Presentations of sequences

The sequences \( \{a_n\}_{n \geq 1} \) that we will study in this course are presented in one of three ways:

1. In list form: \( \{0, 1, -1, 2, -2, 3, -3, \ldots\} \)
2. Closed form, where \( a_n = f(n) \): \( \{\cos(n\pi)\}_{n \geq 1} = \{-1, 1/2, -1/3, 1/4, \ldots\} \).
3. Recursively, for example \( a_n = 2 - a_{n-1}/2 + a_{n-2}, a_0 = a_1 = 1 \) leads to \( \{1, 1/2, 7/4, 21/8, \ldots\} \).

Exercises:

1. Write the following sequences (a) in closed form and (b) recursively.
   
a) \( \{4, 7, 10, 13, 16, 19, \ldots\} \)  
b) \( \{0, 2, 5, 9, 14, 20, \ldots\} \)  
c) \( \{0, 1/2, 3/4, 7/8, 15/16, \ldots\} \)  
d) \( \{2, 6, 18, 54, \ldots\} \)  
e) \( \{-1, 2, -3, 4, -5, \ldots\} \)

2. Use sin(n) or cos(n) to write \( \{0, 1/2, 0, -1/4, 0, 1/8, 0, 1/16, \ldots\} \) in closed form. For a challenge write it recursively.

3. Find a simpler closed form expression for \( \{\cos(n\pi)\}_{n \geq 1} \).

4. Write the following recursively defined sequences in closed form. Note, this is not possible for every recursively defined sequence!
   
a) \( a_n = -19, a_n = a_{n-1} + 6 \).  
b) \( a_0 = 3, a_n = 2 - a_n \).
   
c) \( x_1 = 5, x_2 = 15, x_n = 2x_{n-1} + 3x_{n-2} \).  
d) \( b_0 = 7, b_n = b_{n-1}/2 \)

Convergence of Sequences

We have two convergence tests that work for all presentations of sequences:

**The Squeeze Theorem:** Let \( \{a_n\}_{n \geq 1}, \{b_n\}_{n \geq 1} \) and \( \{c_n\}_{n \geq 1} \) be sequences such that \( a_n \leq b_n \leq c_n \) for each integer \( n \geq 1 \). If \( a_n \) and \( c_n \) converge to the same limit \( L \) then \( b_n \) converges to \( L \).

Exercises:

5. Use the Squeeze Theorem to show each sequences converges:
   
a) \( \left\{ \frac{\sin(n) + 1}{n^2 + 1} \right\}_{n=1}^\infty \)  
b) \( \left\{ \frac{1}{n!} \right\}_{n=1}^\infty \)  
c) \( \left\{ \frac{(-2)^{-n}}{n + 1} \right\}_{n=1}^\infty \)  
d) \( a_0 = 4, a_n = a_{n-1}/n \)
Monotonicity: A sequence \( \{a_n\}_{n \geq 1} \) is monotone increasing or increasing monotonically if \( a_{n+1} \geq a_n \) for each integer \( n \geq 1 \). We say that it is monotone decreasing or deceasing monotonically if \( a_{n+1} \leq a_n \) for each integer \( n \geq 1 \).

Monotone and Bounded: A sequence is bounded if \( |a_n| \leq M \) for all integers \( n \geq 0 \). If a sequence is bounded and monotone it has a limit.

Exercises:

6. Determine which of the following sequences is (a) monotone, and if so if it is increasing or decreasing and (b) bounded.

   a) \( \left\{ \frac{\ln(n)}{n} \right\}_{n=1}^{\infty} \)
   
   b) \( \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \)
   
   c) \( a_0 = 1, a_1 = 1, a_n = 2a_{n-1} - a_{n-2} \)
   
   d) \( \left\{ \frac{\sin(n)}{n} \right\}_{n=2}^{\infty} \)
   
   e) \( \left\{ \frac{\sin(\pi n)}{n} \right\}_{n=2}^{\infty} \)
   
   f) \( a_0 = 1, a_n = n^2 - a_{n-1} \)

Sequences expressed in closed form

For sequences expressed in closed form we can use the tools we’ve developed for smooth functions: If \( a_n = f(n) \) and \( \lim_{x \to \infty} f(x) = L \), then \( a_n \) converges to \( L \). In particular, one can often use L’Hopital’s Rule to determine \( \lim_{x \to \infty} f(x) \). In addition, if \( \lim_{x \to \infty} f(x) = \pm \infty \) then \( a_n = f(n) \) diverges.

Exercises:

7. Find the limits of the following sequences.

   a) \( \{n \sin(1/n)\}_{n=1}^{\infty} \)
   
   b) \( \left\{ \frac{(2n - 7)^2}{n^3 - 4n + 1} \right\}_{n=1}^{\infty} \)
   
   c) \( \{n \ln n\}_{n=1}^{\infty} \)
   
   d) \( \left\{ \frac{2^n - n}{2^{n+1} + n} \right\}_{n=1}^{\infty} \)

8. It is possible for \( f(x) \) to have no limit but for \( a_n = f(n) \) to still converge. Take for example \( a_n = f(n) \) where \( f(x) = x^2 \sin(\pi x) \). Show that \( a_n \) is convergent even though \( f(x) \) does not have a limit.

Limits of recursive sequences

If a sequence is recursively defined we another tactic for determining it’s limits. Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence defined recursively by \( a_n = f(a_{n-1}) \). Then the only possible limits of \( \lim_{n \to \infty} a_n = a \) are the solutions to \( a = f(a) \). Note, for different starting values \( a_0 \), different limits will be reached. It is possible that for certain starting values there is no limit.

Exercises:

9. Find the limits of the following recursively defined sequence:

   a) \( a_0 = 1 \) and \( a_n = \sqrt{6 + a_{n-1}} \)
   
   b) \( a_0 = 2 \) and \( a_n = \frac{1}{4} a_{n-1} + \frac{3}{4} \)
   
   c) \( a_0 = 4 \) and \( a_n = \sqrt{3a_{n-1}} \)
   
   d) \( a_0 = \frac{3}{2} \) and \( a_n = a_{n-1}^2 - 2a_{n-1} + 2 \)
Hints:

1. (a) Look at the difference between each term
   (b) Write the recursive form first, then use use the formulas from Chapter 1 to find the closed form expression.
   (c) Think about how the element of this sequence differ from 1.
   (d) Try comparing the terms using addition and multiplication
2. Remember that \( \cos(n\pi/2) = 0, \pm 1 \) depending on the value of \( n \).
3. Write out the first couple of terms, what do you find?
4. Write out the first few terms of each sequence, for (b) using something like \((-1)^n\) might be useful.
5. (a) We know that \( 0 < \sin(x) + 1 < 2 \), so how can we bound the bottom?
   (b) Try bounding by a power of 2?
   (c) Remember that you have to find a lower bound for this sequence as well
   (d) Write this sequence in closed form
6. (a) Think about the first terms as well as the later ones
   (b) Write out a few terms to get an idea of what this sequence is doing
   (c) Think about the definition of monotone and bounded
   (d) What are reasonable bounds, if not the closest?
   (e) What are the terms of this sequence?
7. (a) Use L’Hopitals Rule
   (b) Find the highest power in the numerator and denominator
   (c)
   (d) When \( n \) is large, which terms will dominate the numerator and denominator?
8. Work out the first few terms of the sequence and find a rule
9. Find the possible limits and classify them as stable or unstable. A cobweb can help you here.
Answers:

1. 

   a) \( \{1 + 3n\}_{n=1}^{\infty} \), \( a_1 = 4 \), \( a_n = 3 + a_{n-1} \)  
   b) \( \{-1 + n(n+1)/2\}_{n=1}^{\infty} \), \( a_1 = 0 \), \( a_n = n + a_{n-1} \)  
   c) \( \left\{1 - \frac{1}{2^n}\right\}_{n=0}^{\infty} \), \( a_0 = 0 \), \( a_n = \frac{1}{2^n} + a_{n-1} \)  
   d) \( \{2 \cdot 3^n\}_{n=0}^{\infty} \), \( a_0 = 2 \), \( a_n = 3a_{n-1} \)  
   e) \( \{(−1)^n \cdot n\}_{n=0}^{\infty} \), \( a_1 = -1 \), \( a_n = (−1)^n \cdot (1 + 2n) + a_{n-1} \)

2. \( \left\{\frac{\sin(n\pi/2)}{n}\right\}_{n=1}^{\infty} \)

3. \( \left\{(\frac{2}{7})^n\right\}_{n=1}^{\infty} \)

4. 

   a) \( \{-19 + 6n\}_{n=0}^{\infty} \)  
   b) \( \left\{\frac{3}{2} + (-1)^n \frac{3}{2}\right\}_{n=0}^{\infty} \)  
   c) \( \{(3)^n \cdot 5\}_{n=0}^{\infty} \)  
   d) \( \left\{\frac{7}{2^n}\right\}_{n=0}^{\infty} \)

5. 

   a) \( 0 \leq \frac{\sin(n) + 1}{n^2 + 1} \leq \frac{2}{n} \)  
   b) \( 0 \leq \frac{1}{n!} \leq \frac{1}{n^2} \)  
   c) \( -\frac{1}{2^n} \leq \frac{(-2)^{-n}}{n + 1} \leq \frac{1}{2^n} \)  
   d) \( 0 \leq \frac{4}{n!} \leq \frac{4}{n} \)

6. (a) Not monotone. Bounded below by 0 and above by 1.  
   (b) Monotone increasing and bounded below by 0 and above by 1.  
   (c) The sequence is constant, so it’s both monotone increasing and decrease. It is bounded above and below by 1.  
   (d) Not monotone, but bounded below by 1 and above by -1 for example. Note, these are not the most exact bounds, but they are sufficient to say the sequence is bounded.  
   (e) Constant, so monotone and bounded.  
   (f) Neither monotone or bounded.

7. 

   a) 1  
   b) 0  
   c) \( \infty \)  
   d) \( \frac{1}{2} \)

8. The sequence is constantly 0 and so converges trivially.

9. (a) Fixed point: \( a = 3 \) is stable, \( a_{\infty} = 3 \). Note: \( a = -2 \) is not a fixed point since it satisfies the equation \( a = -\sqrt{6 + a} \), not the equation \( a = \sqrt{6 + a} \). It would be a fixed point of the recursively defined sequence \( a_n = -\sqrt{6 + a_{n-1}} \).
(b) Fixed point: $a = 1$ is stable, $a_\infty = 1$.

(c) Fixed points: $a = 0$ is unstable, $a = 3$ is stable. Sequence is monotone and bounded and converges to $a_\infty = 3$.

(d) Fixed points: $a = 1$ is stable, $a = 2$ is unstable. Starting at $a_0 = 3/2$ the sequence will limit to $a = 1$. 