Lecture 13 (Oct. 04, 2013)

Learning Goal: (1) Logistic growth in population

(2) solve optimization problem

- Logistic Growth: the growth rate of a population depends on the density of the population.

G - growth rate

N - density of the population

$G(N) = rN \cdot (\frac{K-N}{N})$

$r > 0$, intrinsic growth rate

$K > 0$, carrying capacity

Example 1: Given the logistic growth function, find N that leads to the maximal growth rate

$$G'(N) = [rN \cdot (\frac{K-N}{K})]' = [rN(1-\frac{N}{K})]' = [rN - \frac{r}{K}N^2]' = r - \frac{2r}{K} \cdot N = 0$$

$\Rightarrow$ critical point $N = \frac{K}{2}$

Compare $G(\frac{K}{2}) = r \cdot \frac{K}{2} \cdot \frac{1}{2} = \frac{1}{4} rK$

with the function's behavior when N approaches to the boundary of the domain

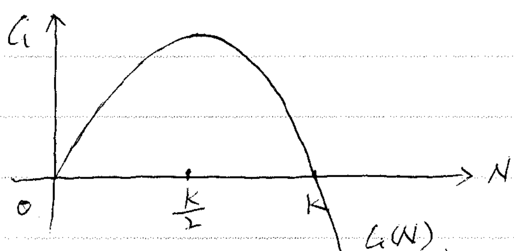
Information about the domain is "buried" in the biological background, as the density of the population is expected to be non-negative $\Rightarrow N \geq 0$

Then $G(0) = 0$

$\lim_{N \to +\infty} G(N) = -\infty$

$\Rightarrow$ at $N = \frac{K}{2}$, $G(N)$ arrives the maximum

Sketch the logistic growth rate:



Notice that when $N < K$, $G(N) > 0$ indicates the density of the population increases

when $N > K$, $G(N) < 0$ indicates the density of the population decreases

- Optimization Problem in applications of Biology and Life Science

Example 2: A spherical shaped cell with absorption rate $A(r) = 4\pi k_1 r^2$ and consumption rate $C(r) = \frac{4}{3}\pi k_2 r^3$. Define the net rate of increase $N(r) = A(r) - C(r) = 4\pi k_1 r^2 - \frac{4}{3}\pi k_2 r^3$

Find the size of the cell to maximize the net rate of increase

$N'(r) = 8k_1\pi r - 4k_2\pi r^2 = 4\pi r(2k_1 - k_2 r) = 0$

$\Rightarrow$ critical points : $r = 0$ or $r = \frac{2k_1}{k_2}$

Then $N(0) = 0$

$N\left(\frac{2k_1}{k_2}\right) = 4\pi k_1 \left(\frac{2k_1}{k_2}\right)^2 - \frac{4}{3}\pi k_2 \left(\frac{2k_1}{k_2}\right)^3 = \frac{16}{3}\pi \cdot \frac{k_1^3}{k_2^2} > 0$

The domain is defined by the size of the cell that a cell can exist : $0 < r \leq \frac{3k_1}{k_2}$

Then $N\left(\frac{3k_1}{k_2}\right) = 0$

$\lim_{r \to 0} N(r) = 0$

$\Rightarrow r = \frac{2k_1}{k_2}$ can maximize the net rate of increase