Limits, Continuous Functions, and the Derivative
Math 102 Section 106

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Math 102: Announcements

- First full week done!
- What's next? [https://wiki.math.ubc.ca/mathbook/M102/Course_calendar](https://wiki.math.ubc.ca/mathbook/M102/Course_calendar)
I rate my presentation (neatness, style, clarity, sentences):

A. Excellent
B. Very good
C. Good
D. Could be better
E. Not so great
I rate the correctness of my work (neatness, style, clarity, sentences):

A. Excellent
B. Very good
C. Good
D. Could be better
E. Not so great
OSH Self-assessment: Content

I rate the content of my OSH as (Explanation of set-up, logical flow, correct conclusions, and interpretation)

A. Excellent
B. Very good
C. Good
D. Could be better
E. Not so great
Last time

- Limits
  - plug-in for continuous functions
  - factoring
  - Some limits DNE

- To get the instantaneous rate of change, take $h \to 0$ in the average velocity

- The derivative of $y = f(x)$ at $x_0$ is

  \[ f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}. \]

- As $h \to 0$, the secant line approaches a tangent line
Today...

- Continuous Functions
- Types of discontinuities
- Limits
- Limits at infinity
- Limits and the derivative

I have many Clicker questions for you. Please ask many questions in return!
Continuous function

- A function \( f(x) \) is **continuous** at a point \( x = a \) in its domain if

\[
\lim_{x \to a} f(x) = f(a).
\]

- For \( f(x) \) to be continuous at \( x = a \), three things need to be true:
  1. \( f(x) \) needs to be defined at the point
  2. the limit exists as \( x \) approaches the point
  3. the limit value is the same as the value of the function
Types of discontinuities

https://www.desmos.com/calculator/viztz0quu8
Types of discontinuities

Q1. The function

\[ f(x) = \frac{x^3 - ax^2}{x - a} \]

is undefined at \( x = a \). What value should be assigned to \( f(a) \) such that \( f(x) \) is continuous at \( x = a \)?

A. 0
B. \( a \)
C. \( a^2 \)
D. \( a^3 \)
Types of discontinuities

Q1. The function

\[ f(x) = \frac{x^3 - ax^2}{x - a} \]

is undefined at at \( x = a \). What value should be assigned to \( f(a) \) such that \( f(x) \) is continuous at \( x = a \)?

Define

\[
\begin{align*}
    f(a) &= \lim_{x \to a} f(x) = \lim_{x \to a} \frac{x^3 - ax^2}{x - a} \\
    &= \lim_{x \to a} x^2 \frac{x - a}{x - a} = \lim_{x \to a} x^2 = a^2
\end{align*}
\]
Types of discontinuities

Q2. Consider the function

\[ f(x) = \begin{cases} 
-1, & x < a, \\
1, & x \geq a. 
\end{cases} \]

What is \( \lim_{x \to a} f(x) \)?

A. 1  
B. −1  
C. 0  
D. The limit does not exist
Types of discontinuities

Q2. Consider the function

\[ f(x) = \begin{cases} 
  -1, & x < a, \\
  1, & x \geq a. 
\end{cases} \]

\[ \lim_{x \to a} f(x) \text{ DNE since the right-side and left-side limits are not equal:} \]

- Approach \( x = a \) from above (right-side limit):

\[ \lim_{x \to a^+} f(x) = 1 \]

- Approach \( x = a \) from below (left-side limit):

\[ \lim_{x \to a^-} f(x) = -1 \]
Types of discontinuities

Q3. What is

\[ \lim_{x \to a} \frac{1}{x - a} \]?

A. $\infty$
B. $-\infty$
C. DNE
D. More than one answer is correct
Types of discontinuities

Q3. What is

\[ \lim_{x \to a} \frac{1}{x - a} \]?

- The function \( f(x) = \frac{1}{x-a} \) is not defined at \( x = a \).

- \( \lim_{x \to a} \frac{1}{x-a} \) DNE:
  - Right-side limit:
    \[ \lim_{x \to a^+} \frac{1}{x-a} = +\infty \quad (x > a \text{ means } x - a > 0) \]
  - Left-side limit:
    \[ \lim_{x \to a^-} \frac{1}{x-a} = -\infty \quad (x < a \text{ means } x - a < 0) \]
Q4. Which of the following limits DNE?

A. \[ \lim_{x \to 0} \frac{1}{x + 1} \]

B. \[ \lim_{x \to -1} \frac{1}{x + 1} \]

C. \[ \lim_{x \to -1} \frac{x + 1}{x + 1} \]

D. more than one of the above limits DNE
Q5. What value of $b$ makes the following function continuous?

$$f(x) = \begin{cases} 
ax^2 + 1, & x < 1 \\
-x^3 + x, & x \geq 1.
\end{cases}$$

A. $a = 1$
B. $a = 0$
C. $a = -1$
D. no value of $a$ works

https://www.desmos.com/calculator/v8hfkabrry
Limits at infinity

Q6. Which of the following limits exist?

A. \( \lim_{x \to \infty} x^3 \)

B. \( \lim_{x \to \infty} x^{\frac{1}{3}} \)

C. \( \lim_{x \to \infty} 3^x \)

D. \( \lim_{x \to \infty} x^{-3} \)

https://www.desmos.com/calculator/ihvwwf4zwh
Q7. What is the following limit?

\[ \lim_{{x \to \infty}} \frac{x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6}{6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1} \]

A. 0
B. 6
C. \( \frac{1}{6} \)
D. DNE

Two methods: asymptotic approximation or factoring
The derivative of a function $y = f(x)$ at $x_0$ is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$ 

You can also write $\frac{dy}{dx} \bigg|_{x_0}$ to denote $f'(x_0)$. 


Derivative

Q8. To compute the derivative of the function $f(x) = (x + 3)^2$, we need to compute:

A. 
$$\lim_{h \to 0} \frac{(x + 3 + h)^2 - (x + 3)^2}{h}$$

B. 
$$\lim_{x \to 1} \frac{(x + 3)^2 - x^2}{x}$$

C. 
$$\lim_{h \to 1} \frac{(x + 3 + h)^2 - (x + 3)^2}{h}$$

D. 
$$\lim_{h \to 0} \frac{(x + 3 + h)^2 - (x + 3)^2}{x}$$
Continuous functions, limits, and the derivative.

See end of the slides for a related exam problem.

**Bonus Challenge** Write the following limit as the derivative of a function:

\[
limit_{x \to 2} \frac{x^2 - 4}{x - 2}\]
Answers

1. C
2. D
3. C
4. B
5. C
6. D
7. C
8. A
1. Use the definition of the derivative and the following hint to calculate the derivative of the function \( f(x) = \sqrt{x} \). Hint:

\[
\frac{(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})}{(\sqrt{a} + \sqrt{b})} = \frac{a - b}{(\sqrt{a} + \sqrt{b})}
\]