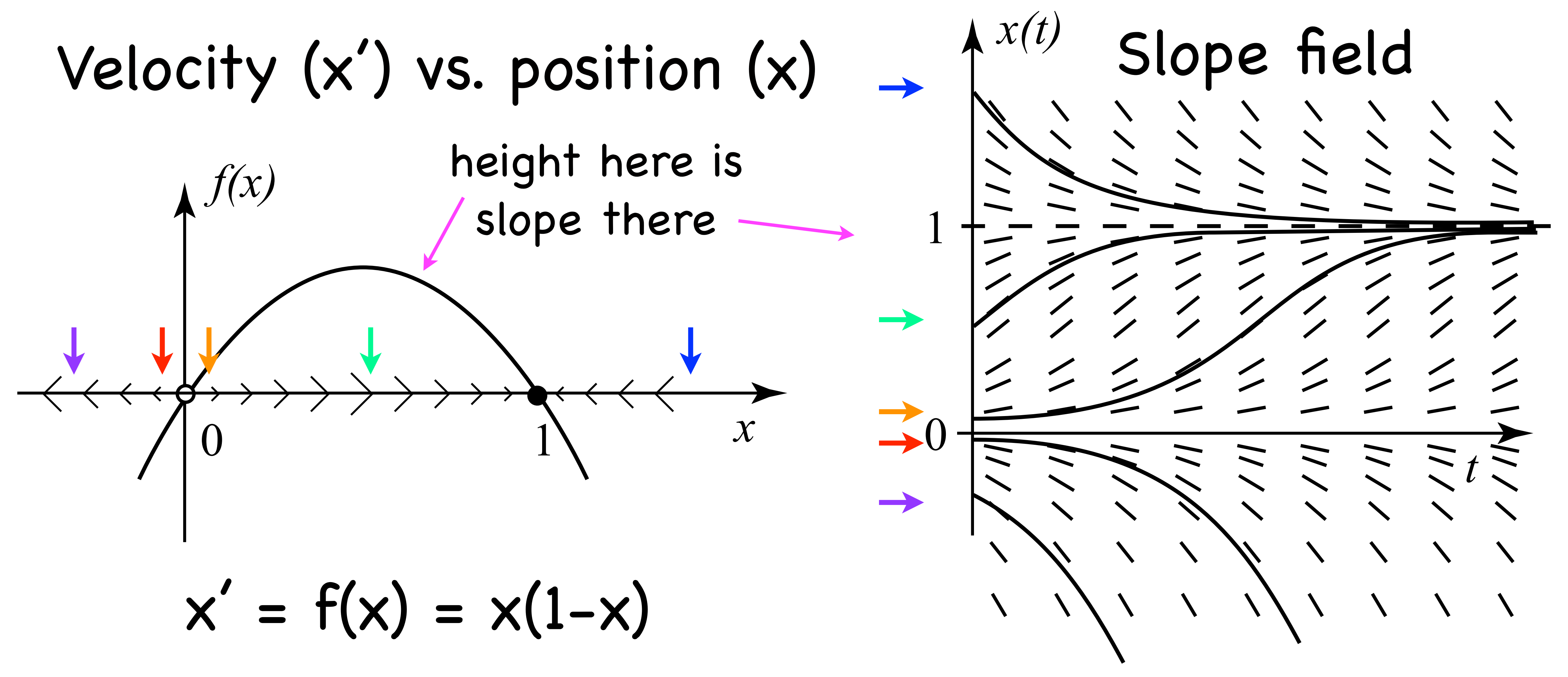
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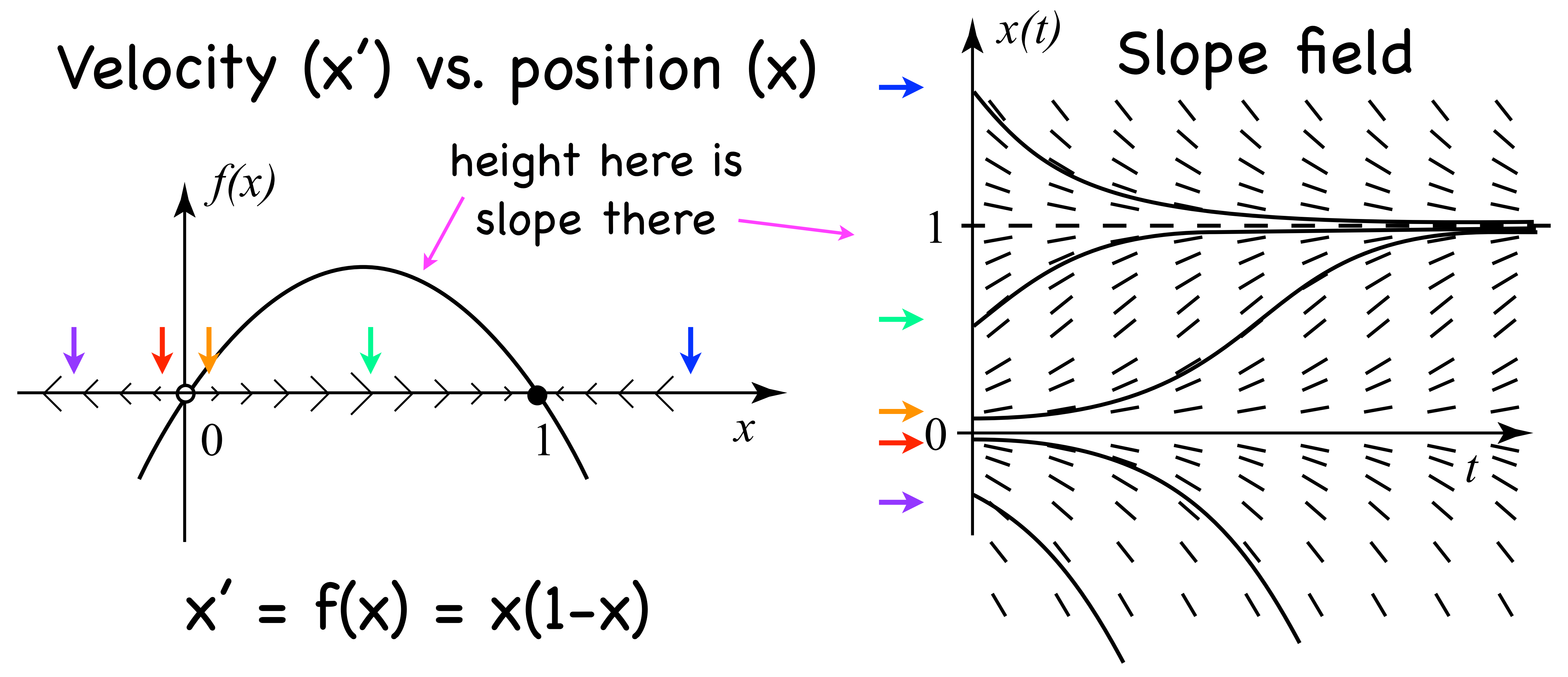
Velocity versus position



Velocity (x′) vs. position (x)

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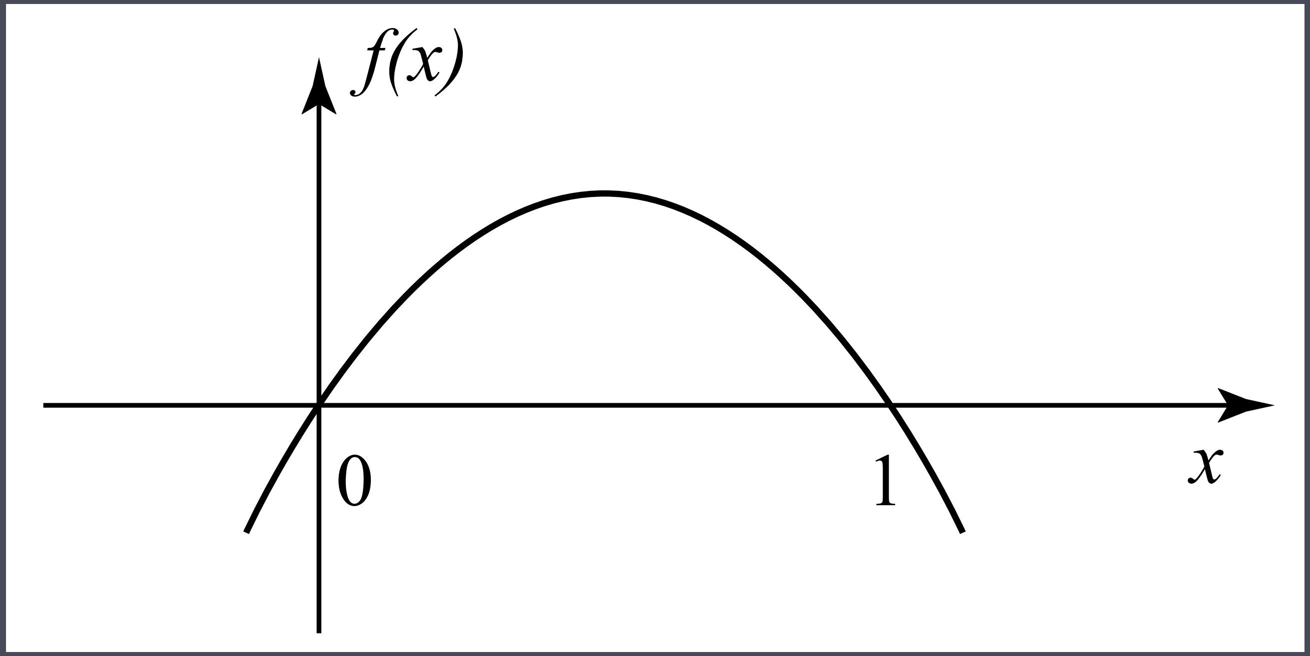
$x' = f(x) = x(1-x)$

Stable steady state- all nearby solutions approach

Unstable steady state - not stable

Determine stability

$$x' = x(1 - x)$$



- If you start at x(0) = -0.01, the solution

(A) increases

(B) decreases

Determine stability

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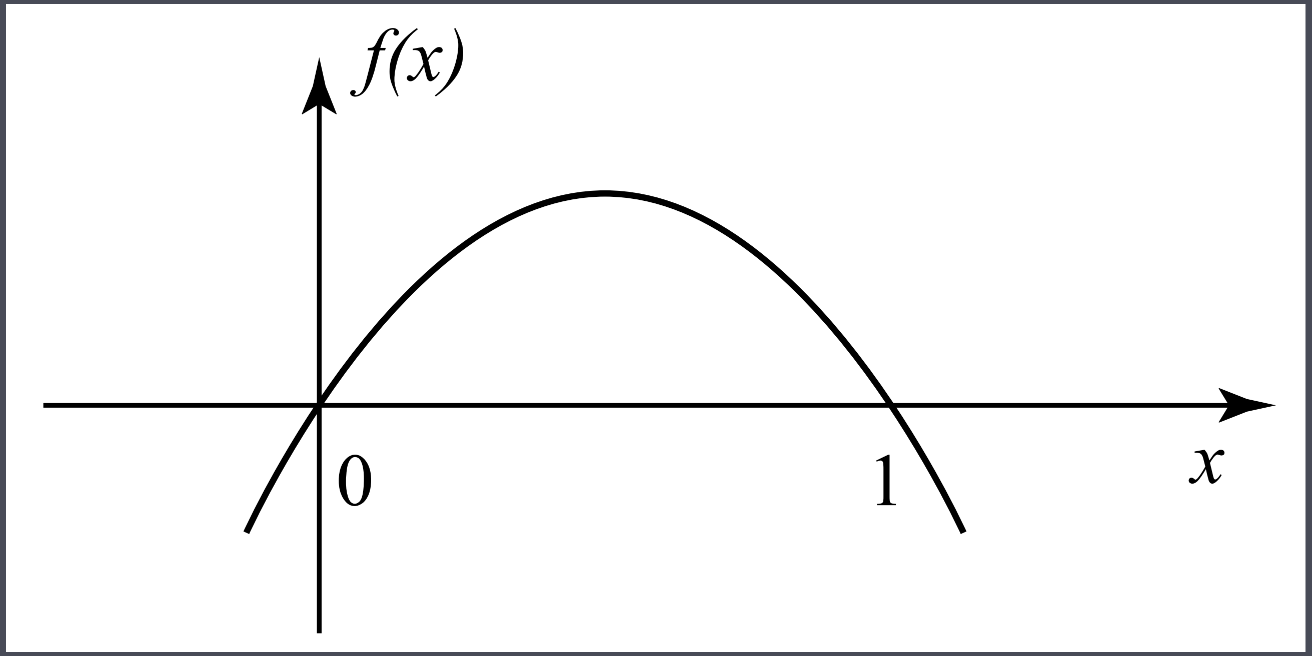

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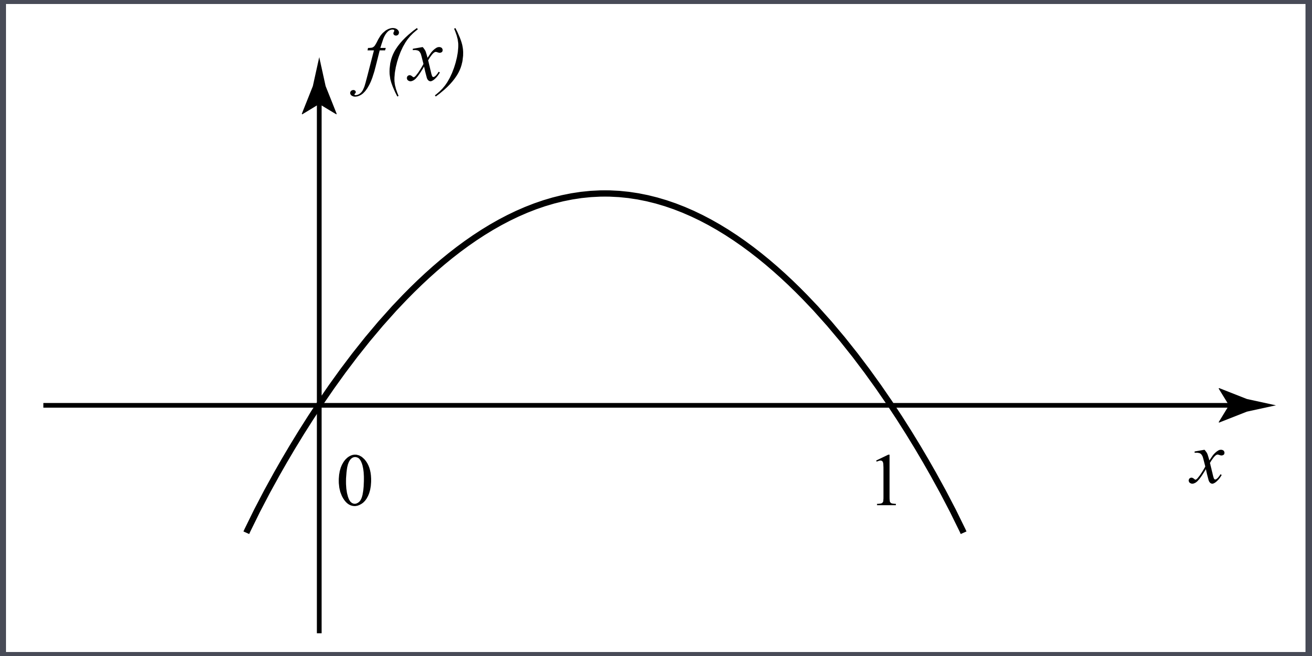

- If you start at x(0) = 0.99, the solution

(A) increases

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Determine stability

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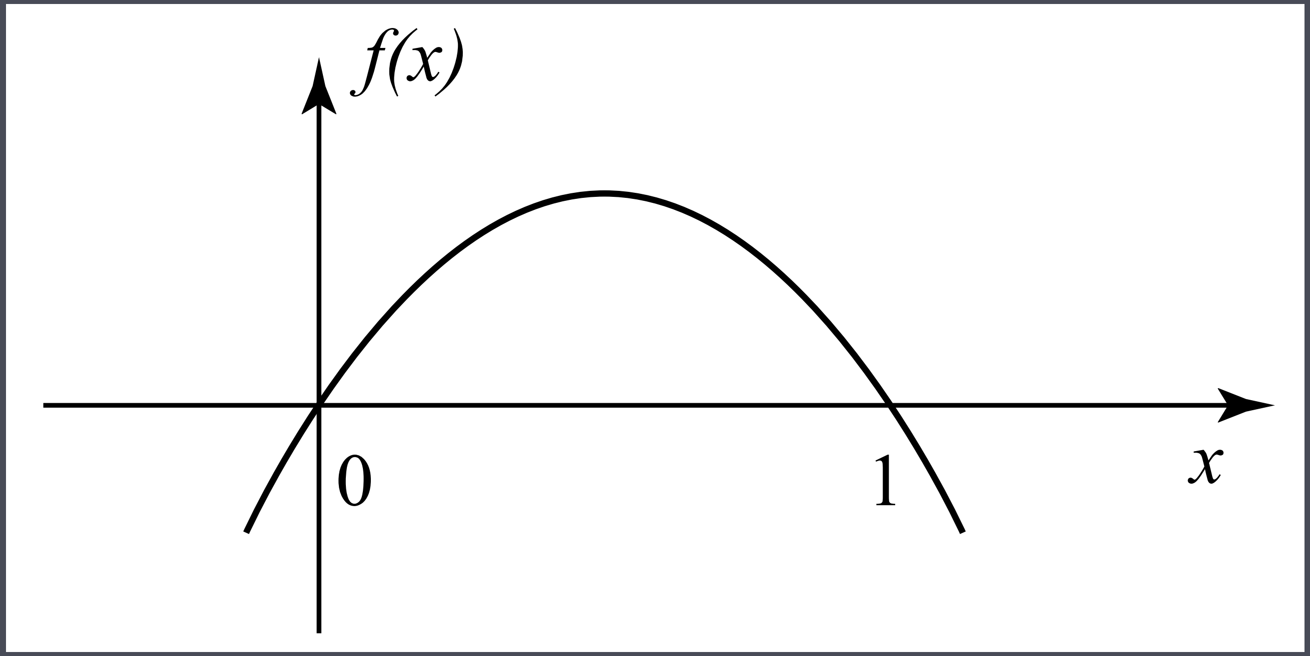


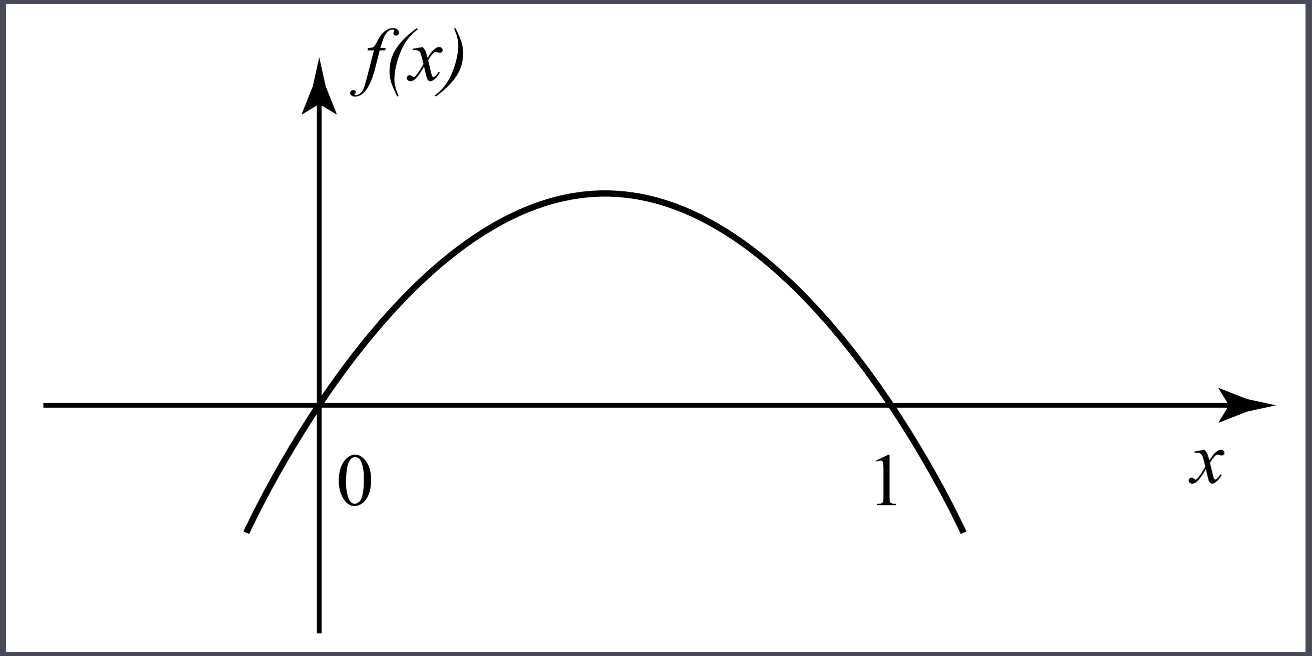
- If you start at x(0) = 1.01, the solution

(A) increases

(B) decreases

Determine stability

$$x' = x(1-x)$$



(A) Both x(t)=0 and x(t)=1 are stable steady states.

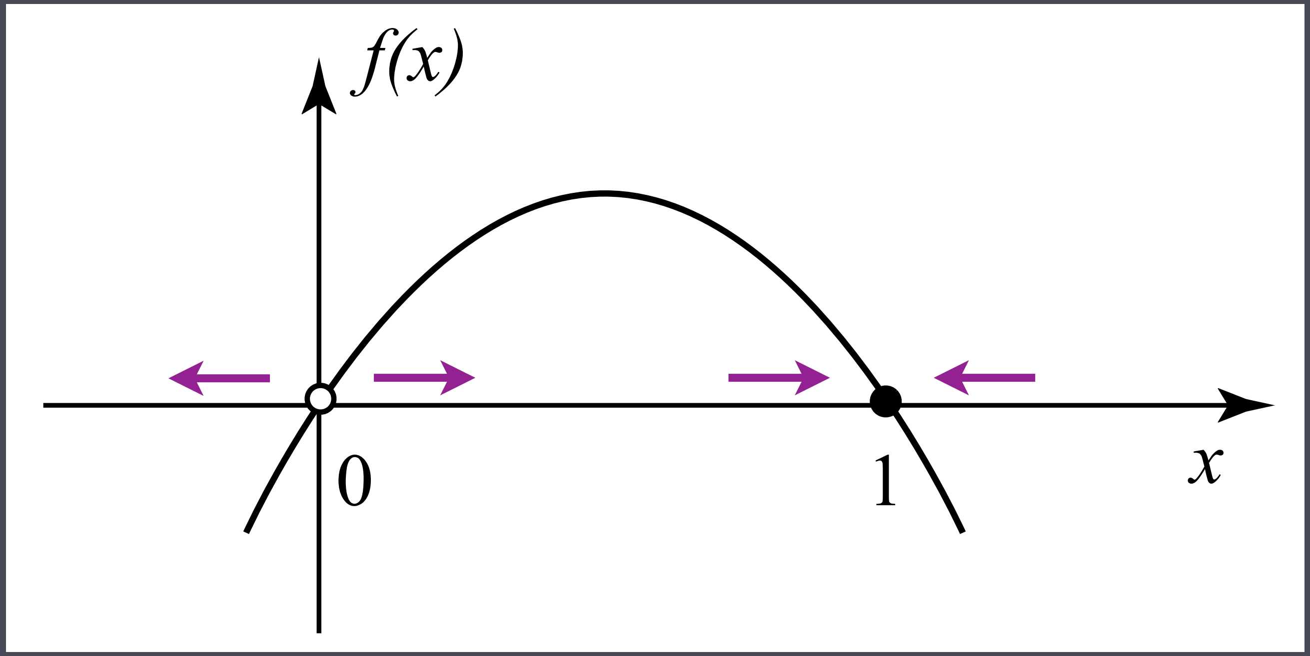
(B) x(t)=0 is stable and x(t)=1 is unstable.

(C) x(t)=0 is unstable and x(t)=1 is stable.

(D) Both x(t)=0 and x(t)=1 are unstable steady states.

Determine stability

$$x' = x(1-x)$$



(A) Both x(t)=0 and x(t)=1 are stable steady states.

(B) x(t)=0 is stable and x(t)=1 is unstable.

(C) x(t)=0 is unstable and x(t)=1 is stable.

(D) Both x(t)=0 and x(t)=1 are unstable steady states.

Stable - solid dot. Unstable - empty dot.