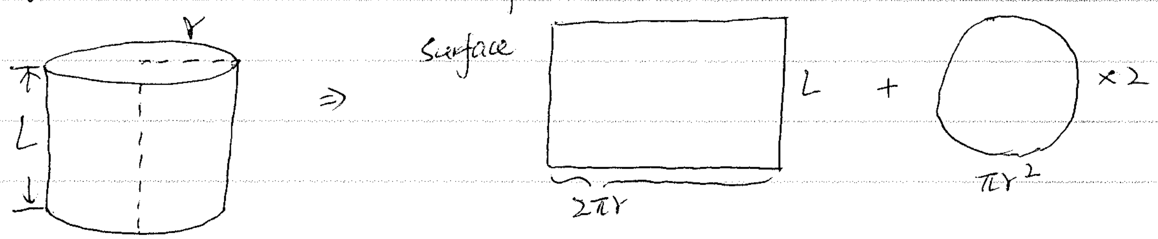
Lecture 14 (Oct. 07, 2013)

Learning Goals: Solve optimization problems

Example 1: Consider a cell shaped like a cylinder with fixed volume. Find the ratio of length to radius to minimize the surface area.



Volume $V = \pi r^2 \cdot L$ is fixed (V - constant)

Surface area $S = 2\pi r \cdot L + \pi r^2 \cdot 2$ (as a function of two unknowns)

Transform $S(r, L)$ to be a function of one variable as $S(L)$ or $S(r)$

Plug in $L = \frac{V}{\pi r^2}$ or $r = \sqrt{\frac{V}{\pi L}}$ (can try by yourself)

$\Rightarrow S(r) = 2\pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}$

$\Rightarrow S'(r) = 4\pi r + (-\frac{2V}{r^2}) = 0 \Rightarrow r = (\frac{V}{2\pi})^{1/3}$ as a critical point

Find the domain for r as $r > 0 \Leftrightarrow r \in (0, +\infty)$

$\lim_{r \to 0^+} S(r) = +\infty$, $\lim_{r \to +\infty} S(r) = +\infty$

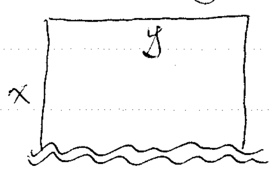
$S(\sqrt[3]{\frac{V}{2\pi}}) = 2\pi (\frac{V}{2\pi})^{2/3} + 2V \cdot (\frac{V}{2\pi})^{-1/3}$ = finite $\Rightarrow$ global minimum (* don't stop here)

To find the ratio of length to radius, $L = \frac{V}{\pi r^2} = \frac{V}{\pi} \cdot (\frac{V}{2\pi})^{-2/3} = (\frac{4V}{\pi})^{1/3}$

Then ratio $\frac{L}{r} = \frac{(\frac{4V}{\pi})^{1/3}}{(\frac{V}{2\pi})^{1/3}} = 8^{1/3} = 2$

- Procedures to Solve Optimization Problems:
  (1) Read the problem carefully (unknown and known "pieces", quantity that needs to be optimized)
  (2) introduce variables and list the information in math. language (geometric knowledge)
  (3) draw a picture to help for (2)
  (4) write the function of only one variable
  (5) Find global extrema, find final answer to the problem

Example 2: A rectangular shaped farmland will be bounded on one side by a river and on the other three sizes by a elect. fence. With 800 meters of wire at your disposal, what is the largest area you can enclose and what are the dimensions?



Total length of the fence $L = 2x + y = 800$

Arear of the farmland $S = xy = x(800 - 2x) = 800x - 2x^2$

$\Rightarrow S'(x) = 800 - 4x = 0 \Rightarrow x = 200$ as critical point

Domain $0 < x < 400$:

$$\left.\begin{array}{l} S(200) = 200 \cdot 400 = 8 \times 10^4 \\ \lim_{x \to 0^+} S(x) = 0, \quad \lim_{x \to 400^-} S(x) = 0 \end{array}\right\} \Rightarrow S(200) \text{ is the global maximum}$$

$\Rightarrow$ Dimensions as width $x = 200$ and length $y = 400$ (meters)