Lecture 20 (Oct. 23, 2013)

Learning Goals: ① relationship between (natural) exponential function and (natural) logarithm
② properties of natural logarithm (including derivative)

· Natural exponential function: $y = f(x) = e^x$ with $\frac{dy}{dx} = f'(x) = e^x$

Properties: ① $e^b \cdot e^c = e^{b+c}$, $b, c \in \mathbb{R}$

② $(e^b)^c = e^{b \cdot c}$

③ $f(0) = 1$ & $f(1) = e$

④ $e^x \to 0$ for $x \to -\infty$; $e^x \to +\infty$ for $x \to +\infty$

⑤ e^x is continuous and differentiable for all $x \in \mathbb{R}$

⑥ $e^x > 0$ for all $x \in \mathbb{R}$

· Natural Logarithm: $y = f(x) = e^x$ | $y = f(x) = a^x$, $a > 0$

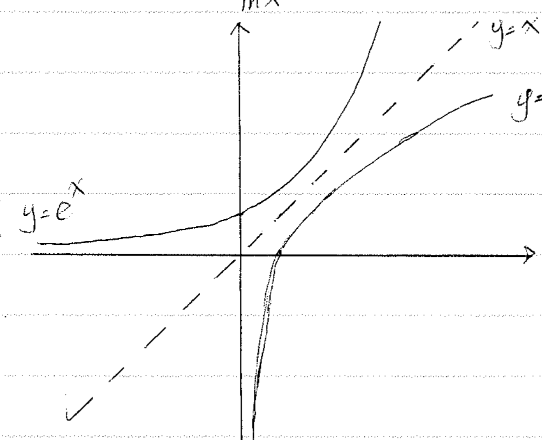
$\boxed{y = f^{-1}(x) = \ln x \quad \Updownarrow \quad x = e^y}$ | $y = f^{-1}(x) = \log_a x$

$\Rightarrow \underbrace{y = \ln(e^y)}, \underbrace{x = e^{\ln x}}$

used to change any variable/constant into log/exponent form

sketch: The graph of $f^{-1}(x)$ is a reflection of $f(x)$ about the line $y = x$

" " " $\ln x$ " " " " e^x " " " "



* use Geometric transformation to sketch
$y = e^{x+2}$ or $y = \ln(2x)$ or $y = e^x - 5$ ..
(Review Appendix C.B in LKN)

Properties. ① $\ln(a \cdot b) = \ln a + \ln b$, $a, b > 0$

② $\ln(a^b) = b \cdot \ln a$

③ Domain & Range: $x > 0$, $y \in \mathbb{R}$

④ $\ln x \to -\infty$ for $x \to 0$; $\ln x \to +\infty$ for $x \to +\infty$

⑤ $\ln x$ is continuous and differentiable for all $x > 0$

Derivative of $y = \ln x$: $\Rightarrow$ same as $e^y = x$

use implicit differentiation $e^y \cdot \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$

Example 1: Find the derivative of $y = a^x$, $a > 0$

Transform the function to be a composite exponential function by $a = e^{\ln a}$

$\Rightarrow y = a^x = (e^{\ln a})^x = e^{x \cdot \ln a}$, $*$ $\ln a$ is a constant

$\Rightarrow \frac{dy}{dx} = e^{x \cdot \ln a} \cdot (x \ln a)' = e^{x \cdot \ln a} \cdot \ln a = \ln a \cdot a^x$

Extra: Question (optional): Find the derivative of $y = \log_a x$

Similar to finding the derivative of $y = \ln x$

$y = \log_a x \Leftrightarrow x = a^y$

apply implicit differentiation: $1 = \ln a \cdot a^y \cdot \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\ln a \cdot a^y} = \frac{1}{(\ln a) x}$

Example 2: Find the derivative of $y = \ln\left(\frac{e^x}{e^x+1}\right)$

Assume $u = \frac{e^x}{e^x+1}$ then $y = \ln u$, and $\frac{du}{dx} = \frac{e^x(e^x+1) - e^x \cdot e^x}{(e^x+1)^2} = \frac{e^x}{(e^x+1)^2}$

then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{e^x}{(e^x+1)^2} = \frac{e^x+1}{e^x} \cdot \frac{e^x}{(e^x+1)^2} = \frac{1}{e^x+1}$

Example 3: Find the derivative of $y = x^x$

#1 transform the function to be a composite exponential function

$y = x^x = (e^{\ln x})^x = e^{x \cdot \ln x}$

$\frac{dy}{dx} = e^{x \cdot \ln x} \cdot (x \cdot \ln x)' = e^{x \ln x} (\ln x + x \cdot \frac{1}{x}) = x^x (\ln x + 1)$

#2 take logarithm on both sides, $\ln y = x \cdot \ln x$

apply implicit differentiation $\frac{1}{y} \cdot \frac{dy}{dx} = \ln x + x \cdot \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = y \cdot (\ln x + 1) = x^x (\ln x + 1)$