Today

- log graph and semi-log plots
- Exponential derivative (what is $C_a$?)
- Bacterial growth example
- Doubling time, half life, characteristic time
- Exponential behaviour as solution to DE
Which of following is the graph of $\ln(x)$?
Which of following is the graph of \( \ln(x) \)?

- \( \ln(1) = 0 \)
- \( \ln(0^+) \to -\infty \)
- Keeps growing but slowly
Log-log and semi-log plots

- A log-log plot is a plot on which you plot \( \log(y) \) versus \( \log(x) \) instead of \( y \) versus \( x \).

- A semi-log plot is a plot on which you plot \( \log(y) \) versus \( x \) instead of \( y \) versus \( x \).

\[
\text{----this is what OSH 5 asks you to use}
\]
Suppose $y = ae^{kx}$. $a$ and $k$ are constants.

Define new variable $V = \ln(y)$.

$V = \ln(y) = \ln(ae^{kx}) = \ln(a) + kx$.

$V = A + kx$ where $A = \ln(a)$.

On a semi-log plot, $y = ae^{kx}$ looks linear.
Suppose $y = ax^p$.

Define new variable $V = \ln(y)$.

$V = \ln(y) = \ln(ax^p) = \ln(a) + p \ln(x)$.

$V = A + pU$ where $A = \ln(a)$, $U = \ln(x)$.

On a log-log plot, $y = ax^p$ looks linear.
Regular, log-log and semi-log plots

Two data sets.

Power function?

Exponential function?

Regular x-y plot.

Slide not shown in class but include for interest.
Plot $Y_i = \ln(y_i)$ versus $x_i$.

Conclude that:

(A) Blue is power function.
(B) Blue is exponential.
(C) Orange is power function.
(D) Orange is exponential.

Semi-log plot.

Slide not shown in class but include for interest.
Plot $Y_i = \ln(y_i)$ versus $x_i$.

Conclude that:

(A) Blue is power function.
(B) Blue is exponential.
(C) Orange is power function.
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Semi-log plot.

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Plot $Y_i = \ln(y_i)$ versus $X_i = \ln(x_i)$.

Conclude that:

(A) Blue is power function.
(B) Blue is exponential.
(C) Orange is power function.
(D) Orange is exponential.

Log-log plot.

Slide not shown in class but include for interest.
Plot $Y_i = \ln(y_i)$ versus $X_i = \ln(x_i)$. 

Conclude that:

(A) Blue is power function.
(B) Blue is exponential.
(C) Orange is power function.
(D) Orange is exponential.

Slide not shown in class but include for interest.
\[ f(x) = a^x. \quad f'(x) = C_a a^x. \quad C_a = ?? \]

- Recall that we got stuck on this derivative.
- Time to get unstuck...
\[ f(x) = e^{\ln(2)x}. \]

(A) \[ f'(x) = e^{\ln(2)x}. \]

(B) \[ f'(x) = \ln(2)e^{\ln(2)x}. \]

(C) \[ f'(x) = \ln(2) \cdot 1/2 \cdot e^{\ln(2)x}. \]

(D) \[ f'(x) = \ln(2)x e^{\ln(2)x-1}. \]
\[ f(x) = e^{\ln(2)x}. \]

(A) \[ f'(x) = e^{\ln(2)x}. \]

(B) \[ f'(x) = \ln(2)e^{\ln(2)x}. \]

(C) \[ f'(x) = \ln(2) \cdot \frac{1}{2} \cdot e^{\ln(2)x}. \]

(D) \[ f'(x) = \ln(2)x e^{\ln(2)x-1}. \]
\( f(x) = e^{\ln(2)x} \).

(A) \( f(x) = 2x \).

(B) \( f(x) = (e^{\ln(2)})^x = 2^x \).

(C) \( f(x) = e^{\ln(2)}e^x = 2e^x \).

(D) \( f(x) = e^{\ln(x^2)} = x^2 \).
\( f(x) = e^{\ln(2)x} \).

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(D) \( f(x) = e^{\ln(x^2)} = x^2 \).
From the last two clicker Qs...

- \( f(x) = e^{\ln(2)x} \rightarrow f'(x) = \ln(2)e^{\ln(2)x} \).
- \( f(x) = e^{\ln(2)x} \rightarrow f(x) = 2^x \).
- So \( f(x) = 2^x \rightarrow f'(x) = 2^x \ln(2) \).
- In general, \( f(x) = a^x \rightarrow f'(x) = a^x \ln(a) \).
What value of $k$ makes $a^x = e^{kx}$?

(A) $k = e^a$

(B) $k = e^{-a}$

(C) $k = \ln(a)$

(D) $k = -\ln(a)$

(E) $k = \ln(-a)$
What value of $k$ makes $a^x = e^{kx}$?

(A) $k = e^a$

(B) $k = e^{-a}$

(C) $k = \ln(a)$

(D) $k = -\ln(a)$

(E) $k = \ln(-a)$

\[a^x = (e^k)^x\]

\[a = e^k\]

\[\ln(a) = \ln(e^k)\]

\[\ln(a) = k \ln(e)\]

\[\ln(a) = k\]

\[f(x) = a^x = e^{\ln(a)x}\]

---\> $f'(x) = a^x \ln(a)$. 

---
A single cell is placed in a dish containing a growth medium. The dish can support 100,000 cells. If each cell divides once every 24 hours, how long before the dish is full?

Rough estimate (no calcs, just intuition):

(A) ~1 week.
(B) ~2 weeks.
(C) ~1 month.
(D) ~1 year.
(E) ~$10^4$ days $\approx 27$ years.
A single cell is placed in a dish containing a growth medium. The dish can support 100,000 cells. If each cell divides once every 24 hours, how long before the dish is full?

Rough estimate (no calcs, just intuition):

(A) ~1 week.
(B) ~2 weeks.
(C) ~1 month.
(D) ~1 year.
(E) $\sim 10^4$ days $\approx 27$ years.
A single cell is placed in a dish containing a growth medium. The dish can support 100,000 cells. If each cell divides once every 24 hours, how long before the dish is full?

(A) \( p(t) = e^{t/24} \).

(B) \( p(t) = 100,000 \ 2^{t/24} \).

(C) \( p(t) = e^{\ln(2)\ t} \).

(D) \( p(t) = 2^{-t/24} \).

(E) \( p(t) = 2^{t/24} \).
A single cell is placed in a dish containing a growth medium. The dish can support 100,000 cells. If each cell divides once every 24 hours, how long before the dish is full?

(A) \( p(t) = e^{t/24} \).

(B) \( p(t) = 100,000 \cdot 2^{t/24} \).

(C) \( p(t) = e^{\ln(2)t} = 2^t \quad \text{--- \( t \) measured in days.} \)

(D) \( p(t) = 2^{-t/24} \).

(E) \( p(t) = 2^{t/24} \). \quad \text{--- \( t \) measured in hours.}
A single cell is placed in a dish containing a growth medium. The dish can support 100,000 cells. If each cell divides once every 24 hours, how long before the dish is full?

(A) \( t = \frac{\ln(10^5)}{\ln(2)} \)

(B) \( t = \frac{10^5}{\ln(2)} \)

(C) \( t = \frac{\ln(10^5)}{2} \)

(D) \( t = \frac{100,000}{24 \text{ days}} \)
A single cell is placed in a dish containing a growth medium. The dish can support 100,000 cells. If each cell divides once every 24 hours, how long before the dish is full?

(A) \( t = \frac{\ln(10^5)}{\ln(2)} \)  

(B) \( t = \frac{10^5}{\ln(2)} \)  

(C) \( t = \frac{\ln(10^5)}{2} \)  

(D) \( t = \frac{100,000}{24} \) days  

\( p(t) = 2^t \)  

\( 10^5 = 2^t \)  

\( \ln(10^5) = \ln(2^t) \)  

\( \ln(10^5) = t \ln(2) \)  

\( t \approx 16.6 \) days
Doubling time

Let $c(t) = c_0e^{kt}$. At $t=0$, $c(0) = c_0e^0 = c_0$.

If $k>0$, $c(t)$ is increasing and doubles when $c_0e^{kt} = 2c_0$.

That is when $t=\ln(2)/k$.

This is called the doubling time.
Half-life

Let \( c(t) = c_0 e^{kt} \). At \( t=0 \), \( c(0) = c_0 e^0 = c_0 \).

If \( k<0 \), \( c(t) \) is decreasing and halves when \( c_0 e^{kt} = c_0/2 \).

That is when \( t=-\ln(2)/k \).

If written \( c(t) = c_0 e^{-kt} \) with \( k>0 \) then \( t=\ln(2)/k \) (same as doubling time).

This is called the half-life.
Let $c(t) = c_0e^{-kt}$. At $t=0$, $c(0) = c_0e^0 = c_0$.

If $k>0$, $c(t)$ is decreasing and reaches $1/e$ its original value when $c_0e^{kt} = c_0/e$.

That is when $t=1/k$.

This is called the characteristic time or mean life. Just like half-life but replace 2 with $e$ (could be called $1/e$-life).