Lecture 32 (Nov. 22, 2013)

Learning Goals: ① Effect of initial guess
② Qualitative methods for differential equations

- (Continue) Newton's method

Example 1: Estimate the root of $\sin(x)=0$ by ① $x_0=4$ ② $x_0=4.4$

① $x_0=4$. $f(x_0)=\sin(4)$, $f'(x_0)=\cos(4)$

$$x_1 = x_0 - \frac{\sin(x_0)}{\cos(x_0)} = 4-\tan(4) \approx 2.842178$$

$$x_2 = x_1 - \frac{\sin(x_1)}{\cos(x_1)} \approx 3.150873$$

$$x_3 = x_2 - \tan(x_2) \approx 3.141592 \quad \rightarrow \text{approach } \pi \approx 3.141592$$

② $x_0 = 4.4$

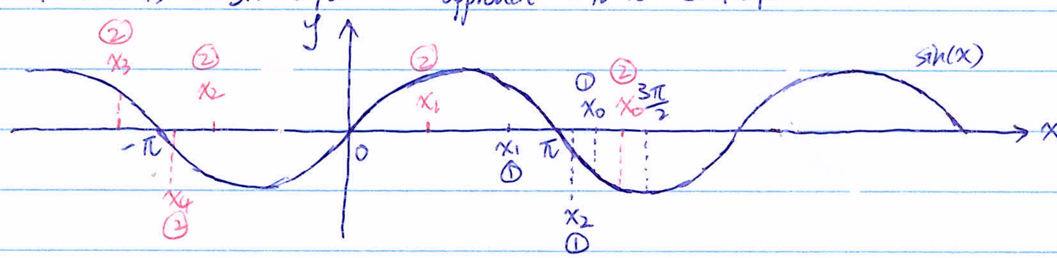
$$x_1 = x_0 - \tan(x_0) \approx 1.303676$$

$$x_2 = x_1 - \tan(x_1) \approx -2.350491$$

$$x_3 = x_2 - \tan(x_2) \approx -3.361963$$

$$x_4 = x_3 - \tan(x_3) \approx -3.137955$$

$$x_5 = x_4 - \tan(x_4) \approx -3.141592 \quad \rightarrow \text{approach } -\pi \approx -3.141592$$



Notice: ① To make Newton's method work, $f'(x_{k-1}) \neq 0$ for $k=1,2,3\cdots$
② Different initial guess may end up with different roots.

- Qualitative methods

① $\frac{dy}{dt} = ay+b$, derive the DE based on some word description
find the solution $y(t)$ analytically

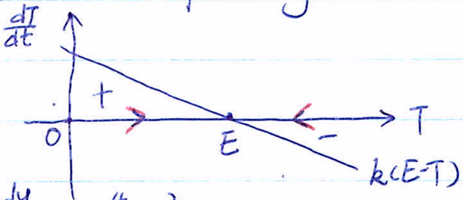
② $\frac{dy}{dt} = f(y)$, $f(y)$ - nonlinear. Euler's method
Qualitative methods to understand the behaviour of $y(t)$

(i) Sketch $\frac{dy}{dt}$ vs. y: $\frac{dy}{dt}$ - rate of change of $y(t)$

the sign indicates that $y(t)$ increases or decreases

$f(y)$ - a function of y (implicitly related to time)

Example 2: Newton's law of cooling $\frac{dT}{dt} = k(E-T)$, $k>0$

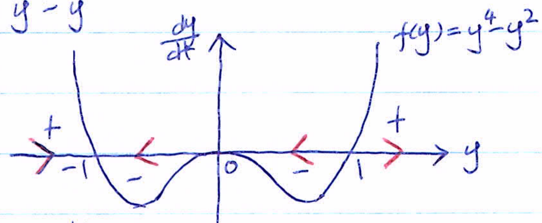


$T<E$, $\frac{dT}{dt} > 0$, $\Rightarrow$ $T(t)$ increases

$T>E$, $\frac{dT}{dt} < 0$, $\Rightarrow$ $T(t)$ decreases

$T=0$, $\frac{dT}{dt} = 0$, $\Rightarrow$ no change in $T(t)$

Example 3: $\frac{dy}{dt} = y^4 - y^2$



$y<-1$, $\frac{dy}{dt} > 0$ $\Rightarrow$ $y(t)$ increases

$-1<y<0$, $\frac{dy}{dt} < 0$ $\Rightarrow$ $y(t)$ decreases

$0<y<1$, $\frac{dy}{dt} < 0$ $\Rightarrow$ $y(t)$ decreases

$y>1$, $\frac{dy}{dt} > 0$ $\Rightarrow$ $y(t)$ increases

use an arrow on y-axis in each interval indicating the trend of $y(t)$

$y=-1, 0, 1$, $\frac{dy}{dt} = 0$ $\Rightarrow$ no change in $y(t)$

(ii) Steady State: a state in which a system is not changing

i.e. $\frac{dy}{dt} = f(y^*) = 0$, find y^*

Stability: The steady state is stable if states that are initially close to it will get closer with time.

The steady state is unstable if states that are initially close to it will move away from it over time.

(Continue) Example 2: Steady state T^* satisfies $k(E-T^*)=0$ $\Rightarrow$ $T^*=E$

arrows in the sketch indicate that $T^*=E$ is stable

(Continue) Example 3: Steady states y^* satisfies $y^{*4} - y^{*2} = 0$ $\Rightarrow$ $y^* = -1, 0, 1$

arrows in the sketch indicates that

$y^*=-1$ is stable.

$y^*=0$ is half stable

$y^*=1$ is unstable