M 103-Lecture January 28, 2016

Contents:

From Rates Of Change To Total Change

In this section, we examine several examples in which the rate of change of some process is specified. We use this information to obtain the total change that occurs over some time period.

Ex: Birth rates:

After WW II, the birth rate in western countries increased dramatically. Suppose that the number of babies born (in millions per year) was given by

$$b(t) = 5 + 2t \quad 0 \leq t \leq 10;$$

where t is time in years after the end of the war.

(a) How many babies in total were born during this time period?

(b) Find the time t_0 such that the total number of babies born from the end of the war up to the time t_0 was precisely 14 million.

Solution: Let us introduce the function $B(t)$ which gives the total number of babies born (in millions) from the end of the war up to time t, where $0 < t \leq 10$. From the information given notice that $B'(t) = b(t)$, since $b(t)$ describes the rate at which babies are born during this period. Furthermore, notice that $B(0) = 0$ since we do not consider babies born before the end of the war (i.e., when $t < 0$). By the FTC, we thus have

$$\int_0^t b(x)dx = B(t) - B(0) = B(t),$$

or in words: The number of babies born up to time t is given by the sum of (infinitesimal) changes in baby population during the time interval.

For (a), notice we are being asked for the value $B(10)$. Therefore, the total number of babies is given as

$$B(10) = \int_0^{10} b(t)dt = \int_0^{10} (5+2t)dt = 5t + t^2|_0^{10} = 50 + 100 = 150 \text{ million babies.}$$

For part (b), we need to find t_0 such that $B(t_0) = 14$. In terms of the definite integral, this means

$$B(t_0) = \int_0^{t_0} b(t)dt = 14 \text{ million babies.}$$

So by the FTC, the definite integral becomes

$$14 = \int_0^{t_0} b(t)dt = \int_0^{t_0} (5+2t)dt = 5t + t^2|_0^{t_0} = 5t_0 + t_0^2.$$

So we are left to solve for t_0 such that $t_0^2 + 5t_0 - 14 = 0$. Since $t_0^2 + 5t_0 - 14 = (t_0+7)(t_0-2)$, this quadratic equation has solutions $t_0 = 2$ and $t_0 = -7$. Since $t_0 = -7$ is not in our domain of interest (this was during or before the war), we obtain that after $t_0 = 2$ years there wer 14 million babies born.

Production and Removal

In some processes, two or more independent factors can affect the net change of the process in a given time interval. Such a situation is similar in concept to the previous example, but we must consider such independent factors separately in order to obtain the net change of the process.

Ex: A new shoe factory monitors its shoe production rate in its first year of operation. Each undefective shoe produced gives the company a $8\$$ profit. The rate at which undefectice shoes are produced (or production rate) is given by

$$p(t) = -\frac{(t-3)^2}{4} + 8 \text{ thousand shoes per month } 0 \le t \le 12.$$

However, in the production process many defective shoes are produced and must be discarded. Each defective shoe costs the company $4\$$ dollars. The rate at which faulty shoes are produced (or removal rate) during this time period is given by

$$r(t) = \frac{(t-2)^2}{4} \text{ thousand shoes per month } 0 \le t \le 12.$$

(a) How much money did the company make/loose during the first year of production?

(b) At what time did the company begin to loose money?

Solution: Notice that in this process, the production rate benefits the company and the removal rate harms the company, so both of these components need to be considered. To this end, let us write the expressions that give the profit and losses of the company. Since the profit is 8 dollars times the total number of undefective shoes produced, we need to compute the total number of undefective shoes produced. Let $P(t)$ be the function that gives the number of undefective shoes produced up to time t. Then notice that $\frac{dP}{dt} = p(t)$ since $p(t)$ gives the rate of production and $P(0) = 0$. By the FTC

$$\int_0^t p(x)dx = P(t) - P(0) = P(t).$$

Therefore the total profit during the year is given by $8P(12)$.

Similarly, the loss is 4 dollars times the total number of defectice shoes produced. If we let $R(t)$ be the function that gives the number of defective shoes produced up to time t, then as before

$$\int_0^t r(x)dx = R(t) - R(0) = R(t).$$

Therefore the total loss during the year is given by $4R(12)$.

To see how much money the company made/lost during the first year of production we must compute the profit minus the loss:

$$8P(12) - 4R(12) = 8\int_0^t p(x)dx - 4\int_0^t r(x)dx = \int_0^t (8p(x) - 4r(x))dx$$

$$\int_0^t \left(8(-\frac{(x-3)^2}{4} + 8) - 4(\frac{(x-2)^2}{4})\right) dx$$

$$= \int_0^t \left(-3x^2 + 16x + 42\right) dx = -x^3 + 8x^2 + 42x|_0^{12}$$

$$= -12^3 + 8 \cdot 12^2 + 42 \cdot 12 = -72 \text{ thousand dollars.}$$

So the company ended up loosing money.

For part (b), notice that as before the total money made by the company up to time t is given by $8P(t) - 4R(t)$. Therefore we must find t_0 such that $8P(t_0) - 4R(t_0) = 0$. Solving as before gives

$$0 = 8P(t_0) - 4R(t_0) = -x^3 + 8x^2 + 42x|_0^{t_0} = -t_0^3 + 8t_0^2 + 42t_0.$$

Solving for roots of this quadratic equation $0 = -t_0^3 + 8t_0^2 + 42t_0$, we obtain $t_0 = -3.6158, 11.6158$. Since $t_0 = -3.6158$ is not in our interval of interest, we see that the company began to loose money after $t = 11.6158$ (around December 19th?).

Average Value

In many situations, the average value of a function in a given interval is an important quantity to measure. We have the following definition:

Definition: The average value of $f(x)$ over the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Notice, this is just the sum of all the function values in the interval (given by the definite integral) divided by the length of the interval. Let us consider a simple example

Ex: Find the average value of the function $f(x) = x^2$ over the interval $2 < x < 4$.

As defined, this is given by

$$\bar{f} = \frac{1}{4-2}\int_2^4 x^2 dx = \frac{1}{2}[\frac{1}{3}x^3]_2^4 = \frac{1}{2}\frac{4^3}{3} - \frac{2^3}{3} = \frac{28}{3}.$$

Now let us consider a more interesting example.

Ex: Data collected twice per day, suggests that the temperature at a UBC weather station is given by (in °Celcius)

$$T(t) = 10\cos(t) + 10\sin(t/100) + 15 \qquad 1 \le t \le 730$$

where summer corresponds to $50 \le t \le 250$ and winter corresponds to $400 \le t \le 600$. What is the average temperature in summer and winter?

To answer this we use our definition of average value in the indicated intervals. Let $\bar{f}_s$, $\bar{f}_w$ be the average value of the temperature in summer and winter respectively. Then

$$\bar{f}_s = \frac{1}{200}\int_{50}^{250}(10\cos(t) + 10\sin(t/100) + 15)dt = \frac{1}{200}[10\sin(t) - 1000\cos(t/100) + 15t]_{50}^{250} = 23.25$$

°Celcius, and

$$\bar{f}_w = \frac{1}{200}\int_{400}^{600}(10\cos(t) + 10\sin(t/100) + 15)dt = \frac{1}{200}[10\sin(t) - 1000\cos(t/100) + 15t]_{400}^{600} = 6.9$$

°Celcius.