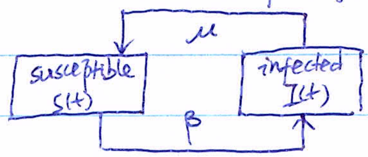
Lecture 34 (Nov. 27, 2013)

Learning Goals: an example of nonlinear DE in epidemiology

- SIS model

Define: $S(t)$ = the number of susceptible people in the population

$I(t)$ = the number of infected people in the population



$I + S \to 2I$

$I \to S$

$N = S + I$, the whole population.

Assumptions: ① the population is well mixed

one individual has equal probability to contact with any other individuals

β – contact rate

$\Rightarrow \beta \cdot N$ is how many people one individual can make contact with per unit time

$\Rightarrow \beta N \cdot \frac{S}{N}$ is how many susceptible people one infected individual can make contact with and make infected

$\Rightarrow \beta N \cdot \frac{S}{N} \cdot I = \beta\ I$ is how many susceptible individuals become infected

② μ – recovery rate

$\Rightarrow \mu I$ is how many sick people become healthy (susceptible) per unit time

③ No other factors are considered.

Derive the model: How does $S(t)/I(t)$ change at one moment?

$$\left.\begin{aligned} \frac{dI}{dt} &= \beta S I - \mu I \\ \frac{dS}{dt} &= \mu I - \beta S I \end{aligned}\right\} \Rightarrow$$

$\frac{dN}{dt} = \frac{d}{dt}(S+I) = \frac{dS}{dt} + \frac{dI}{dt} = 0$

then $I = N - S$

$\frac{dS}{dt} = \mu(N-S) - \beta S(N-S) = \mu N - (\mu + \beta N)S + \beta N^2$

μ, β, N are positive constants

Analysis: (i) Steady States: $\frac{dS}{dt} = \mu(N-S) - \beta S(N-S) = (\mu - \beta S)(N-S) = 0$

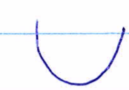
$\Rightarrow S = N$ or $S = \mu/\beta$

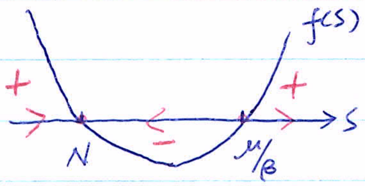
$S = N$: no infected individuals

$S = \mu/\beta > 0$, depends on μ & β

(ii) Sketch $\frac{dS}{dt}$ vs. S. $\frac{dS}{dt} = f(S) = \mu N - (\mu + \beta N)S + \beta S^2$

the concavity of $f(S)$ depends on $f''(S) = 2\beta > 0$

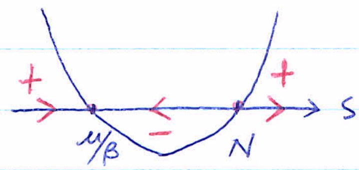




(a) $\mu/\beta > N$

$S = N$ is stable

$S = \mu/\beta$ is unstable



(b) $\mu/\beta < N$

$S = N$ is unstable

$S = \mu/\beta$ is stable

(iii) μ v.s. βN : (a) $\mu > \beta N$, S in $[0, N)$ approaches to N

$\mu/\beta > N$ is the steady state that never shows up because $S < N$

(b) $\mu < \beta N$, S in $[0, N)$ approaches to μ/β

β & N depend on the population.

μ depends on the type of disease and how strong every individual is.