Exercises for Math 102

University of British Columbia
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8. Differential equations & Trigonometric functions

Exercise 1: Suppose a patient with a bacteria infection is treated with antibiotics. Assume that the reproduction rate of the bacteria per hour is 0.3 and their mortality rate per hour is $0.5 \times A^2$, where A is the amount of antibiotics (in ml). Find how much of antibiotics is necessary to treat this patient so that the bacteria population be reduced to 1% within 48 hours.
Note: instead of decimals, use expressions like $\ln 2$, $\sqrt{e}$.

Exercise 2: Let $f(t) = 2\sin\left(\frac{\pi}{6}(t-7)\right)$. Find the period of this function and find the value of $f(20000)$.
Note: do not use a calculator.

Exercise 3: Which differential equation does the function $y = e^{2x} + 1$ satisfy?

(a) $y' = 2y - 2$, (b) $y' = 2y + 2$, (c) $y' = 2y$, (d) $y' = 2x$.

Exercise 4: Find the derivative of $e^{-\frac{2}{x^2+1}}$.

Exercise 5: Let $a > 0$. Given a constant C with $0 < C < 1$, consider the function

$$y = f(t) = \frac{Ce^{at}}{(1-C) + Ce^{at}}, \quad -\infty < t < \infty.$$

(a) Show that $y = f(t)$ satisfies the differential equation

$$y' = ay[1-y]$$

and interpret the constant C in terms of $f(0)$.

(b) Find the critical points and inflection points of $f(t)$, if any.

(c) Analyze the asymptotic behavior of $f(t)$ as $t \to \infty$ and as $t \to -\infty$.

(d) Use your answers in (b) and (c) to sketch the graph of $f(t)$ for $C = 1/2$ *with asymptotes.*

Exercise 6: Find the 2011-th derivative of $\sin x$:

$$\frac{d^{2011}}{dx^{2011}} \sin(x).$$

Exercise 7: A wheel of radius 5m is rolling on the ground towards the right so that its trace on the ground grows at a constant speed 2m/second. Assume that the wheel is placed on the x-axis and touches $(0, 0)$ at some marked point P on the wheel at time 0. Also, let

(i) $P(t) = (x(t), y(t))$ be the coordinate of the marked point at time t,

(ii) C be the center of the wheel,

(iii) G be the point on the ground which touches the wheel at time t,

(iv) $\phi(t)$ be the angle which the two rays $\overrightarrow{CP}$ and $\overrightarrow{CG}$ make measured *clockwise*.

The distance D of the point P from the origin is given by $D^2 = x^2(t) + y^2(t)$. Find the rate of change of D^2, i.e. $\frac{d}{dt}[x^2(t) + y^2(t)]$, and express your answer in terms of $\phi(t)$.

Exercise 8: Differentiate the following functions:

(a) $y = \sin^2 x + 10 \tan x$

(b) $g(t) = e^{t^3 \cos t}$

(c) $y = \sin(\tan(\sqrt{\sin x}))$

Exercise 9: An observer watches a rocket launch vertically off the ground from a distance of 2km away. The angle of elevation θ is increasing at 3 degrees per second at the instant when $\theta = 45^o$. How fast is the rocket climbing at that instant?

Exercise 10: State whether each function is odd, even or neither.

(a) $f(x) = \sin x$

(b) $f(x) = \tan x$

(c) $f(x) = \frac{\sin x}{x}$

(d) $f(x) = \sin^2 x$

Exercise 11: The temperature of a lake varies sinusoidally over the year. It is hottest on August 1st when its temperature is 20°C. It is coldest on February 13th when its temperature is 4°C. Write the temperature of the lake, $T(t)$ where t is days since the start of the year.