M 103-Lecture January 26, 2016

Contents:

1) Applications Of The Definite Integral To Velocities And Rates
2) From rates of change to total change

We now focus on applications of the definite integral to practical problems. We begin by showing that the notion of antiderivatives and integrals allows us to deduce details of the motion of an object from underlying Laws of Motion.

Displacement, velocity and acceleration

Let $x(t)$ describe the position of some particle at time t, $v(t)$ its velocity, and $a(t)$ the acceleration. From the study of derivatives, the following relationships hold:

$$\frac{dx}{dt} = v(t)$$

and

$$\frac{dv}{dt} = a(t)$$

This means that position is an anti-derivative of velocity and velocity is an anti-derivative of acceleration. Therefore, applying the FTC in some interval $T_1 \leq t \leq T_2$

$$\int_{T_1}^{T_2} v(t)dt = x(T_2) - x(T_1).$$

The quantity on the right hand side is a displacement, i.e., the difference between the position at time T_1 and the position at time T_2.

Similarly, since velocity is the antiderivative of acceleration, the FTC gives

$$\int_{T_1}^{T_2} a(t)dt = v(T_2) - v(T_1).$$

This last term is the net change in velocity (we don't have a special term to describe this quantity).

Ex: The acceleration in m/s^2 and initial velocity of a particle moving along a line is given by

$$a(t) = -2t - 12, \quad v(0) = -15, t \leq 10.$$

Find (a) an expression for the velocity of the particle at time T and (b) the total distance traveled by the particle.

Warning: In these types of problems, we must be careful to distinguish between "displacement of position" and "distance traveled". What is the difference between these two concepts? When do these values coincide?

Sol: For (a) we may use the expression above for velocity in terms of acceleration to obtain

$$v(T) - v(0) = v(T) + 15 = \int_0^T a(t)dt = \int_0^T (-2t - 12)dt = [-t^2 - 12t]_0^T = -T^2 - 12T.$$

So that $v(T) = -T^2 - 12T - 15$ m/s.

For part (b), notice that the particle begins with negative velocity and continues to have negative velocity (from the expression we just obtained). Therefore the total distace traveled is given by $|x(10) - x(0)|$ (in this case coincides with the absolute displacement since the particle travels in only one direction). We use our expression for position in terms of velocity and the equation of velocity we just obtained to write

$$\text{Distance traveled} = |x(10) - x(0)| = \left|\int_0^{10} v(t)dt\right| = \left|\int_0^{10} (-t^2 - 12t - 15)dt\right| = \left|-\frac{1}{3}t^3 - 6t^2 - 15t|_0^{10}\right|$$

$$\left|-\frac{1}{3}10^3 - 600 - 150\right| = 1083.3$$

meters.

Constant velocity

Let us look at the simplest case when we have uniform motion, i.e., the velocity is constant and for simplicity let $T_1 = 0$ and let us denote the final time as T. Then $v(t) = v$ (where v is a constant) and notice $a = \frac{dv}{dt} = 0$. As before we have

$$\int_0^T v(t)dt = x(T) - x(0),$$

but on the other hand, using our known constant velocity

$$\int_0^T v(t)dt = \int_0^T vdt = [ct]_0^T = vT.$$

Therefore, we must have

$$vT = x(T) - x(0),$$

or

$$x(T) = x(0) + vT,$$

for any given time T. Thus, in this case we have a nice expression for the position of the particle at any desired time in terms of the initial position ($x(0)$) and the constant velocity. Notice this is a linear relation in terms of t.

Constant acceleration

Let us now consider a slightly more complicated case, when instead the accerelation is constant $a(t) = a$ (where a is a constant). Similar to the previous case we obtain

$$\int_0^T a(t)dt = v(T) - v(0),$$

but on the other hand, using our known constant velocity

$$\int_0^T a(t)dt = \int_0^T adt = [at]_0^T = aT.$$

Therefore, we must have

$$aT = v(T) - v(0),$$

or

$$v(T) = v(0) + aT.$$

Applying this result and the FTC we obtain

$$\int_0^T v(t)dt = \int_0^T (v(0) + at)dt = [v(0)t + \frac{a}{2}t^2]_0^T = v(0)T + \frac{a}{2}T^2.$$

On the other hand, as before

$$\int_0^T v(t)dt = x(T) - x(0),$$

so that

$$x(T) = x(0) + v(0)T + \frac{a}{2}T^2.$$

This expression represents the position of a particle at time T given that it experienced a constant acceleration. The initial velocity $v(0)$, initial position $x(0)$ and acceleration a allowed us to predict the position of the object $x(t)$ at any later time t.

Ex: A car braked with a constant acceleration of -16 ft/s^2, producing skid marks of lenght 200 ft. How fast was the car traveling when the brakes where applied?

To answer this question let us look at our derived expressions

$$v(t) = v(0) + at$$

and

$$x(t) = x(0) + v(0)t + \frac{a}{2}t^2.$$

Let $t = 0$ be the time when the car hit the brakes and denote by t^* as the point in time when the car came to a stop. Notice that $a = -16$, $v(t^*) = 0$ and $x(t^*) - x(0) = 200$ (this is the total displacement of the car), we obtain from the velocity expression above

$$0 = v(t^*) = v(0) + at^* = v(0) - 16t^* \Rightarrow v(0) = 16t^*$$

and we obtain from the position expression above

$$200 = x(t^*) - x(0) = v(0)t^* + \frac{a}{2}t^{*2} = v(0)t^* - 8t^{*2} \Rightarrow 200 = v(0)t^* - 8t^{*2}.$$

Substituting $v(0)$ from the first expression into the second expression gives

$$200 = v(0)t^* - 8t^{*2} = (16t^*)t^* - 8t^{*2} = 8t^{*2}$$

so that $t^* = 5$ s, and so

$$v(0) = 16t^* = 80.$$

ft/s.