MATH 102 - MIDTERM TEST 2

University of British Columbia

Last name (print): First name (print):

ID number:

Section number:

Date: *November 4, 2014.*
Time: *6:00 p.m. to 7:00 p.m.*
Number of pages: *8 (including cover page)*
Exam type: *Closed book*
Aids: *No calculators or other electronic aids*

Rules governing formal examinations:

Each candidate must be prepared to produce, upon request, a UBC card for identification.

Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:

- *Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/record-ers/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;*
- *Speaking or communicating with other candidates;*
- *Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.*

Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

For examiners' use only		
Section	Mark	Possible marks
MC		6
SAP		16
LAP 1		7
LAP 2		14
LAP 3		7
Total		50

Multiple choice (MC)

No partial points will be given for work shown.

1. The tangent line to the function $f(x) = \ln(x)$ at the point $(x_0, \ln(x_0))$ . . .

   (a) . . . lies above the function for all values of $x_0 > 0$.
   (b) . . . lies below the function for all values of $x_0 > 0$.
   (c) . . . lies above the function for $x_0 < 1$ and below the function for $0 < x_0 < 1$.
   (d) . . . lies above the function for $x_0 < 0$ and below the function for $x_0 > 0$.

2. A 99 bus leaves the UBC Bus Loop full of passengers. Along the route, people get on the bus at a constant rate and get off the bus at a rate proportional to the number of people on the bus. Which of the following Initial Value Problems is the best model for how many people are on the bus counted as a fraction of the total number that can fit on the bus? Assume that a and b are both positive constants and that $b > a$.

   (a) $\dfrac{dP}{dt} = a + bP, \quad P(0) = 0$
   (b) $\dfrac{dP}{dt} = a - bP, \quad P(0) = 0$
   (c) $\dfrac{dP}{dt} = -a + bP, \quad P(0) = 0$
   (d) $\dfrac{dP}{dt} = a + bP, \quad P(0) = 1$
   (e) $\dfrac{dP}{dt} = a - bP, \quad P(0) = 1$
   (f) $\dfrac{dP}{dt} = -a + bP, \quad P(0) = 1$

3. Which of the following is a true statement about a function $f(x)$? Assume that $f(x)$ has well-defined first and second derivatives and that c is a constant.

   (a) If $f'(c) = 0$, then $f(x)$ has either a local maximum or a local minimum at $x = c$.
   (b) If $f'(c) = 0$ and $f''(c) = 0$, then $f(x)$ has an inflection point at $x = c$.
   (c) If $f'(x) < 0$ for all x in the interval $(0, 1)$, then $f(x)$ is decreasing on $(0, 1)$.
   (d) If $f(x)$ and $g(x)$ are increasing on an interval I, then $f(x)g(x)$ is increasing on I.
   (e) More than one of the above options are correct.

Enter your answers to these questions here:

MC.1 [**2 pts**]	MC.2 [**2 pts**]	MC.3 [**2 pts**]
a	e	c

Short-answer problems

A correct answer in the box will get full points. Partial marks might be given for work shown.

1. **[2 pt]** Calculate the derivative of $f(x) = \pi^x + x^\pi$.

$$f'(x) = \pi^x \ln(\pi) + \pi x^{\pi-1}$$

1 1

$f'(x) =$ $\boxed{\pi^x \ln(\pi) + \pi x^{\pi-1}}$

2. **[3 pt]** Suppose that $f(x) = \ln\left(e^{g(x)} + x\right)$ and that $g(2) = 3$, $g'(2) = 5$. Compute $f'(2)$.

$$f'(x) = \frac{1}{e^{g(x)} + x} \cdot \left(e^{g(x)} g'(x) + 1\right)$$

1 1

$$f'(2) = \frac{1}{e^3 + 2} \cdot \left(5e^3 + 1\right)$$

1

$f'(2) =$ $\boxed{\frac{5e^3+1}{e^3+2}}$

3. **[3 pt]** Oil is leaking out of a cargo ship at the rate of 1 m^3/hr, forming a circular patch on the surface of the water (an "oil slick"). The radius $r(t)$ of the oil slick increases while its thickness, $\tau = 0.01$ m, is constant. Find the rate of change of the radius, at the moment when $r = 10$ m. Reminder: The volume of a cylinder is $V = \pi r^2 h$.

$$V(t) = \pi r(t)^2 \tau$$

$$1 m^3/hr = V'(t) = 2\pi\tau\, r(t) r'(t)$$

1

$$r'(t) = \frac{1}{2\pi\tau r(t)}$$

1

$$r'(t_0) = \frac{1}{2\pi \frac{1}{100} 10} = \frac{5}{\pi} \; m/hr$$

1

$\frac{dr}{dt} =$ $\boxed{\frac{5}{\pi} \text{ m/hr}}$

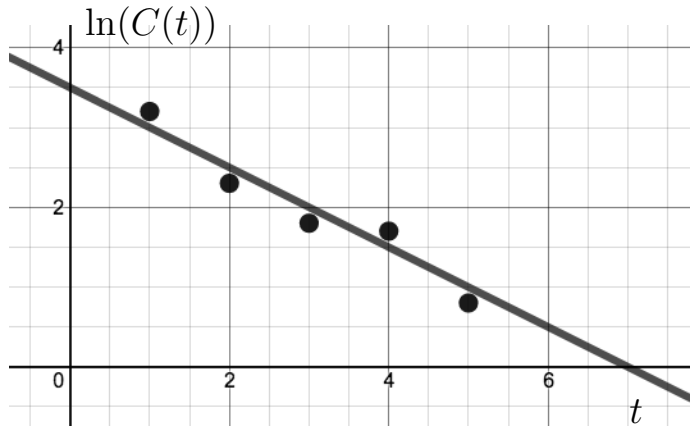
4. **[3 pt]** The figure below shows $\ln(C(t))$ plotted against t as the best fit line through the data points. What is $C(t)$? Note that the straight line in the figure goes through the points (1,3) and (3,2).

$\ln(C(t)) = A + Bt$

$A = 3.5$ 1

$B = -\frac{1}{2}$ 1

$C(t) = e^{A+Bt} = e^{3.5} e^{-\frac{1}{2}t}$ 1



$C(t) =$ $e^{3.5}e^{-\frac{1}{2}t}$

5. **[5 pt]** Find the point(s) at which the tangent line(s) to the ellipse defined by $x^2 + xy + y^2 = 9$ are parallel to the line $y = x + 3$.

$2x + y + xy' + 2yy' = 0$ 1

$y' = \frac{-2x-y}{x+2y} = 1$ 1

$-2x - y = x + 2y$

$3y = -3x$

$y = -x$ 1

$x^2 - x^2 + x^2 = 9$

$x = \pm 3$ 1

$y = \mp 3$

$(3,-3), (-3,3)$ 1

List points in the form (x_1, y_1): (3,-3),(-3,3)

Long-Answer Problems

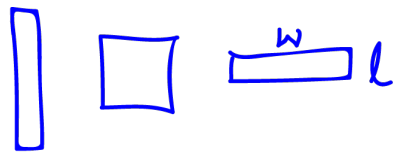
In this section, you must show all work or reasoning necessary for justifying your answers.

1. **[7 pt]** A farmer is building a rectangular enclosure for a petting zoo. She wants the enclosure to have an area of 100 m^2 and as large a perimeter as possible. For the comfort of the animals, both the width and the length of the enclosure should be no less then to 2 m. What is the maximum possible perimeter?

$P = 2w + 2l$

$A = wl = 100$ 1

$l = \frac{100}{w}$



$P(w) = 2\left(w + \frac{100}{w}\right)$ 1

$w \geq 2$ 1, $l \geq 2 \Rightarrow w \leq 50$ 1

$P'(w) = 2\left(1 - \frac{100}{w^2}\right) = 0 \Rightarrow w = 10$ 1

$P(2) = 2(2+50) = 104$ 1 max

$P(50) = 2(50+2) = 104$

$P(10) = 2(10+10) = 40$ 1 (or show $P''(10) > 0$)

The absolute max of $\boxed{104\text{m}}$ occurs at the extremes ($w=2$ and $w=5$).

$P =$ 104

2. **[14 pt]** Sketch the graph of the function $f(x) = xe^{-x^2}$. All zeros, minima, maxima and inflection points should be calculated and labeled on the graph. Note that

$$f'(x) = (1-2x^2)e^{-x^2} \quad \text{and} \quad f''(x) = 2x(2x^2-3)e^{-x^2}.$$

zeros: $xe^{-x^2} = 0$ at $x = 0$. 1

crit. pts: $1 - 2x^2 = 0$ at $x = \pm\frac{1}{\sqrt{2}} \approx \pm 0.7$ 1

potential IPs: $2x(2x^2-3) = 0$ at $x = 0, \pm\sqrt{\frac{3}{2}} \approx 1.2$ 2

Check crit pts: 1 $f''(\frac{1}{\sqrt{2}}) = \frac{2}{\sqrt{2}}(1-3)e^{-\frac{1}{2}} < 0$ so $\frac{1}{\sqrt{2}}$ is a max 1

$f''(-\frac{1}{\sqrt{2}}) = -\frac{2}{\sqrt{2}}(1-3)e^{-\frac{1}{2}} > 0$ so $-\frac{1}{\sqrt{2}}$ is a min 1

Check IPs: $f''(x) = 2x(\sqrt{2}x - \sqrt{3})(\sqrt{2}x + \sqrt{3})e^{-x^2}$

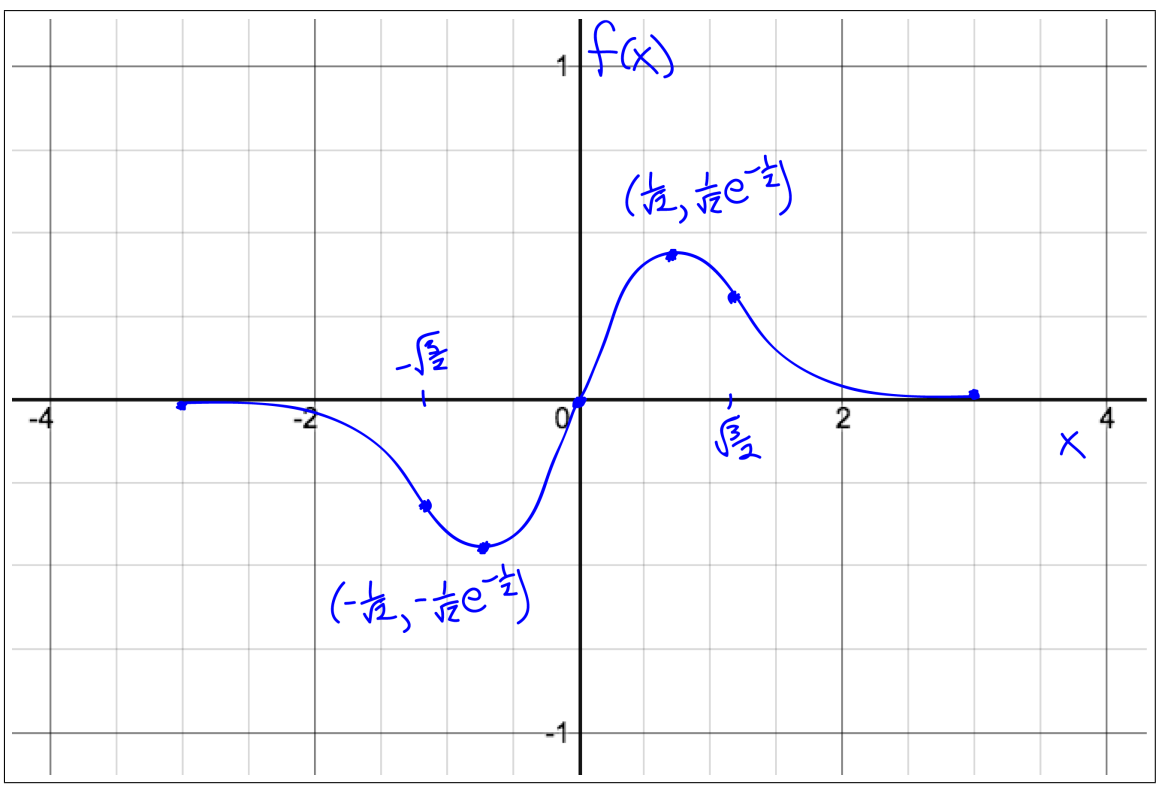
3

		$-\sqrt{\frac{3}{2}}$		0		$\sqrt{\frac{3}{2}}$	
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$

The sign changes at each one so all of them are IPs.

$f(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}e^{-\frac{1}{2}} \approx 0.7 \cdot 0.6 = 0.42$, 1 $f(-\frac{1}{\sqrt{2}}) \approx -0.42$

Graph points: 2 pts for the overall shape. 1 pt for denoting the key points with a dot (or otherwise) and labelling the min and max. The heights of the IPs are not required.



Note: it may be useful to know that $\sqrt{3} \approx 1.7$, $\frac{1}{\sqrt{2}} \approx 0.7$ and $e^{-\frac{1}{2}} \approx 0.6$.

3. **[7 pt]** The prevalence of Ebola has been increasing in Africa. Starting with 10 cases on March 1, 2014 ($t = 0$), it has increased to 12,000 cases 200 days later. Let $E(t)$ be the total number of cases at time t.

(a) Using one clearly worded sentence, explain the model

$$\frac{dE}{dt} = kE.$$

The rate of change of the number of cases of Ebola is proportional to the number of cases. (2 pts.)

(b) Using the information provided, determine the value of the constant k. (Your answer will contain a natural logarithm and can be left in that form.)

The solution is $E(t) = 10e^{kt}$. (2 pts.)
To find k, set $E(200) = 12000$ so that $12000 = 10e^{200k}$. (1 pt.)
Solving for k gives us

$$k = \frac{1}{200}\ln(1200). \text{ (1 pt.)}$$

(c) What is the doubling time of the disease according to this model?

Set $E(t_{\text{double}}) = 10e^{kt_{\text{double}}} = 20$ and find

$$t_{\text{double}} = \frac{\ln(2)}{k} = 200\frac{\ln(2)}{\ln(1200)}. \text{ (1 pt.)}$$

This page may be used for rough work. It will not be marked.